ON THE DEGREE OF CONVERGENCE OF BOREL AND EULER MEANS FOR DOUBLE FOURIER SERIES OF FUNCTIONS OF BOUNDED VARIATION IN HARDY SENSE

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Summary. For real functions of bounded variation in the Hardy sense, 2π-periodic in each variable, the rates of pointwise convergence of the Borel and Euler means of their Fourier series are estimated.

Keywords: double trigonometric series, Borel means, Euler means, rate of convergence

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Let f be a function defined in the rectangle $P = \langle a, b; c, d\rangle$. We shall use the notation

$$\Delta(f; P') = f(b', d') - f(a', d') - f(b', c') + f(a', c')$$

for any rectangle $P' = \langle a', b'; c', d'\rangle \subset P$.

We will call the *variation of f in the rectangle P* the value $V(f; a, b; c, d)$ defined as the supremum of all numbers of the form

$$\sum_{j=0}^{m-1} \sum_{k=0}^{n-1} |\Delta(f; P_{jk})|$$

where the supremum is taken over all possible partitions π,

$$\pi\colon \begin{cases} a = x_0 < x_1 < x_2 < \ldots < x_m = b \\ c = y_0 < y_1 < y_2 < \ldots < y_n = d \end{cases}$$

of the rectangle P into subrectangles $P_{jk} = \langle x_j, x_{j+1}; y_k, y_{k+1}\rangle$.

We say that f is a function of bounded variation in Hardy sense in the rectangle P if

$$V(f;a,b;c,d)<\infty, \quad V\big(f(\cdot,c);a,b\big)<\infty \quad \text{and} \quad V\big(f(a,\cdot);c,d\big)<\infty,$$

where $V\big(f(\cdot,c);a,b\big)$ denotes the total variation of f with respect to the first variable on the interval $\langle a,b\rangle$ for some fixed $y=c$. The variation $V\big(f(a,\cdot);c,d\big)$ is defined analogously.

Let H be the class of all real-valued functions of two variables, 2π-periodic in each variable, of bounded variation in Hardy sense in the square $Q=\langle 0,2\pi;0,2\pi\rangle$.

For every $f\in H$ the following properties
(1) $V\big(f(\cdot,y);0,2\pi\big)<\infty$ for all $y\in\langle 0,2\pi\rangle$,
(2) $V\big(f(x,\cdot);0,2\pi\big)<\infty$ for all $x\in\langle 0,2\pi\rangle$
can be easily deduced.

The following two lemmas will be of importance in our further considerations.

Lemma 1. *If $f\in H$, $0<x<\bar{x}\leqslant\pi$ and $0<y<\bar{y}\leqslant\pi$, then*
(1) $V\big(f(x,\cdot);y,\pi\big)\leqslant V(f;0,x;y,\pi)+V\big(f(0,\cdot);y,\pi\big)$,
(1′) $V\big(f(\cdot,y);x,\pi\big)\leqslant V(f;x,\pi;0,y)+V\big(f(\cdot,0);x,\pi\big)$,
(2) $V\big(f(x,\cdot);y,\bar{y}\big)\leqslant V(f;0,x;y,\bar{y})+V\big(f(0,\cdot);y,\bar{y}\big)$,
(2′) $V\big(f(\cdot,y);x,\bar{x}\big)\leqslant V(f;x,\bar{x};0,y)+V\big(f(\cdot,0);x,\bar{x}\big)$,
(3) $V\big(f(\cdot,y);0,x\big)\leqslant V(f;0,x;0,y)+V\big(f(\cdot,0);0,x\big)$,
(3′) $V\big(f(x,\cdot);0,y\big)\leqslant V(f;0,x;0,y)+V\big(f(0,\cdot);0,y\big)$.

Lemma 2. *Suppose that $f\in H$ and m, n are the positive integers. Then*
(1) $V\big(f;0,\frac{\pi}{m};0,\frac{\pi}{n}\big)\leqslant\frac{1}{mn}\sum\limits_{k=1}^{m}\sum\limits_{l=1}^{n}V\big(f;0,\frac{\pi}{k};0,\frac{\pi}{l}\big)$,
(2) $V\big(f(\cdot,0);0,\frac{\pi}{m}\big)\leqslant\frac{1}{m}\sum\limits_{k=1}^{m}V\big(f(\cdot,0);0,\frac{\pi}{k}\big)$,
(3) $V\big(f(0,\cdot);0,\frac{\pi}{n}\big)\leqslant\frac{1}{n}\sum\limits_{k=1}^{n}V\big(f(0,\cdot);0,\frac{\pi}{k}\big)$,
(4) $\sum\limits_{k=1}^{n}V\big(f;0,\frac{\pi}{m};0,\frac{\pi}{k}\big)\leqslant\frac{1}{m}\sum\limits_{k=1}^{n}\sum\limits_{l=1}^{m}V\big(f;0,\frac{\pi}{l};0,\frac{\pi}{k}\big)$,
(5) $\sum\limits_{k=1}^{m}V\big(f;0,\frac{\pi}{k};0,\frac{\pi}{n}\big)\leqslant\frac{1}{n}\sum\limits_{k=1}^{m}\sum\limits_{l=1}^{n}V\big(f;0,\frac{\pi}{k};0,\frac{\pi}{l}\big)$.

The proofs of Lemmas 1 and 2 are omitted.

Remark 1. If $f\in H$ is continuous at the point $(0,0)$, then

$$\lim_{s,t\to 0+}V(f;0,s;0,t)=0.$$

To prove this remark it is convenient to apply Lemma 2 of [4] (with $p=1$) and its two-dimensional analogue.

Let $f \in H$. Then the partial sums $S_{mn}[f]$ of the Fourier series of f have the form

$$S_{mn}[f](x,y) = \frac{1}{\pi^2} \iint_Q f(x+u, y+v) D_m(u) D_n(v) \, \mathrm{d}u \, \mathrm{d}v,$$

where $D_k(t)$ is the Dirichlet kernel, i.e.

$$D_k(t) = \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{1}{2}t}.$$

As is well known,

$$|D_k(t)| \leqslant k + \tfrac{1}{2}, \quad k = 0, 1, 2, \dots .$$

The next theorem is a two-dimensional analogue of the theorem of Bojanić ([1]).

Theorem 1. *For any $f \in H$ and all positive integers m, n, we have*

$$\begin{aligned}(1) \qquad &\Big| S_{mn}[f](x,y) - \tfrac{1}{4}\big(f(x+,y+) + f(x-,y+) + f(x+,y-) + f(x-,y-)\big)\Big| \\ &\leqslant \frac{13}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} V\big(g; 0, \tfrac{\pi}{k}; 0, \tfrac{\pi}{l}\big) + \frac{6}{n} \sum_{k=1}^{n} V\big(g(0,\cdot); 0, \tfrac{\pi}{k}\big) + \frac{6}{m} \sum_{k=1}^{m} V\big(g(\cdot,0); 0, \tfrac{\pi}{k}\big),\end{aligned}$$

where $g(s,t) = g_{xy}(s,t)$ is a function which is defined by the following conditions:

(i) *for $s,t \neq 0$, $g_{xy}(s,t) = f(x+s,y+t) + f(x+s,y-t) + f(x-s,y+t) + f(x-s,y-t) - f(x+,y+) - f(x+,y-) - f(x-,y+) - f(x-,y-)$,*

(ii) *for $s = t = 0$, $g_{xy}(s,t) = 0$,*

(iii) *for $s = 0$ and $t \neq 0$, $g_{xy}(s,t) = f(x+,y+t) + f(x+,y-t) + f(x-,y+t) + f(x-,y-t) - f(x+,y+) - f(x+,y-) - f(x-,y+) - f(x-,y-)$,*

(iv) *for $s \neq 0$ and $t = 0$, $g_{xy}(s,t) = f(x+s,y+) + f(x+s,y-) + f(x-s,y+) + f(x-s,y-) - f(x+,y+) - f(x+,y-) - f(x-,y+) - f(x-,y-)$.*

P r o o f. Denote by I the difference from the left-hand side of the inequality (1). To estimate I we split the square $\langle 0,\pi; 0,\pi\rangle$ into four parts as follows:

$$\begin{aligned} I &= \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} g(s,t) D_m(s) D_n(t) \, \mathrm{d}s \, \mathrm{d}t \\ &= \frac{1}{\pi^2} \Big(\int_0^{\frac{\pi}{m}} \int_0^{\frac{\pi}{n}} + \int_0^{\frac{\pi}{m}} \int_{\frac{\pi}{n}}^{\pi} + \int_{\frac{\pi}{m}}^{\pi} \int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{m}}^{\pi} \int_{\frac{\pi}{n}}^{\pi} \Big) g(s,t) D_m(s) D_n(t) \, \mathrm{d}s \, \mathrm{d}t \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

First we consider I_1,

$$(2)\quad |I_1| \leqslant \frac{1}{\pi^2}\int_0^{\frac{\pi}{m}}\int_0^{\frac{\pi}{n}} \Big\{|g(s,t)-g(s,0)-g(0,t)+g(0,0)|+|g(s,0)-g(0,0)|$$
$$+|g(0,t)-g(0,0)|\Big\}|D_m(s)||D_n(t)|\,\mathrm{d}s\,\mathrm{d}t$$
$$\leqslant \frac{1}{\pi^2}\int_0^{\frac{\pi}{m}}\int_0^{\frac{\pi}{n}} \Big\{V(g;0,s;0,t)+V\big(g(\cdot,0);0,s\big)$$
$$+V\big(g(0,\cdot);0,t\big)\Big\}\big(m+\tfrac{1}{2}\big)\big(n+\tfrac{1}{2}\big)\,\mathrm{d}s\,\mathrm{d}t$$
$$\leqslant \frac{9}{4}\Big\{V\big(g;0,\tfrac{\pi}{m};0,\tfrac{\pi}{n}\big)+V\big(g(\cdot,0);0,\tfrac{\pi}{m}\big)+V\big(g(0,\cdot);0,\tfrac{\pi}{n}\big)\Big\}.$$

Now, let

$$\Lambda_{mn}(x,y)=\int_0^x\int_y^{\pi} D_m(s)D_n(t)\,\mathrm{d}s\,\mathrm{d}t.$$

Clearly,

$$(3)\qquad |\Lambda_{mn}(x,y)| \leqslant \big(m+\tfrac{1}{2}\big)x\frac{\pi}{ny}$$

(see [1], p. 59). Making use of Theorems 2.5 and 2.3 of [5] it is easy to see that I_2 may be written by means of the Stieltjes integral in the form

$$I_2=-\frac{1}{\pi^2}\int_0^{\frac{\pi}{m}}\int_{\frac{\pi}{n}}^{\pi} g(s,t)\,\mathrm{d}\Lambda_{mn}(s,t).$$

Integrating by parts (Theorem 2.2 of [5]), we obtain

$$I_2 = -\frac{1}{\pi^2}\int_0^{\frac{\pi}{m}}\int_{\frac{\pi}{n}}^{\pi}\Lambda_{mn}(s,t)\,\mathrm{d}g(s,t)+\frac{1}{\pi^2}\int_0^{\frac{\pi}{m}}\Lambda_{mn}\big(s,\tfrac{\pi}{n}\big)\,\mathrm{d}g\big(s,\tfrac{\pi}{n}\big)$$
$$+\frac{1}{\pi^2}\int_{\frac{\pi}{n}}^{\pi}\Lambda_{mn}\big(\tfrac{\pi}{m},t\big)\,\mathrm{d}g\big(\tfrac{\pi}{m},t\big)+\frac{1}{\pi^2}\Lambda_{mn}\big(\tfrac{\pi}{m},\tfrac{\pi}{n}\big)g\big(\tfrac{\pi}{m},\tfrac{\pi}{n}\big)$$
$$=A_1+A_2+A_3+A_4.$$

Now, let us remark that in view of (3) the inequality

$$(4)\quad |A_4|\leqslant \frac{3}{2\pi}\big|g\big(\tfrac{\pi}{m},\tfrac{\pi}{n}\big)\big| \leqslant \frac{3}{2\pi}\Big\{V\big(g;0,\tfrac{\pi}{m};0,\tfrac{\pi}{n}\big)+V\big(g(\cdot,0);0,\tfrac{\pi}{m}\big)+V\big(g(0,\cdot);0,\tfrac{\pi}{n}\big)\Big\}$$

holds.

Applying (3) and integrating by parts, we get

$$
\begin{aligned}
|A_3| &\leqslant \frac{3}{2n}\int_{\frac{\pi}{n}}^{\pi}\frac{1}{t}\,\mathrm{d}V\Big(g\big(\tfrac{\pi}{m},\cdot\big);\tfrac{\pi}{n},t\Big)\\
&\leqslant \frac{3}{2n\pi}V\Big(g\big(\tfrac{\pi}{m},\cdot\big);\tfrac{\pi}{n},\pi\Big)+\frac{3}{2n}\int_{\frac{\pi}{n}}^{\pi}\frac{1}{t^2}V\Big(g\big(\tfrac{\pi}{m},\cdot\big);\tfrac{\pi}{n},t\Big)\,\mathrm{d}t.
\end{aligned}
$$

Replacing the variable t in the last integral by $\frac{\pi}{u}$, we observe that

$$
\int_{\frac{\pi}{n}}^{\pi}\frac{1}{t^2}V\Big(g\big(\tfrac{\pi}{m},\cdot\big);\tfrac{\pi}{n},t\Big)\,\mathrm{d}t\leqslant\frac{1}{\pi}\sum_{k=1}^{n-1}V\Big(g\big(\tfrac{\pi}{m},\cdot\big);\tfrac{\pi}{n},\tfrac{\pi}{k}\Big).
$$

According to Lemma 1, we have

$$
\begin{aligned}
|A_3| &\leqslant \frac{1}{2n\pi}V\big(g;0,\tfrac{\pi}{m};\tfrac{\pi}{n},\pi\big)+\frac{3}{2n\pi}V\big(g(0,\cdot);\tfrac{\pi}{n},\pi\big)\\
&\quad+\frac{3}{2n\pi}\sum_{k=1}^{n}V\big(g;0,\tfrac{\pi}{m};\tfrac{\pi}{n},\tfrac{\pi}{k}\big)\\
&\leqslant \frac{3}{n\pi}\sum_{k=1}^{n}V\big(g;0,\tfrac{\pi}{m};\tfrac{\pi}{n},\tfrac{\pi}{k}\big)+\frac{3}{n\pi}\sum_{k=1}^{n}V\big(g(0,\cdot);\tfrac{\pi}{n},\tfrac{\pi}{k}\big).
\end{aligned}
\tag{5}
$$

Now let us estimate A_2. Using (3) and partial integration, we find that

$$
|A_2|\leqslant\frac{1}{\pi^2}\big(m+\tfrac{1}{2}\big)\int_0^{\frac{\pi}{m}}s\,\mathrm{d}V\Big(g\big(\cdot,\tfrac{\pi}{n}\big);0,s\Big)\leqslant\frac{3}{\pi}V\Big(g\big(\cdot,\tfrac{\pi}{n}\big);0,\tfrac{\pi}{m}\Big).
$$

Moreover, the inequality (3) of Lemma 1 implies

$$
|A_2|\leqslant\frac{3}{\pi}V\big(g;0,\tfrac{\pi}{m};0,\tfrac{\pi}{n}\big)+\frac{3}{\pi}V\Big(g\big(\cdot,0\big);0,\tfrac{\pi}{m}\Big).
\tag{6}
$$

In order to estimate A_1 we apply (3) and then integration by parts. Then

$$
\begin{aligned}
|A_1| &\leqslant \frac{1}{\pi n}\big(m+\tfrac{1}{2}\big)\int_0^{\frac{\pi}{m}}\int_{\frac{\pi}{n}}^{\pi}\frac{s}{t}\,\mathrm{d}V\big(g;0,s;\tfrac{\pi}{n},t\big)\\
&=\frac{1}{\pi n}\big(m+\tfrac{1}{2}\big)\Big\{\int_0^{\frac{\pi}{m}}\int_{\frac{\pi}{n}}^{\pi}V\big(g;0,s;\tfrac{\pi}{n},t\big)\,\mathrm{d}\big(\tfrac{s}{t}\big)-\int_0^{\frac{\pi}{m}}V\big(g;0,s;\tfrac{\pi}{n},\pi\big)\,\mathrm{d}\big(\tfrac{s}{\pi}\big)\\
&\quad-\int_{\frac{\pi}{n}}^{\pi}V\big(g;0,\tfrac{\pi}{m};\tfrac{\pi}{n},t\big)\,\mathrm{d}\big(\tfrac{\pi}{mt}\big)+\frac{1}{m}V\big(g;0,\tfrac{\pi}{m};\tfrac{\pi}{n},\pi\big)\Big\}.
\end{aligned}
$$

Since $\mathrm{d}\bigl(\frac{s}{t}\bigr) = -\,\mathrm{d}s\frac{1}{t^2}\,\mathrm{d}t$, we have

$$\Bigl|\int_0^{\frac{\pi}{m}}\int_{\frac{\pi}{n}}^{\pi} V\bigl(g;0,s;\tfrac{\pi}{n},t\bigr)\,\mathrm{d}\bigl(\tfrac{s}{t}\bigr)\Bigr|$$

$$= \Bigl|\int_0^{\frac{\pi}{m}}\mathrm{d}s\int_{\frac{\pi}{n}}^{\pi}\frac{1}{t^2}V\bigl(g;0,s;\tfrac{\pi}{n},t\bigr)\,\mathrm{d}t\Bigr| \leqslant \frac{1}{m}\sum_{k=1}^{n} V\bigl(g;0,\tfrac{\pi}{m};\tfrac{\pi}{n},\tfrac{\pi}{k}\bigr).$$

The last inequality was obtained by the substitution $t = \frac{\pi}{u}$. In a similar way we obtain

$$\Bigl|\int_{\frac{\pi}{n}}^{\pi} V\bigl(g;0,\tfrac{\pi}{m};\tfrac{\pi}{n},t\bigr)\,\mathrm{d}\bigl(\tfrac{1}{t}\bigr)\Bigr| \leqslant \frac{1}{\pi}\sum_{k=1}^{n} V\bigl(g;0,\tfrac{\pi}{m};\tfrac{\pi}{n},\tfrac{\pi}{k}\bigr).$$

Consequently,

$$|A_1| \leqslant \frac{6}{n\pi}\sum_{k=1}^{n} V\bigl(g;0,\tfrac{\pi}{m};\tfrac{\pi}{n},\tfrac{\pi}{k}\bigr). \tag{7}$$

Collecting (4), (5), (6) and (7) we get

$$|I_2| \leqslant \frac{9}{n\pi}\sum_{k=1}^{n} V\bigl(g;0,\tfrac{\pi}{m};\tfrac{\pi}{n},\tfrac{\pi}{k}\bigr) + \frac{9}{2\pi}V\bigl(g;0,\tfrac{\pi}{m};0,\tfrac{\pi}{n}\bigr) + \frac{9}{2\pi}V\bigl(g(\cdot,0);0,\tfrac{\pi}{m}\bigr)$$
$$+ \frac{3}{n\pi}\sum_{k=1}^{n} V\bigl(g(0,\cdot);\tfrac{\pi}{n},\tfrac{\pi}{k}\bigr) + \frac{3}{2\pi}V\bigl(g(0,\cdot);0,\tfrac{\pi}{n}\bigr).$$

Applying inequalities (1), (2) and (4) of Lemma 2, we obtain

$$|I_2| \leqslant \frac{27}{2\pi mn}\sum_{k=1}^{m}\sum_{l=1}^{n} V\bigl(g;0,\tfrac{\pi}{k};0,\tfrac{\pi}{l}\bigr) + \frac{9}{2n\pi}\sum_{k=1}^{n} V\bigl(g(0,\cdot);0,\tfrac{\pi}{k}\bigr)$$
$$+ \frac{9}{2\pi}V\bigl(g(\cdot,0);0,\tfrac{\pi}{m}\bigr). \tag{8}$$

Putting

$$\Lambda^*_{mn}(x,y) = \int_x^{\pi}\int_0^{y} D_m(s)D_n(t)\,\mathrm{d}s\,\mathrm{d}t$$

and reasoning as in the case of the integral I_2, we observe that

$$|I_3| \leqslant \frac{9}{m\pi}\sum_{k=1}^{m} V\bigl(g;\tfrac{\pi}{m},\tfrac{\pi}{k};0,\tfrac{\pi}{n}\bigr) + \frac{9}{2\pi}V\bigl(g;0,\tfrac{\pi}{m};0,\tfrac{\pi}{n}\bigr) + \frac{9}{2\pi}V\bigl(g(0,\cdot);0,\tfrac{\pi}{n}\bigr) \tag{9}$$
$$+ \frac{3}{m\pi}\sum_{k=1}^{m} V\bigl(g(\cdot,0);\tfrac{\pi}{m},\tfrac{\pi}{k}\bigr) + \frac{3}{2\pi}V\bigl(g(0,\cdot);0,\tfrac{\pi}{m}\bigr)$$
$$\leqslant \frac{27}{2\pi m}\sum_{k=1}^{m}\sum_{l=1}^{n} V\bigl(g;0,\tfrac{\pi}{k};0,\tfrac{\pi}{l}\bigr) + \frac{9}{2m\pi}\sum_{k=1}^{m} V\bigl(g(\cdot,0);0,\tfrac{\pi}{k}\bigr)$$
$$+ \frac{9}{2\pi}V\bigl(g(0,\cdot);0,\tfrac{\pi}{n}\bigr).$$

To estimate I_4 we use the function

$$\overline{\Lambda}_{mn}(x,y) = \int_x^{\pi}\int_y^{\pi} D_m(s)D_n(t)\,\mathrm{d}s\,\mathrm{d}t.$$

Let us remark that

$$|\overline{\Lambda}_{mn}(x,y)| \leqslant \frac{\pi^2}{mnxy} \tag{10}$$

and

$$I_4 = \frac{1}{\pi^2}\int_{\frac{\pi}{m}}^{\pi}\int_{\frac{\pi}{n}}^{\pi} g(s,t)\,\mathrm{d}\overline{\Lambda}_{mn}(s,t).$$

Integrating by parts, we get

$$\begin{aligned}
I_4 &= \frac{1}{\pi^2}\int_{\frac{\pi}{m}}^{\pi}\int_{\frac{\pi}{n}}^{\pi} \overline{\Lambda}_{mn}(s,t)\,\mathrm{d}g(s,t) + \frac{1}{\pi^2}\int_{\frac{\pi}{m}}^{\pi} \overline{\Lambda}_{mn}\big(s,\tfrac{\pi}{n})\big)\,\mathrm{d}g\big(s,\tfrac{\pi}{n}\big)\\
&\quad + \frac{1}{\pi^2}\int_{\frac{\pi}{n}}^{\pi} \overline{\Lambda}_{mn}\big(\tfrac{\pi}{m},t\big)\,\mathrm{d}g\big(\tfrac{\pi}{m},t\big) + \frac{1}{\pi^2}\overline{\Lambda}_{mn}\big(\tfrac{\pi}{m},\tfrac{\pi}{n}\big)g\big(\tfrac{\pi}{m},\tfrac{\pi}{n}\big)\\
&= H_1 + H_2 + H_3 + H_4.
\end{aligned}$$

The estimate (10) yields

$$|H_4| \leqslant \frac{1}{\pi^2}V\big(g;0,\tfrac{\pi}{m};0,\tfrac{\pi}{n}\big) + \frac{1}{\pi^2}V\big(g(\cdot,0);0,\tfrac{\pi}{m}\big) + \frac{1}{\pi^2}V\big(g(0,\cdot);0,\tfrac{\pi}{n}\big). \tag{11}$$

Reasoning analogously as in the case A_3, we obtain estimates

$$|H_3| \leqslant \frac{2}{n\pi^2}\sum_{k=1}^{n} V\big(g;0,\tfrac{\pi}{m};0,\tfrac{\pi}{k}\big) + \frac{2}{n\pi^2}\sum_{k=1}^{n} V\big(g(0,\cdot);\tfrac{\pi}{n},\tfrac{\pi}{k}\big) \tag{12}$$

and

$$|H_2| \leqslant \frac{2}{m\pi^2}\sum_{k=1}^{m} V\big(g;\tfrac{\pi}{m},\tfrac{\pi}{k};0,\tfrac{\pi}{n}\big) + \frac{2}{m\pi^2}\sum_{k=1}^{m} V\big(g(\cdot,0);\tfrac{\pi}{m},\tfrac{\pi}{k}\big). \tag{13}$$

Using the inequality (10), after partial integration we find

$$\begin{aligned}
|H_1| \leqslant \frac{1}{mn}\int_{\frac{\pi}{m}}^{\pi}\int_{\frac{\pi}{n}}^{\pi} V\big(g;\tfrac{\pi}{m},s;\tfrac{\pi}{n},t\big)\frac{1}{s^2t^2}\,\mathrm{d}s\,\mathrm{d}t + \frac{1}{mn\pi}\int_{\frac{\pi}{m}}^{\pi} V\big(g;\tfrac{\pi}{m},s;\tfrac{\pi}{n},\pi\big)\frac{1}{s^2}\,\mathrm{d}s\\
+ \frac{1}{mn\pi}\int_{\frac{\pi}{n}}^{\pi} V\big(g;\tfrac{\pi}{m},\pi;\tfrac{\pi}{n},t\big)\frac{1}{t^2}\,\mathrm{d}t + \frac{1}{mn\pi^2}V\big(g;\tfrac{\pi}{m},\pi;\tfrac{\pi}{n},\pi\big).
\end{aligned}$$

Integration by substitution with $s=\frac{\pi}{u}$ and $t=\frac{\pi}{v}$ gives

$$\begin{aligned}(14)\qquad |H_1| &\leqslant \frac{1}{mn\pi^2}\sum_{k=1}^{m}\sum_{l=1}^{n} V\big(g;\tfrac{\pi}{m},\tfrac{\pi}{k};\tfrac{\pi}{n},\tfrac{\pi}{l}\big)+\frac{1}{mn\pi^2}\sum_{k=1}^{m} V\big(g;\tfrac{\pi}{m},\tfrac{\pi}{k};\tfrac{\pi}{n},\pi\big)\\ &\quad+\frac{1}{mn^2}\sum_{k=1}^{n} V\big(g;\tfrac{\pi}{m},\pi;\tfrac{\pi}{n},\tfrac{\pi}{k}\big)+\frac{1}{mn\pi^2} V\big(g;\tfrac{\pi}{m},\pi;\tfrac{\pi}{n},\pi\big)\\ &\leqslant \frac{4}{mn\pi^2}\sum_{k=1}^{m}\sum_{l=1}^{n} V\big(g;\tfrac{\pi}{m},\tfrac{\pi}{k};\tfrac{\pi}{n},\tfrac{\pi}{l}\big).\end{aligned}$$

As a consequence of (11)–(14) and Lemma 2 we obtain the estimate

$$\begin{aligned}(15)\qquad |I_4| &\leqslant \frac{9}{mn\pi^2}\sum_{k=1}^{m}\sum_{l=1}^{n} V\big(g;0,\tfrac{\pi}{k};0,\tfrac{\pi}{l}\big)+\frac{3}{n\pi^2}\sum_{k=1}^{n} V\big(g(0,\cdot);0,\tfrac{\pi}{k}\big)\\ &\quad+\frac{3}{m\pi^2}\sum_{k=1}^{m} V\big(g(\cdot,0);0,\tfrac{\pi}{k}\big).\end{aligned}$$

Collecting the inequalities (2), (8), (9), (15) and using Lemma 2 once more, we obtain (1) and thus our proof is completed. □

R e m a r k 2. Continuity of the functions g at the point $(x,y)=(0,0)$ implies

$$\lim_{t\to 0+} V\big(g(0,\cdot);0,t\big)=0 \quad\text{and}\quad \lim_{s\to 0+} V\big(g(\cdot,0);0,s\big)=0.$$

Then by the well known theorem of Cauchy the second and the third term on the right side of inequality (1) converges to zero as $m,n\to\infty$. For function g the equality $\lim\limits_{s,t\to 0+} V(g;0,s;0,t)=0$ is true, too. Hence the first term on the right-hand side of (1) tends to zero when $m,n\to\infty$. We conclude that

$$\lim_{m,n\to\infty} S_{mn}[f](x,y)=\tfrac{1}{4}\big\{f(x+,y+)+f(x+,y-)+f(x-,y+)+f(x-,y-)\big\}.$$

Thus we arrive at a theorem concerning the convergence of partial sums of double Fourier series ([6], §176, p. 472).

Let us introduce the Euler mean $E^{pq}_{mn}[f]$ $(p,q>0)$ of the partial sums of double Fourier series of a function f defined in the square $Q=\langle-\pi,\pi;-\pi,\pi\rangle$ and 2π-periodic in each variable by the formula

$$E^{pq}_{mn}[f](x,y)=\frac{1}{(1+p)^m}\frac{1}{(1+q)^n}\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}p^{m-j}q^{n-k}S_{jk}[f](x,y).$$

In case $p = q = 0$ we obtain

$$E_{mn}^{pq}[f](x,y) = S_{mn}[f](x,y).$$

For the operator $E_{mn}^{pq}[f]$ the following equality is true:

$$E_{mn}^{pq}[f](x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u, y+v) K_m^p(u) K_n^q(v)\,\mathrm{d}u\,\mathrm{d}v$$

where

$$\begin{aligned} K_n^q(t) &= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} D_k(t) \\ &= \Big(\frac{q^2+2q\cos t+1}{q^2+2q+1}\Big)^{n/2}\frac{\sin(n\theta_t + t/2)}{2\sin(t/2)}. \end{aligned}$$

θ_t is uniquely determined by the relations

(1) $\theta_t \in (-\pi, \pi)$,

(2) $\operatorname{sgn}\theta_t = \operatorname{sgn} t$,

(3) $|\theta_t| < |t| \leqslant \pi$,

(4) $q\sin\theta_t = \sin(t-\theta_t)$ (see [2]).

In the proof of the next theorem we will need

Lemma 3. *Let $0 < x < \delta \leqslant \pi$ and $q > 0$. Then*

$$\Big|\int_x^\delta K_n^q(t)\,\mathrm{d}t\Big| \leqslant \frac{2\pi}{nx}(1+q)$$

for every positive integer n.

Proof. Using the second mean-value theorem we obtain

$$\begin{aligned} \Big|\int_x^\delta K_n^q(t)\,\mathrm{d}t\Big| &\leqslant \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{2\sin(x/2)}\max_{x\leqslant\varrho\leqslant\delta}\Big|\int_x^\varrho \sin\big(k+\tfrac12\big)t\,\mathrm{d}t\Big| \\ &\leqslant \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{\sin(\frac{x}{2})}\cdot\frac{1}{k+\frac12} \\ &\leqslant \frac{2\pi}{(1+q)^n(n+1)x}\sum_{k=1}^{n+1}\binom{n+1}{k}q^{n-k+1}. \end{aligned}$$

Hence, we get the desired assertion. □

Retaining the notation of Theorem 1, we have

Theorem 2. *If $f \in H$ and $p, q > 0$, then, for every x, y and all natural numbers m, n,*

$$\Big|E_{mn}^{pq}[f](x,y) - \tfrac{1}{4}\{f(x+,y+) + f(x+,,y-) + f(x-,y+) + f(x-,y-)\}\Big|$$
$$\leqslant 12(1+p)(1+q)\Big\{\frac{3}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} V\big(g;0,\tfrac{\pi}{k};0,\tfrac{\pi}{l}\big) + \frac{1}{m}\sum_{k=1}^{m} V\big(g(\cdot,0);0,\tfrac{\pi}{k}\big)$$
$$+ \frac{1}{n}\sum_{k=1}^{n} V\big(g(0,\cdot);0,\tfrac{\pi}{k}\big)\Big\}.$$

P r o o f. A simple calculation shows that

$$\begin{aligned}
&E_{mn}^{pq}[f](x,y) - \tfrac{1}{4}\{f(x+,y+) + f(x+,y-) + f(x-,y+) + f(x-,y-)\}\\
&= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi g(s,t)K_m^p(s)K_n^q(t)\,\mathrm{d}s\,\mathrm{d}t\\
&= \frac{1}{\pi^2}\Big(\int_0^{\frac{\pi}{m}}\int_0^{\frac{\pi}{n}} + \int_0^{\frac{\pi}{m}}\int_{\frac{\pi}{n}}^{\pi} + \int_{\frac{\pi}{m}}^{\pi}\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{m}}^{\pi}\int_{\frac{\pi}{n}}^{\pi}\Big)g(s,t)K_m^p(s)K_n^q(t)\,\mathrm{d}s\,\mathrm{d}t\\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

We can see that the kernel K_n^q has the same estimate as the Dirichlet one, i.e.

$$|K_n^q(t)| \leqslant n + \tfrac{1}{2} \tag{16}$$

for $n \geqslant 1$, $q > 0$ and every t.

Hence the inequality (2) is true with J_1 instead of I_1. So, we have

$$|J_1| \leqslant \frac{9}{4}\Big\{V\big(g;0,\tfrac{\pi}{m};0,\tfrac{\pi}{n}\big) + V\big(g(\cdot,0);0,\tfrac{\pi}{m}\big) + V\big(g(0,\cdot);0,\tfrac{\pi}{n}\big)\Big\}. \tag{17}$$

Now, we introduce the function

$$F_{mn}^{pq}(x,y) = \int_0^x\int_y^\pi K_m^p(s)K_n^q(t)\,\mathrm{d}s\,\mathrm{d}t.$$

It follows immediately from (16) and Lemma 3 that

$$|F_{mn}^{pq}(x,y)| \leqslant \big(m+\tfrac{1}{2}\big)x(1+q)\frac{2\pi}{ny}.$$

Using this estimate for J_2 and an argument similar to that from the proof of Theorem 1 for integral I_2, we obtain

$$|J_2| \leqslant \frac{27}{n\pi}(1+q)\sum_{k=1}^{n} V\big(g;0,\tfrac{\pi}{m};0,\tfrac{\pi}{k}\big) + \frac{9}{\pi}(1+q)V\big(g(\cdot,0);0,\tfrac{\pi}{m}\big) + \frac{9}{n\pi}(1+q)\sum_{k=1}^{n} V\big(g(0,\cdot);\tfrac{\pi}{n},\tfrac{\pi}{k}\big). \tag{18}$$

In the case of integral J_3 we proceed analogously. The only difference is in the construction of the function used in the Stieltjes integral. Namely, let

$$\tilde{F}^{pq}_{mn}(x,y) = \int_x^{\pi}\int_0^{y} K^p_m(s)K^q_n(t)\,\mathrm{d}s\,\mathrm{d}t.$$

According to (16) and Lemma 3, we have

$$|\tilde{F}^{pq}_{mn}(x,y)| \leqslant \frac{2\pi(1+p)}{mx}\big(n+\tfrac{1}{2}\big)y.$$

Consequently,

$$|J_3| \leqslant \frac{27}{m\pi}(1+p)\sum_{k=1}^{m} V\big(g;0,\tfrac{\pi}{k};0,\tfrac{\pi}{n}\big) + \frac{9}{\pi}(1+p)V\big(g(0,\cdot);0,\tfrac{\pi}{n}\big) + \frac{9}{m\pi}(1+p)\sum_{k=1}^{m} V\big(g(\cdot,0);\tfrac{\pi}{m},\tfrac{\pi}{k}\big). \tag{19}$$

In order to estimate J_4 we introduce

$$\overline{F}^{pq}_{mn}(x,y) = \int_x^{\pi}\int_y^{\pi} K^p_m(s)K^q_n(t)\,\mathrm{d}s\,\mathrm{d}t.$$

Hence

$$|\overline{F}^{pq}_{mn}(x,y)| \leqslant \frac{4\pi^2(1+p)(1+q)}{mnxy}$$

and therefore

$$|J_4| \leqslant \frac{36}{mn\pi^2}(1+p)(1+q)\sum_{k=1}^{m}\sum_{l=1}^{n} V\big(g;0,\tfrac{\pi}{k};0,\tfrac{\pi}{l}\big) + \frac{12}{m\pi^2}(1+p)(1+q)\sum_{k=1}^{m} V\big(g(\cdot,0);0,\tfrac{\pi}{k}\big) + \frac{12}{n\pi^2}(1+p)(1+q)\sum_{k=1}^{n} V\big(g(0,\cdot);0,\tfrac{\pi}{k}\big). \tag{20}$$

Combining (17), (18), (19), (20) and using Lemma 2, we get our thesis. □

R e m a r k 3. Reasoning analogously as in Remark 2 we obtain that, for $f \in H$,

$$\lim_{m,n\to\infty} E_{mn}^{pq}[f](x,y) = \tfrac{1}{4}\big\{f(x+,y+)+f(x+,y-)+f(x-,y+)+f(x-,y-)\big\}.$$

The Borel means $B_{pr}[f]$ $(p,r>0)$ of partial sums of double Fourier series of a function f defined in the square $Q=\langle -\pi,\pi;-\pi,\pi\rangle$ and 2π-periodic in each variable can be written in the form

$$B_{pr}[f](x,y) = \mathrm{e}^{-p}\mathrm{e}^{-r}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{p^j}{j!}\frac{r^k}{k!}S_{jk}[f](x,y).$$

This operator has the integral representation

$$B_{pr}[f](x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,y+v)Q_p(u)Q_r(v)\,\mathrm{d}u\,\mathrm{d}v,$$

where

$$Q_r(t) = \mathrm{e}^{-2r\sin^2(t/2)}\frac{\sin(r\sin t + t/2)}{2\sin(t/2)}$$

(see [3], p. 364).

The next theorem may be proved analogously to Theorem 2.

Theorem 3. *Let $f\in H$ and $p,r\geqslant 2$. Then, for every x, y,*

$$\Big|B_{pr}[f](x,y) - \tfrac{1}{4}\big\{f(x+,y+)+f(x+,y-)+f(x-,y+)+f(x-,y-)\big\}\Big|$$
$$\leqslant \frac{39}{r}\sum_{k=1}^{[r]} V\big(g(0,\cdot);0,\tfrac{\pi}{k}\big) + \frac{39}{p}\sum_{k=1}^{[p]} V\big(g(\cdot,0);0,\tfrac{\pi}{k}\big) + \frac{86}{pr}\sum_{k=1}^{[p]}\sum_{l=1}^{[r]} V\big(g;0,\tfrac{\pi}{k};0,\tfrac{\pi}{l}\big),$$

where $[r]$ denotes the integer part of r.

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