WEAK AVERAGING OF STOCHASTIC EVOLUTION EQUATIONS

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Summary. A theorem on continuous dependence of solutions to stochastic evolution equations on coefficients is established, covering the classical averaging procedure for stochastic parabolic equations with rapidly oscillating both the drift and the diffusion term.

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1. Introduction and Main Results

This paper is devoted to integral continuity type results on continuous dependence of solutions of stochastic evolution equations on coefficients, in which, roughly speaking, the convergence of coefficients is defined via pointwise convergence of their indefinite integrals. Such theorems are known to be fairly general and, in particular, they provide justification for the averaging procedures for equations with rapidly oscillating coefficients. As a motivation let us consider a stochastic differential equation

$$\mathrm{d}x_\varepsilon = a_\varepsilon(t,x_\varepsilon)\,\mathrm{d}t + b_\varepsilon(t,x_\varepsilon)\,\mathrm{d}w(t), \qquad x_\varepsilon(0)=x_0 \tag{1.1}$$

in $\mathbb{R}^d$, $w(t)$ being a standard Wiener process in $\mathbb{R}^d$ defined on a probability space $(\Omega,\mathscr{F},\mathsf{P})$, and a_ε, b_ε being, roughly speaking, lipschitzian in the space variable uniformly in $\varepsilon \geqslant 0$. It was proved in [12] and [4] that if

$$\lim_{\varepsilon\to 0+}\int_0^t a_\varepsilon(s,x)\,\mathrm{d}s = \int_0^t a_0(s,x)\,\mathrm{d}s, \qquad x\in\mathbb{R}^d,\ t\geqslant 0, \tag{1.2}$$

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and

$$\lim_{\varepsilon\to 0+}\int_0^t \|b_\varepsilon(s,x)-b_0(s,x)\|^2\,\mathrm{d}s=0,\qquad x\in\mathbb{R}^d,\ t\geqslant 0,\tag{1.3}$$

then $x_\varepsilon(t)\longrightarrow x_0(t)$ in $L^2(\Omega)$. Obviously, the assumption (1.3) excludes rapidly oscillating diffusion coefficients, but, as shown in the quoted papers, is essentially indispensable for the L^2-convergence. On the other hand, as shown by R. Z. Khas'minskiĭ, under (1.2) and

$$\lim_{\varepsilon\to 0+}\int_0^t b_\varepsilon(s,x)b_\varepsilon^*(s,x)\,\mathrm{d}s=\int_0^t b_0(s,x)b_0^*(s,x)\,\mathrm{d}s,\qquad x\in\mathbb{R}^d,\ t\geqslant 0,\tag{1.4}$$

the finite-dimensional distributions of the process x_ε tend to those of x_0 in law (see [5], [6]; cf. [11] for additional information).

The results on L^2-convergence were extended to semilinear stochastic evolution equations in [10], [7], [8]. In the present paper we establish a result on the convergence in law under hypotheses similar to (1.2) and (1.4), thus covering, in particular, the case of stochastic parabolic equations with both the drift and the diffusion coefficient rapidly oscillating.

To state our results let us introduce some notation. Let U, H be real separable Hilbert spaces with norms $|\cdot|_H$, $|\cdot|_U$, respectively, let $\langle\cdot,\cdot\rangle$ denote the inner product in H. (If there is no danger of confusion we will omit the subscript H.) Let $L(U,H)$ be the space of all bounded linear mappings from U to H, whose norm will be denoted by $|\cdot|_{L(U,H)}$. If $A\in L(H)$ then A^* denotes the adjoint operator. Further, $|A|_{\mathscr{N}}$ stands for the nuclear norm of $A\in L(H)$, provided A is a nuclear operator, that is

$$|A|_{\mathscr{N}}=\sup\Big\{\sum_i|\langle Ae_i,f_i\rangle|;\ \{e_i\},\ \{f_i\}\text{ orthonormal bases of }H\Big\}.$$

The space of all H-valued continuous functions on $[0,T]$ will be denoted by $C([0,T],H)$. As usual, if $(\mu_\alpha)_{\alpha\in\Gamma}$ is a net of Borel probabilities on a separable metric space M, Γ a right directed ordered set, we say that $\mu_\alpha\longrightarrow\mu$ weakly in M provided

$$\int_M f\,\mathrm{d}\mu_\alpha\xrightarrow[\alpha\in\Gamma]{}\int_M f\,\mathrm{d}\mu$$

for any bounded continuous function $f\colon M\longrightarrow\mathbb{R}$. If $\xi_n\colon(\Omega_n,\mathscr{F}_n,\mathsf{P}_n)\longrightarrow M$ are M-valued random variables, then $\xi_n\longrightarrow\xi_0$ weakly in M means that $\xi_n(\mathsf{P}_n)\longrightarrow\xi_0(\mathsf{P}_0)$ weakly in M, where the probability measures $\xi_n(\mathsf{P}_n)$ are defined by

$$\xi_n(\mathsf{P}_n)(C)=\mathsf{P}_n\{\xi_n\in C\}$$

for any Borel set $C \subseteq M$.

We will consider stochastic evolution equations

$$
\begin{cases} \mathrm{d}x_\varepsilon(t) = (Ax_\varepsilon(t) + a_\varepsilon(t, x_\varepsilon(t)))\,\mathrm{d}t + b_\varepsilon(t, x_\varepsilon(t))\,\mathrm{d}w_\varepsilon(t), \\ x_\varepsilon(0) = \eta_\varepsilon \end{cases} \tag{1.5}
$$

for $\varepsilon \in [0, \varepsilon_0)$, assuming

(A1) $A\colon \mathrm{Dom}(A) \longrightarrow H$ is an infinitesimal generator of an analytic C_0-semigroup (e^{At}) on H.

(A2) The mappings $a_\varepsilon\colon \mathbb{R}_+ \times H \longrightarrow H$, $b_\varepsilon\colon \mathbb{R}_+ \times H \longrightarrow L(U, H)$ are measurable and there exists a constant L such that

$$
|a_\varepsilon(t, x)|_H + |b_\varepsilon(t, x)|_{L(U,H)} \leqslant L(1 + |x|_H)
$$

for all $t \geqslant 0$, $x \in H$, and $\varepsilon \in [0, \varepsilon_0)$.

(A3) The mappings a_ε, b_ε are Lipschitz continuous uniformly in ε, i.e.

$$
|a_\varepsilon(t, x) - a_\varepsilon(t, y)|_H + |b_\varepsilon(t, x) - b_\varepsilon(t, y)|_{L(U,H)} \leqslant L|x - y|_H
$$

for all $t \geqslant 0$, $x, y \in H$, and $\varepsilon \in [0, \varepsilon_0)$.

(A4) $w_\varepsilon(t)$ are U-valued Wiener processes with nuclear covariance operators W_ε such that

$$
\sup_{\varepsilon \in [0, \varepsilon_0)} \operatorname{tr} W_\varepsilon < \infty,
$$

w_ε being defined on a filtered probability space $(\Omega^\varepsilon, \mathscr{F}^\varepsilon, (\mathscr{F}^\varepsilon_t)_{t \geqslant 0}, \mathsf{P}_\varepsilon)$ and $(\mathscr{F}^\varepsilon_t)$-adapted.

(A5) η_ε are H-valued $\mathscr{F}^\varepsilon_0$-measurable random variables, $\varepsilon \in [0, \varepsilon_0)$.

The solutions of (1.5) will be understood in the mild sense, i.e. as the solutions of the integral equations

$$
x_\varepsilon(t) = \mathrm{e}^{At}\eta_\varepsilon + \int_0^t \mathrm{e}^{A(t-s)} a_\varepsilon(s, x_\varepsilon(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)} b_\varepsilon(s, x_\varepsilon(s))\,\mathrm{d}w_\varepsilon(s).
$$

By Theorem 7.4 in [2] or Theorem 1.4 of [9] there exists a unique mild solution $x_\varepsilon \equiv x_\varepsilon(\cdot, \eta_\varepsilon)$ of (1.5) and $x_\varepsilon \in C([0, T], H)$ almost surely.

Theorem 1.1. *Let the assumptions* (A1)–(A5) *be fulfilled. Let $T > 0$ be arbitrary but fixed and suppose that*

$$
\lim_{\varepsilon \to 0+} \int_0^t \mathrm{e}^{A(t-s)} a_\varepsilon(s, x)\,\mathrm{d}s = \int_0^t \mathrm{e}^{A(t-s)} a_0(s, x)\,\mathrm{d}s \qquad \text{in } H \tag{1.6}
$$

for any $x \in H$, $t \in [0,T]$; *and*

$$\lim_{\varepsilon \to 0+} \left| \int_0^t \mathrm{e}^{A(t-s)} U_\varepsilon(s,x) \mathrm{e}^{A^*(t-s)} \, \mathrm{d}s \right|_{\mathscr{N}} = 0 \tag{1.7}$$

for all $x \in H$, $t \in [0,T]$, *where*

$$U_\varepsilon(t,x) = b_\varepsilon(t,x) W_\varepsilon b_\varepsilon^*(t,x) - b_0(t,x) W_0 b_0^*(t,x).$$

If $\eta_\varepsilon \longrightarrow \eta_0$ *weakly in* H *as* $\varepsilon \to 0+$ *then* $x_\varepsilon(\cdot,\eta_\varepsilon) \longrightarrow x_0(\cdot,\eta_0)$ *weakly in* $C([0,T],H)$ *as* $\varepsilon \to 0+$.

Due to the analyticity of the semigroup (e^{At}) the rather complicated assumptions (1.6), (1.7) may be replaced by more restrictive but verifiable hypotheses.

Proposition 1.2. *Assume* (A1) *and* (A2). *If*

$$\lim_{\varepsilon \to 0+} \int_0^t a_\varepsilon(s,x) \, \mathrm{d}s = \int_0^t a_0(s,x) \, \mathrm{d}s \qquad \text{in } H \tag{1.8}$$

for any $t \in [0,T]$, $x \in H$ *then* (1.6) *is satisfied. Analogously, if*

$$\lim_{\varepsilon \to 0+} \left| \int_0^t b_\varepsilon(s,x) W_\varepsilon b_\varepsilon^*(s,x) \, \mathrm{d}s - \int_0^t b_0(s,x) W_0 b_0^*(s,x) \, \mathrm{d}s \right|_{\mathscr{N}} = 0 \tag{1.9}$$

for all $t \in [0,T]$, $x \in H$ *then* (1.7) *is fulfilled.*

It will be shown in Example 2.9 that the assumptions (1.6), (1.7) are weaker than (1.8), (1.9).

As we have already mentioned, Theorem 1.1 contains the classical averaging procedure as a particular case, as can be shown in a standard way (see e.g. [10], Theorem 4). Because of its importance we state this corollary as a separate theorem. Let us consider equations

$$\begin{cases} \mathrm{d}x_\varepsilon(t) = \varepsilon\big(Ax_\varepsilon(t) + a(t,x_\varepsilon(t))\big)\,\mathrm{d}t + \varepsilon^{1/2} b(t,x_\varepsilon(t))\,\mathrm{d}w(t), \\ x_\varepsilon(0) = \eta_\varepsilon \end{cases} \tag{1.10}$$

for $\varepsilon \in (0,\varepsilon_0)$, where

(B2) the mappings $a\colon \mathbb{R}_+ \times H \longrightarrow H$, $b\colon \mathbb{R}_+ \times H \longrightarrow L(U,H)$ are measurable and there exists a constant L such that

$$|a(t,x)|_H + |b(t,x)|_{L(U,H)} \leqslant L(1+|x|_H),$$
$$|a(t,x) - a(t,y)|_H + |b(t,x) - b(t,y)|_{L(U,H)} \leqslant L|x-y|_H$$

for all $x, y \in H$, $t \geqslant 0$;

(B3) $w(t)$ is a U-valued $(\mathscr{F}_t)$-adapted Wiener process with a nuclear covariance operator W, defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathsf{P})$;

(B4) $\eta_\varepsilon\colon \Omega \longrightarrow H$, $\varepsilon \in [0, \varepsilon_0)$, are $\mathscr{F}_0$-measurable random variables.

Theorem 1.3. *Assume* (A1), (B2), (B3) *and* (B4). *Suppose further that there exist Lipschitz continuous functions $\tilde{a}\colon H \longrightarrow H$ and $\tilde{b}\colon H \longrightarrow L(U, H)$ such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T a(s, x)\,\mathrm{d}s = \tilde{a}(x) \qquad \text{in } H$$

for any $x \in H$, and

$$\lim_{T\to\infty} \left| \frac{1}{T} \int_0^T b(s, x) W b^*(s, x)\,\mathrm{d}s - \tilde{b}(x) W \tilde{b}^*(x) \right|_{\mathscr{N}} = 0$$

for all $x \in H$. Set

$$\tilde{x}_\varepsilon(t, \eta_\varepsilon) = x_\varepsilon\Big(\frac{t}{\varepsilon}, \eta_\varepsilon\Big), \qquad t \geqslant 0,\ \varepsilon \in (0, \varepsilon_0),$$

$x_\varepsilon(\cdot, \eta_\varepsilon)$ being the mild solutions to (1.10). *Let $x_0(\cdot, \eta_0)$ be the mild solution of the problem*

$$\begin{cases} \mathrm{d}x_0(t) = (Ax_0(t) + \tilde{a}(x_0(t)))\,\mathrm{d}t + \tilde{b}(x_0(t))\,\mathrm{d}w(t), \\ x_0(0) = \eta_0. \end{cases} \tag{1.11}$$

If $\eta_\varepsilon \longrightarrow \eta_0$ weakly in H as $\varepsilon \to 0+$ then $\tilde{x}_\varepsilon(\cdot, \eta_\varepsilon) \longrightarrow x_0(\cdot, \eta_0)$ weakly in $C([0, T], H)$ as $\varepsilon \to 0+$ for any fixed $T > 0$.

2. Proofs

To start with, let us recall a few well-known results.

Proposition 2.1. *Let μ_n, $n \geqslant 0$, be centered Gaussian measures on a separable Hilbert space Y with covariance operators Γ_n. Then $\mu_n \longrightarrow \mu_0$ weakly in Y if and only if $|\Gamma_n - \Gamma_0|_{\mathscr{N}} \longrightarrow 0$, $n \to \infty$.*

For the proof, see e.g. [1]. We will need the following criterion for weak convergence of measures. Let (M, d) be a separable metric space, denote by $BL(M)$ the space of all bounded Lipschitz functions on M, that is

$$BL(M) = \Big\{ f\colon M \longrightarrow \mathbb{R};\ \|f\|_{BL} \equiv \sup_{x\in M} |f(x)| + \sup_{\substack{x,y\in M\\ x\neq y}} \frac{|f(x) - f(y)|}{d(x, y)} < \infty \Big\}.$$

Let μ_n, μ be Borel probabilities on M, then $\mu_n \longrightarrow \mu$ weakly in M if and only if

$$\lim_{n\to\infty} \int_M f \,\mathrm{d}\mu_n = \int_M f \,\mathrm{d}\mu$$

for any $f \in BL(M)$, see e.g. [3], Theorem 11.3.3. Moreover, by the same theorem, the metrics β, defined for probability measures on M by

$$\beta(\mu,\nu) = \sup\Big\{ \Big| \int_M f \,\mathrm{d}(\mu-\nu) \Big| ; \ \|f\|_{BL} \leqslant 1 \Big\},$$

metrizes the weak convergence.

The next proposition looks almost obvious, nonetheless, it will be very useful in the sequel.

Let U, V, H be real separable Hilbert spaces, let w be a U-valued $(\mathscr{G}_t)$-adapted Wiener process with a nuclear covariance operator W, defined on a filtered probability space $(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geqslant 0}, \mathsf{P})$, let $0 \leqslant s < t$.

Proposition 2.2. *Let $\alpha\colon H \longrightarrow V$ be a Lipschitz continuous mapping, let $\sigma\colon [s,t] \times H \longrightarrow L(U,V)$ be a measurable mapping such that*

$$|\sigma(r,x)|_{L(U,V)} \leqslant M(1+|x|_H), \qquad |\sigma(r,x)-\sigma(r,y)|_{L(U,V)} \leqslant M|x-y|_H$$

for a constant M and any $r \in [s,t]$, $x, y \in H$. Let $g \in BL(V)$, define

$$\psi(y) = \mathsf{E} g\Big(\alpha(y) + \int_s^t \sigma(r,y)\,\mathrm{d}w(r)\Big), \qquad y \in H.$$

Let $u\colon \Omega \longrightarrow H$ be a $\mathscr{G}_s$-measurable random variable with $\mathsf{E}|u|_H^2 < \infty$. Then

$$\mathsf{E}\Big[g\Big(\alpha(u) + \int_s^t \sigma(r,u)\,\mathrm{d}w(r)\Big) \Big| \mathscr{G}_s \Big] = \psi(u) \qquad \mathsf{P}\text{-almost surely.} \tag{2.1}$$

Proof. To simplify notation, we will treat the case $\alpha \equiv 0$; it can be seen easily that this leads to no loss of generality.

Take an arbitrary $\gamma > 0$, let $\{z_i ; \ i \in \mathbb{N}\}$ be a dense subset of H and define a Borel partition $\{A(i), \ i \in \mathbb{N}\}$ of H by

$$A(1) = \{\xi \in H ; \ |\xi - z_1| < \gamma\}, \quad A(i+1) = \{\xi \in H ; \ |\xi - z_{i+1}| < \gamma\} \setminus \bigcup_{j\leqslant i} A(j).$$

We may assume that $z_j \in A(j)$ for any $j \in \mathbb{N}$. Define

$$u_\gamma(\omega) = \sum_{i=1}^{\infty} \chi_{A(i)}\big(u(\omega)\big) z_i, \qquad \omega \in \Omega,$$
$$\psi_\gamma(\xi) = \sum_{i=1}^{\infty} \chi_{A(i)}(\xi) \mathsf{E} g\Big(\int_s^t \sigma(r, z_i)\,\mathrm{d}w(r)\Big), \qquad \xi \in H,$$

where $\chi_{A(j)}$ stands for the indicator function of the set $A(j)$. Obviously,

$$\mathsf{E}|u - u_\gamma|_H^2 < \gamma^2$$

and

$$g\Big(\int_s^t \sigma(r, u_\gamma)\,\mathrm{d}w(r)\Big) = \sum_{i=1}^{\infty} \chi_{A(i)}(u)\, g\Big(\int_s^t \sigma(r, z_i)\,\mathrm{d}w(r)\Big) \qquad \mathsf{P}\text{-a.s..}$$

Therefore

$$\begin{aligned}
\mathsf{E}\Big[g\Big(\int_s^t \sigma(r, u_\gamma)\,\mathrm{d}w(r)\Big)\Big|\mathscr{G}_s\Big] &= \sum_{i=1}^{\infty} \chi_{A(i)}(u) \mathsf{E}\Big[g\Big(\int_s^t \sigma(r, z_i)\,\mathrm{d}w(r)\Big)\Big|\mathscr{G}_s\Big] \\
&= \sum_{i=1}^{\infty} \chi_{A(i)}(u) \mathsf{E} g\Big(\int_s^t \sigma(r, z_i)\,\mathrm{d}w(r)\Big) \\
&= \psi_\gamma(u).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathsf{E}\Big|g\Big(\int_s^t \sigma(r, u_\gamma)\,\mathrm{d}w(r)\Big) - g\Big(\int_s^t \sigma(r, u)\,\mathrm{d}w(r)\Big)\Big|^2 \\
&\qquad \leqslant \|g\|_{BL}^2 \mathsf{E}\Big|\int_s^t \big[\sigma(r, u_\gamma) - \sigma(r, u)\big]\,\mathrm{d}w(r)\Big|_V^2 \\
&\qquad \leqslant \|g\|_{BL}^2 \operatorname{tr}(W) \int_s^t \mathsf{E}\big|\sigma(r, u_\gamma) - \sigma(r, u)\big|_{L(U,V)}^2\,\mathrm{d}r \\
&\qquad \leqslant \|g\|_{BL}^2 M^2 \operatorname{tr}(W) \int_s^t \mathsf{E}|u - u_\gamma|_H^2\,\mathrm{d}r \\
&\qquad \leqslant \|g\|_{BL}^2 M^2 \operatorname{tr}(W)(t-s)\gamma^2,
\end{aligned}$$

hence

$$\mathsf{E}\Big[g\Big(\int_s^t \sigma(r, u_\gamma)\,\mathrm{d}w(r)\Big)\Big|\mathscr{G}_s\Big] \xrightarrow{\gamma \to 0+} \mathsf{E}\Big[g\Big(\int_s^t \sigma(r, u)\,\mathrm{d}w(r)\Big)\Big|\mathscr{G}_s\Big] \qquad \text{in } L^2(\Omega).$$

Analogously, take $\xi \in H$, then there exists $j_0 \in \mathbb{N}$ such that $\xi \in A(j_0)$, so

$$
\begin{aligned}
|\psi_\gamma(\xi) - \psi(\xi)|^2 &= \left| \mathsf{E} g\Big(\int_s^t \sigma(r, z_{j_0}) \, \mathrm{d}w(r) \Big) - \mathsf{E} g\Big(\int_s^t \sigma(r, \xi) \, \mathrm{d}w(r) \Big) \right|^2 \\
&\leqslant \|g\|_{BL}^2 M^2 \operatorname{tr}(W)(t-s)\gamma^2.
\end{aligned}
$$

This yields

$$
\psi_\gamma(\zeta) \xrightarrow{\gamma \to 0+} \psi(\zeta) \qquad \text{for all } \zeta \in H
$$

and (2.1) follows. $\square$

We are prepared to return to the equation (1.5). From now on, the hypotheses of Theorem 1.1 are assumed to be fulfilled.

Lemma 2.3. *For every $T > 0$ and $p \in [2, \infty)$ there exists a constant K such that for any $\varepsilon \in [0, \varepsilon_0)$ we have*

$$
\mathsf{E}|x_\varepsilon(t)|_H^p \leqslant K(1 + |\eta_\varepsilon|_H^p), \qquad 0 \leqslant t \leqslant T,
$$

provided the right-hand side is finite.

Since the estimate in (A2) is uniform in ε, Lemma 2.3 follows virtually from [2], Theorem 7.4, or [9], Theorem 1.4, as tracing the proofs of these theorems one can observe easily that the constant K can be obtained independent of $\varepsilon \in [0, \varepsilon_0)$.

Given $N \in \mathbb{N}$, let us define

$$
g^N(x) = \begin{cases} x & \text{if } |x| \leqslant N, \\ \dfrac{Nx}{|x|_H} & \text{otherwise,} \end{cases}
$$

and set $\eta_\varepsilon^N = g^N(\eta_\varepsilon)$. By assumption, $\eta_\varepsilon \longrightarrow \eta_0$ weakly in H, so the continuity of g^N yields that $\eta_\varepsilon^N \longrightarrow \eta_0^N$ weakly in H for any $N \in \mathbb{N}$.

Let us fix $T > 0$ and an arbitrary sequence $\varepsilon_n \in (0, \varepsilon_0)$, $\varepsilon_n \searrow 0$. For brevity we set

$$
a_{\varepsilon_n} = a_n, \quad b_{\varepsilon_n} = b_n, \quad w_{\varepsilon_n} = w_n, \quad \mathsf{P}_{\varepsilon_n} = \mathsf{P}_n
$$

and so forth. Further, set

$$
x_n = x_{\varepsilon_n}(\cdot, \eta_{\varepsilon_n}), \quad x_n^N = x_{\varepsilon_n}(\cdot, \eta_{\varepsilon_n}^N), \quad x_0^N = x_0(\cdot, \eta_0^N).
$$

We denote the integral with respect to the measure P_ε simply by E with omitted subscript as it leads to no ambiguity.

Proposition 2.4. *For any $N \in \mathbb{N}$ and $\gamma > 0$ there exists $\varrho > 0$ such that*

$$\sup_{n\geqslant 0} \mathsf{E} \max_{\substack{0\leqslant s,t\leqslant T\\ |s-t|\leqslant\varrho}} \left|x_n^N(t) - x_n^N(s)\right|^2 \leqslant \gamma.$$

P r o o f. Let us denote by K_i generic constants independent of $n \geqslant 0$. Choose $p > 2$ and $\lambda \in (0, \frac{1}{2} - \frac{1}{p})$. By Remark following Theorem 1.1 in [9] we have

$$\begin{aligned}\mathsf{E} \sup_{\substack{0\leqslant s,t\leqslant T\\ |t-s|<\zeta}} &\left|\int_0^t \mathrm{e}^{A(t-r)} b_n(r, x_n^N(r))\,\mathrm{d}w_n(r) - \int_0^s \mathrm{e}^{A(s-r)} b_n(r, x_n^N(r))\,\mathrm{d}w_n(r)\right|^2 \\ &\leqslant K_1 \zeta^{2\lambda}\end{aligned} \tag{2.2}$$

for any $\zeta > 0$. Indeed,

$$\begin{aligned}
&\mathsf{E} \sup_{\substack{0\leqslant s,t\leqslant T\\ |t-s|<\zeta}} \left|\frac{\int_0^t \mathrm{e}^{A(t-r)} b_n(r, x_n^N(r))\,\mathrm{d}w_n(r) - \int_0^s \mathrm{e}^{A(s-r)} b_n(r, x_n^N(r))\,\mathrm{d}w_n(r)}{\zeta^\lambda}\right|^2 \\
&\quad\leqslant \Biggl(\mathsf{E} \sup_{\substack{0\leqslant s,t\leqslant T\\ |t-s|<\zeta}} \biggl|\zeta^{-\lambda}\biggl\{\int_0^t \mathrm{e}^{A(t-r)} b_n(r, x_n^N(r))\,\mathrm{d}w_n(r) \\
&\qquad - \int_0^s \mathrm{e}^{A(s-r)} b_n(r, x_n^N(r))\,\mathrm{d}w_n(r)\biggr\}\biggr|^p\Biggr)^{2/p} \\
&\quad\leqslant \Biggl(\mathsf{E} \sup_{\substack{0\leqslant s,t\leqslant T\\ |t-s|<\zeta}} \biggl||t-s|^{-\lambda}\biggl\{\int_0^t \mathrm{e}^{A(t-r)} b_n(r, x_n^N(r))\,\mathrm{d}w_n(r) \\
&\qquad - \int_0^s \mathrm{e}^{A(s-r)} b_n(r, x_n^N(r))\,\mathrm{d}w_n(r)\biggr\}\biggr|^p\Biggr)^{2/p} \\
&\quad\leqslant \biggl(C(\lambda)\int_0^T \mathsf{E}\bigl(\operatorname{tr}\bigl\{b_n(r, x_n^N(r)) W_n b_n^*(r, x_n^N(r))\bigr\}\bigr)^{p/2}\,\mathrm{d}r\biggr)^{2/p} \\
&\qquad\qquad \leqslant K_2 \operatorname{tr}(W_n)\biggl(\int_0^T \mathsf{E}\bigl(1 + |x_n^N(r)|^p\bigr)\,\mathrm{d}r\biggr)^{2/p}
\end{aligned}$$

and (2.2) follows by (A4) and Lemma 2.3. By an analogous procedure we obtain

$$\begin{aligned}\mathsf{E} \sup_{\substack{0\leqslant s,t\leqslant T\\ |t-s|<\zeta}} &\left|\int_0^t \mathrm{e}^{A(t-r)} a_n(r, x_n^N(r))\,\mathrm{d}r - \int_0^s \mathrm{e}^{A(s-r)} a_n(r, x_n^N(r))\,\mathrm{d}r\right|^2 \\ &\leqslant K_3 \zeta^{2\lambda}.\end{aligned} \tag{2.3}$$

First, we prove that there exists $\varrho_1 > 0$ such that

$$\sup_{n\geqslant 0} \mathsf{E} \sup_{0\leqslant t\leqslant 2\varrho_1} \bigl|x_n^N(t) - \eta_n^N\bigr|^2 \leqslant K_4\gamma \tag{2.4}$$

(with a constant K_4 independent of ϱ_1, of course). In fact, by the definition of a mild solution

$$\begin{aligned} x_n^N(t) - \eta_n^N &= \bigl[\mathrm{e}^{At} - I\bigr]\eta_n^N + \int_0^t \mathrm{e}^{A(t-s)} a_n(s, x_n^N(s))\,\mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{A(t-s)} b_n(s, x_n^N(s))\,\mathrm{d}w_n(s) \\ &\equiv J_1 + J_2 + J_3. \end{aligned}$$

By (2.2) and (2.3) we have

$$\sup_{n\geqslant 0} \mathsf{E} \sup_{0\leqslant t\leqslant \zeta} \bigl\{|J_2|^2 + |J_3|^2\bigr\} \leqslant K_5\zeta^{2\lambda} \tag{2.5}$$

for any $\zeta > 0$. Since the random variables η_n^N weakly converge, they are tight by the Prokhorov theorem, hence there exists a compact set $C \subseteq H$ such that

$$\sup_{n\geqslant 0} \mathsf{P}_n\{\eta_n^N \notin C\} \leqslant \gamma.$$

As is well known, $\mathrm{e}^{At}x \longrightarrow x$ as $t \searrow 0$ uniformly in $x \in C$, hence we can find $\varrho_1 \in (0, \gamma^{1/2\lambda})$ such that

$$\sup_{0\leqslant t\leqslant 2\varrho_1} \sup_{x\in C} \bigl|\mathrm{e}^{At}x - x\bigr| < \sqrt{\gamma}.$$

Then

$$\begin{aligned} \mathsf{E} \sup_{0\leqslant t\leqslant 2\varrho_1} \bigl|\mathrm{e}^{At}\eta_n^N - \eta_n^N\bigr|^2 &\leqslant \gamma + \mathsf{E}\chi_{\{\eta_n^N\notin C\}} \sup_{0\leqslant t\leqslant 2\varrho_1} \bigl|\mathrm{e}^{At}\eta_n^N - \eta_n^N\bigr|^2 \\ &\leqslant \gamma + K_6\mathsf{E}\chi_{\{\eta_n^N\notin C\}}|\eta_n^N|^2 \leqslant \gamma + K_6N^2\mathsf{P}_n\{\eta_n^N \notin C\} \leqslant K_7\gamma. \end{aligned}$$

This estimate together with (2.5) yields (2.4).

The semigroup (e^{At}) is analytic by (A1), hence the $L(H)$-valued function $t \mapsto \mathrm{e}^{At}$ is uniformly continuous on $[\varrho_1, T]$ (cf. e.g. [2], Theorem A.7). So we can find $\varrho_2 \in (0, \gamma^{1/2\lambda})$ such that

$$\sup_{\substack{\varrho_1\leqslant t,s\leqslant T\\ |s-t|\leqslant\varrho_2}} \bigl|\mathrm{e}^{At} - \mathrm{e}^{As}\bigr|_{L(H)} \leqslant \sqrt{\gamma}. \tag{2.6}$$

Take $s,t\in[\varrho_2,T]$, $t>s$, then

$$
\begin{aligned}
x_n^N(t)-x_n^N(s) &= \big(\mathrm{e}^{At}-\mathrm{e}^{As}\big)\eta_n^N \\
&\quad+\left\{\int_0^t \mathrm{e}^{A(t-r)}a_n(r,x_n^N(r))\,\mathrm{d}r-\int_0^s \mathrm{e}^{A(s-r)}a_n(r,x_n^N(r))\,\mathrm{d}r\right\} \\
&\quad+\left\{\int_0^t \mathrm{e}^{A(t-r)}b_n(r,x_n^N(r))\,\mathrm{d}w_n(r)\right. \\
&\quad-\left.\int_0^s \mathrm{e}^{A(s-r)}b_n(r,x_n^N(r))\,\mathrm{d}w_n(r)\right\} \\
&\equiv Q_1+Q_2+Q_3.
\end{aligned}
$$

Obviously,

$$\sup_{n\geqslant 0}\mathsf{E}\sup_{|t-s|\leqslant\varrho_2}\big\{|Q_2|^2+|Q_3|^2\big\}\leqslant K_8\varrho_2^{2\lambda}\leqslant K_8\gamma$$

by (2.2) and (2.3). Finally,

$$\mathsf{E}\sup_{|t-s|\leqslant\varrho_2}\big|\big(\mathrm{e}^{At}-\mathrm{e}^{As}\big)\eta_n^N\big|^2\leqslant N^2\gamma$$

by (2.6), so we have obtained

$$\sup_{n\geqslant 0}\mathsf{E}\sup_{\substack{\varrho_1\leqslant s,t\leqslant T\\ |s-t|\leqslant\varrho_2}}\big|x_n^N(t)-x_n^N(s)\big|^2\leqslant K_9\gamma. \tag{2.7}$$

Combining the estimates (2.7) and (2.4) we complete the proof of Proposition 2.4. □

Corollary 2.5. *For any $N\in\mathbb{N}$ and $\gamma>0$ there exists a partition $\{0=t_0<\ldots<t_k=T\}$ of the interval $[0,T]$ such that*

$$\sup_{n\geqslant 0}\mathsf{E}\left(\max_{i=0,\ldots,k-1}\ \max_{t_i\leqslant t\leqslant t_{i+1}}\big|x_n^N(t)-x_n^N(t_i)\big|^2\right)\leqslant\gamma. \tag{2.8}$$

To proceed further we need a suitable discretization in time of the process x_n^N. Let $\Pi=\{0=t_0<\ldots<t_k=T\}$ be a partition of the interval $[0,T]$. Define

$$
\begin{aligned}
x_n^\Pi(t)=\mathrm{e}^{At}\eta_n^N &+\sum_{i=0}^{k-1}\int_{t_i\wedge t}^{t_{i+1}\wedge t}\mathrm{e}^{A(t-s)}a_n(s,x_n^\Pi(t_i))\,\mathrm{d}s \\
&+\sum_{i=0}^{k-1}\int_{t_i\wedge t}^{t_{i+1}\wedge t}\mathrm{e}^{A(t-s)}b_n(s,x_n^\Pi(t_i))\,\mathrm{d}w_n(s)
\end{aligned} \tag{2.9}
$$

for any $t \in [0,T]$ and $n \geqslant 0$, where, as usual, $a \wedge b = \min(a,b)$. (Note that the process x_n^{Π} depends on N as well.) We have the following estimate:

Lemma 2.6. *For every $N \in \mathbb{N}$ and $\gamma > 0$ there exists a partition $\Pi = \{0 = t_0 < \ldots < t_k = T\}$ of the interval $[0,T]$ such that*

$$\sup_{n \geqslant 0} \max_{i=0,\ldots,k} \mathsf{E}\big|x_n^N(t_i) - x_n^{\Pi}(t_i)\big|^2 \leqslant \gamma.$$

Proof. Take $\gamma > 0$ and find a partition $\Pi = \{0 = t_0 < \ldots < t_k = T\}$ such that (2.8) holds. By (2.9) we obtain

$$\begin{aligned}
&\mathsf{E}|x_n^{\Pi}(t_{j+1}) - x_n^N(t_{j+1})|^2 \\
&\qquad \leqslant 2T \int_0^{t_{j+1}} \big|\mathrm{e}^{A(t_{j+1}-s)}\big|_{L(H)}^2 \\
&\qquad\qquad\qquad \times \mathsf{E}\Big|\sum_{i=0}^{j} \chi_{[t_i,t_{i+1})}(s)\{a_n(s,x_n^{\Pi}(t_i)) - a_n(s,x_n^N(s))\}\Big|^2 \,\mathrm{d}s \\
&\qquad\quad + 2\operatorname{tr}(W_n) \int_0^{t_{j+1}} \big|\mathrm{e}^{A(t_{j+1}-s)}\big|_{L(H)}^2 \\
&\qquad\qquad\qquad \times \mathsf{E}\Big|\sum_{i=0}^{j} \chi_{[t_i,t_{i+1})}(s)\{b_n(s,x_n^{\Pi}(t_i)) - b_n(s,x_n^N(s))\}\Big|^2 \,\mathrm{d}s \\
&\qquad \leqslant K_{10} \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \mathsf{E}\big|x_n^{\Pi}(t_i) - x_n^N(s)\big|^2 \,\mathrm{d}s \\
&\qquad \leqslant 2K_{10} \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \mathsf{E}\big|x_n^{\Pi}(t_i) - x_n^N(t_i)\big|^2 \,\mathrm{d}s \\
&\qquad\quad + 2K_{10} \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \mathsf{E}\big|x_n^N(t_i) - x_n^N(s)\big|^2 \,\mathrm{d}s \\
&\qquad \leqslant 2K_{10}T\gamma + 2K_{10} \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \mathsf{E}\big|x_n^{\Pi}(t_i) - x_n^N(t_i)\big|^2 \,\mathrm{d}s.
\end{aligned}$$

Define

$$f_n(t) = \mathsf{E}\big|x_n^{\Pi}(t_i) - x_n^N(t_i)\big|^2, \qquad t_i \leqslant t < t_{i+1},\ i = 0,\ldots,k-1.$$

The above estimate implies

$$f_n(t) \leqslant 2K_{10}T\gamma + 2K_{10} \int_0^t f_n(s)\,\mathrm{d}s,$$

and K_{10} is independent of n, so Lemma 2.6 follows by the Gronwall inequality. $\square$

The crucial step in the proof of Theorem 1.1 is to establish that the finite dimensional distributions of the process x_n^{Π} converge weakly as $n \to \infty$.

Proposition 2.7. *Let $N \in \mathbb{N}$ and a partition $\Pi = \{0 = t_0 < \ldots < t_k = T\}$ be given. Let ν_n, $n \geqslant 0$, be Borel probabilities on H^{k+1} defined by*

$$\nu_n = \big(x_n^{\Pi}(t_0), \ldots, x_n^{\Pi}(t_k)\big)(\mathsf{P}_n),$$

that is

$$\nu_n(Q) = \mathsf{P}_n\{\omega;\ \big(x_n^{\Pi}(t_0), \ldots, x_n^{\Pi}(t_k)\big) \in Q\}$$

for any Borel set Q in H^{k+1}. Then $\nu_n \longrightarrow \nu_0$ weakly in H^{k+1} as $n \to \infty$.

Proof. The proof proceeds by induction. By assumptions of Theorem 1.1 we have $x_n^{\Pi}(t_0) \longrightarrow x_0^{\Pi}(t_0)$ weakly in H. Assume that for some l, $0 \leqslant l \leqslant k-1$, the convergence

$$\mu_n \equiv u_n(\mathsf{P}_n) \longrightarrow \mu_0 \equiv u_0(\mathsf{P}_0) \qquad \text{weakly in } H^{l+1} \tag{2.10}$$

has beeen established, where we set

$$u_n = \big(x_n^{\Pi}(t_0), \ldots, x_n^{\Pi}(t_l)\big).$$

Let us define

$$\begin{aligned} &\alpha_n \colon H^{l+1} \longrightarrow H^{l+2}, \\ &\quad (\xi_0, \ldots, \xi_l) \longmapsto \left(\xi_0, \ldots, \xi_l, \mathrm{e}^{A(t_{l+1}-t_l)}\xi_l + \int_{t_l}^{t_{l+1}} \mathrm{e}^{A(t_{l+1}-s)} a_n(s, \xi_l)\,\mathrm{d}s\right), \end{aligned}$$

and

$$\begin{aligned} &B_n \colon H^{l+1} \longrightarrow L^1(\Omega^n, H^{l+2}), \\ &\quad (\xi_0, \ldots, \xi_l) \longmapsto \left(0, \ldots, 0, \int_{t_l}^{t_{l+1}} \mathrm{e}^{A(t_{l+1}-s)} b_n(s, \xi_l)\,\mathrm{d}w_n(s)\right). \end{aligned}$$

Obviously,

$$\begin{aligned} x_n^{\Pi}(t_{l+1}) = \mathrm{e}^{A(t_{l+1}-t_l)} x_n^{\Pi}(t_l) &+ \int_{t_l}^{t_{l+1}} \mathrm{e}^{A(t_{l+1}-s)} a_n(s, x_n^{\Pi}(t_l))\,\mathrm{d}s \\ &+ \int_{t_l}^{t_{l+1}} \mathrm{e}^{A(t_{l+1}-s)} b_n(s, x_n^{\Pi}(t_l))\,\mathrm{d}w_n(s), \end{aligned}$$

thus

$$\big(x_n^{\Pi}(t_0),\dots,x_n^{\Pi}(t_{l+1})\big)=\alpha_n(u_n)+B_n(u_n).$$

Take $g\in BL(H^{l+2})$ arbitrary and set

$$h_n(y)=\mathsf{E}g(\alpha_n(y)+B_n(y)),\qquad y\in H^{l+1}.$$

Proposition 2.2 yields

$$\mathsf{E}g(x_n^{\Pi}(t_0),\dots,x_n^{\Pi}(t_{l+1}))=\mathsf{E}h_n(u_n)=\int_{H^{l+1}}h_n(\xi)\,\mathrm{d}\mu_n(\xi).$$

It follows

$$\begin{aligned}&\big|\mathsf{E}g(x_n^{\Pi}(t_0),\dots,x_n^{\Pi}(t_{l+1}))-\mathsf{E}g(x_0^{\Pi}(t_0),\dots,x_0^{\Pi}(t_{l+1}))\big|\\&\quad\leqslant\int_{H^{l+1}}\big|h_n(\xi)-h_0(\xi)\big|\,\mathrm{d}\mu_n(\xi)+\left|\int_{H^{l+1}}h_0(\xi)\,\mathrm{d}\mu_n(\xi)-\int_{H^{l+1}}h_0(\xi)\,\mathrm{d}\mu_0(\xi)\right|\\&\qquad\equiv M_1(n)+M_2(n).\end{aligned}$$

To proceed further, we check that $h_n\in BL(H^{l+1})$ and

$$\sup_{n\geqslant 0}\|h_n\|_{BL}<\infty. \tag{2.11}$$

Indeed,

$$\begin{aligned}|h_n(\xi)-h_n(\zeta)|&\leqslant\|g\|_{BL}\mathsf{E}|\alpha_n(\xi)-\alpha_n(\zeta)+B_n(\xi)-B_n(\zeta)|_{H^{l+2}}\\&\leqslant K_{11}\|g\|_{BL}|\xi-\zeta|_{H^{l+1}}\end{aligned}$$

for any $\xi,\zeta\in H^{l+1}$ by (A3). Therefore, $M_2(n)\longrightarrow 0$ as $n\to\infty$ due to (2.10). Now, note that

$$\lim_{n\to\infty}h_n(\xi)=h_0(\xi)\qquad\text{for any }\xi\in H^{l+1}. \tag{2.12}$$

First,

$$\begin{aligned}\alpha_n(\xi)-\alpha_0(\xi)=&\left(0,\dots,0,\int_{t_l}^{t_{l+1}}\mathrm{e}^{A(t_{l+1}-s)}\big[a_n(s,\xi_l)-a_0(s,\xi_l)\big]\,\mathrm{d}s\right)\\&\xrightarrow{n\to\infty}0\qquad\text{in }H^{l+1}\end{aligned}$$

by (1.8). Further,

$$\int_{t_l}^{t_{l+1}}\mathrm{e}^{A(t_{l+1}-s)}b_n(s,\xi_l)\,\mathrm{d}w_n(s)$$

is a centered Gaussian random variable in H with a covariance operator

$$\int_{t_l}^{t_{l+1}} \mathrm{e}^{A(t_{l+1}-s)} b_n(s,\xi_l) W_n b_n^*(s,\xi_l) \mathrm{e}^{A^*(t_{l+1}-s)}\,\mathrm{d}s,$$

so $B_n(\xi) \longrightarrow B_0(\xi)$ weakly in H^{l+2} by (1.9) and Proposition 2.1. Hence $\alpha_n(\xi) + B_n(\xi) \longrightarrow \alpha_0(\xi) + B_0(\xi)$ weakly in H^{l+2} for any $\xi \in H^{l+1}$ and recalling the definition of h_n and of the weak convergence we see that (2.12) holds.

Take $\delta > 0$ arbitrary, there exists a compact set $K \subseteq H^{l+1}$ such that

$$\inf_{n\geqslant 0} \mu_n(K) \geqslant 1-\delta \tag{2.13}$$

since the measures $\{\mu_n\}$ are weakly convergent. Due to the compactness of K, (2.12) and (2.11) the functions h_n converge to h_0 uniformly on K, hence

$$\lim_{n\to\infty} \int_K |h_n(\xi) - h_0(\xi)|\,\mathrm{d}\mu_n(\xi) = 0.$$

Finally,

$$\int_{H^{l+1}\setminus K} |h_n(\xi) - h_0(\xi)|\,\mathrm{d}\mu_n(\xi) \leqslant 2 \sup_{n\geqslant 0} \sup_{H^{l+1}} |h_n| \delta$$

by (2.13), so $M_1(n) \longrightarrow 0$ as well and Proposition 2.7 follows. ☐

Corollary 2.8. *Let $N \in \mathbb{N}$ and a partition $\Lambda = \{0 = s_0 < \ldots < s_q = T\}$ of the interval $[0,T]$ be given. Then*

$$\big(x_n^N(s_0), \ldots, x_n^N(s_q)\big)(\mathsf{P}_n) \xrightarrow{n\to\infty} \big(x_0^N(s_0), \ldots, x_0^N(s_q)\big)(\mathsf{P}_0) \qquad \textit{weakly in } H^{q+1}.$$

In other words, x_n^N's converge in law to x_0^N in the sense of finite dimensional distributions.

Proof. Take $\gamma > 0$ arbitrary. According to Lemma 2.6, there exists a partition $\Pi = \{0 = t_0 < \ldots < t_k = T\}$ refining the partition Λ and such that

$$\sup_{n\geqslant 0} \max_{i=0,\ldots,k} \mathsf{E}\big|x_n^N(t_i) - x_n^\Pi(t_i)\big|^2 \leqslant \frac{\gamma^2}{q+1}.$$

In particular,

$$\begin{aligned}\sup_{n\geqslant 0} \mathsf{E}\big|\big(x_n^N(s_0), \ldots, x_n^N(s_q)\big) - \big(x_n^\Pi(s_0), \ldots, x_n^\Pi(s_q)\big)\big|^2_{H^{q+1}} \\ = \sup_{n\geqslant 0} \mathsf{E} \sum_{i=0}^q \big|x_n^N(s_i) - x_n^\Pi(s_i)\big|^2 \leqslant \gamma^2.\end{aligned} \tag{2.14}$$

Let $g \in BL(H^{q+1})$ be a bounded Lipschitz function, then

$$\begin{aligned}
&\big|\mathsf{E}g(x_n^N(s_0),\dots,x_n^N(s_q)) - \mathsf{E}g(x_0^N(s_0),\dots,x_0^N(s_q))\big| \\
&\qquad\leqslant \mathsf{E}\big|g(x_n^N(s_0),\dots,x_n^N(s_q)) - g(x_n^\Pi(s_0),\dots,x_n^\Pi(s_q))\big| \\
&\qquad\quad + \big|\mathsf{E}g(x_n^\Pi(s_0),\dots,x_n^\Pi(s_q)) - \mathsf{E}g(x_0^\Pi(s_0),\dots,x_0^\Pi(s_q))\big| \\
&\qquad\quad + \mathsf{E}\big|g(x_0^\Pi(s_0),\dots,x_0^\Pi(s_q)) - g(x_0^N(s_0),\dots,x_0^N(s_q))\big| \\
&\qquad\equiv I_1(n) + I_2(n) + I_1(0).
\end{aligned}$$

We have

$$\begin{aligned}
I_1(n) &\leqslant \|g\|_{BL}\Big(\mathsf{E}\big|\big(x_n^N(s_0),\dots,x_n^N(s_q)\big) - \big(x_n^\Pi(s_0),\dots,x_n^\Pi(s_q)\big)\big|^2_{H^{q+1}}\Big)^{1/2} \\
&\leqslant \|g\|_{BL}\gamma
\end{aligned}$$

for any $n \geqslant 0$ by (2.14). Moreover, Proposition 2.7 yields

$$\big(x_n^\Pi(t_0),\dots,x_n^\Pi(t_k)\big)(\mathsf{P}_n) \xrightarrow{n\to\infty} \big(x_0^\Pi(t_0),\dots,x_0^\Pi(t_k)\big)(\mathsf{P}_0) \qquad \text{weakly in } H^{k+1},$$

so also

$$\lim_{n\to\infty} I_2(n) = 0$$

as Π is a refinement of the partition Λ. Therefore, for any $\gamma > 0$ and $g \in BL(H^{q+1})$ we have

$$\big|\mathsf{E}g(x_n^N(s_0),\dots,x_n^N(s_q)) - \mathsf{E}g(x_0^N(s_0),\dots,x_0^N(s_q))\big| \leqslant (2\|g\|_{BL} + 1)\gamma$$

for all n sufficiently large, which proves the corollary. $\square$

Proof of Theorem 1.1. Take an arbitrary $\gamma > 0$. Note that there exists $N \in \mathbb{N}$ such that

$$\sup_{n\geqslant 0} \mathsf{P}_n\Big\{\omega;\ \sup_{0\leqslant t\leqslant T}|x_n(t) - x_n^N(t)| > 0\Big\} \leqslant \gamma. \tag{2.15}$$

Indeed, we use again the Prokhorov theorem and the weak convergence of η_n's to find a ball $K(0,N) = \{z \in H;\ |z| \leqslant N\}$ such that

$$\sup_{n\geqslant 0} \mathsf{P}_n\{\eta_n \notin K(0,N)\} < \gamma.$$

Then a standard local uniqueness argument (see e.g. [9], Lemma 4.2) and the continuity of sample paths yield

$$\mathsf{P}_n\Big(\{\eta_n \in K(0,N)\} \cap \Big\{\sup_{0\leqslant t\leqslant T}|x_n(t) - x_n^N(t)| > 0\Big\}\Big) = 0,$$

hence (2.15) holds true. Using Corollary 2.5 we find a partition $\{0 = t_0 < \ldots < t_k = T\}$ of the interval $[0, T]$ such that

$$\sup_{n\geqslant 0} \mathsf{E}\Big(\max_{i=0,\ldots,k-1} \sup_{t_i\leqslant t<t_{i+1}} \big|x_n^N(t) - x_n^N(t_i)\big|\Big)^2 \leqslant \gamma^2. \tag{2.16}$$

Let $f \in BL(C([0, T], H))$ be a bounded Lipschitz function. Denote by $\mathscr{P}$ the space of all functions $\psi\colon [0, T] \longrightarrow H$ which are right continuous with left limits on $[0, T]$ and continuous on $(0, T) \setminus \{t_1, \ldots, t_{k-1}\}$; endow $\mathscr{P}$ with the sup-norm. There exists a Lipschitz function $f^\natural \in BL(\mathscr{P})$ such that $f^\natural = f$ on $C([0, T], H)$ and $\|f^\natural\|_{BL} = \|f\|_{BL}$ (see e.g. [3], Theorem 6.1.1 and Proposition 11.2.2). Define

$$f^\flat\colon H^{k+1} \longrightarrow \mathbb{R}, \quad y = (y_0, \ldots, y_k) \longmapsto f^\natural(\tilde{y}),$$

where $\tilde{y} \in \mathscr{P}$ is defined by $\tilde{y}(t) = y_i$, $t_i \leqslant t < t_{i+1}$. Then $f^\flat \in BL(H^{k+1})$ and $\|f^\flat\|_{BL} \leqslant \|f\|_{BL}$. Further, set $\tilde{x}_n(t) = x_n^N(t_i)$, $t_i \leqslant t < t_{i+1}$, $i = 0, \ldots, k-1$; obviously, $\tilde{x}_n$ is a stochastic process with paths in $\mathscr{P}$.

Consequently,

$$\begin{aligned}
\big|\mathsf{E}f(x_n) - \mathsf{E}f(x_0)\big| &\leqslant \mathsf{E}\big|f(x_n) - f(x_n^N)\big| + \mathsf{E}\big|f(x_n^N) - f^\natural(\tilde{x}_n)\big| \\
&\quad + \big|\mathsf{E}f^\natural(\tilde{x}_n) - \mathsf{E}f^\natural(\tilde{x}_0)\big| \\
&\quad + \mathsf{E}\big|f^\natural(\tilde{x}_0) - f(x_0^N)\big| + \mathsf{E}\big|f(x_0^N) - f(x_0)\big| \\
&\equiv Z_1(n) + Z_2(n) + Z_3(n) + Z_2(0) + Z_1(0).
\end{aligned}$$

Setting $V(n) = \{\sup_{0\leqslant t\leqslant T} |x_n(t) - x_n^N(t)| > 0\}$ we get

$$Z_1(n) = \mathsf{E}\chi_{V(n)} \big|f(x_n) - f(x_n^N)\big| \leqslant 2\|f\|_{BL}\mathsf{P}_n(V(n)) \leqslant 2\|f\|_{BL}\gamma$$

for any $n \geqslant 0$ by (2.15). Furthermore,

$$\begin{aligned}
Z_2(n) &= \mathsf{E}\big|f^\natural(x_n^N) - f^\natural(\tilde{x}_n)\big| \leqslant \|f^\natural\|_{BL}\mathsf{E} \sup_{0\leqslant t\leqslant T}\big|x_n^N(t) - \tilde{x}_n(t)\big| \\
&\leqslant \|f\|_{BL}\gamma
\end{aligned}$$

for all $n \geqslant 0$ by (2.16). Finally,

$$Z_3(n) = \big|\mathsf{E}f^\flat(x_n^N(t_0), \ldots, x_n^N(t_k)) - \mathsf{E}f^\flat(x_0^N(t_0), \ldots, x_0^N(t_k))\big| \xrightarrow{n\to\infty} 0$$

by Corollary 2.8, so we have established that, given arbitrary $\gamma > 0$ and $f \in BL(C([0, T], H))$, one has

$$\big|\mathsf{E}f(x_n) - \mathsf{E}f(x_0)\big| \leqslant (6\|f\|_{BL} + 1)\gamma$$

for all n sufficiently large. This means that

$$x_{\varepsilon_n}(\cdot,\eta_{\varepsilon_n}) \longrightarrow x_0(\cdot,\eta_0) \qquad \text{weakly in } C([0,T],H) \tag{2.17}$$

for any sequence $\varepsilon_n \searrow 0$. Due to the metrizability of the weak convergence, (2.17) is equivalent to the assertion of Theorem 1.1. The proof is complete. $\square$

Proof of Proposition 1.2. Fix $x \in H$ and set

$$k_\varepsilon(s) = a_\varepsilon(s,x) - a_0(s,x), \qquad K_\varepsilon(s) = \int_0^s k_\varepsilon(u)\,\mathrm{d}u.$$

Note that by (A2) we have

$$L \equiv \sup_{\varepsilon\in[0,\varepsilon_0)} \sup_{s\in[0,T]} |k_\varepsilon(s)| < \infty, \qquad \sup_{\varepsilon\in[0,\varepsilon_0)} \sup_{s\in[0,T]} |K_\varepsilon(s)| \leqslant LT.$$

Take $\delta > 0$, then

$$\begin{aligned}\int_0^t \mathrm{e}^{A(t-s)} k_\varepsilon(s)\,\mathrm{d}s &= \int_0^{t-\delta} \mathrm{e}^{A(t-s)} k_\varepsilon(s)\,\mathrm{d}s + \int_{t-\delta}^t \mathrm{e}^{A(t-s)} k_\varepsilon(s)\,\mathrm{d}s\\ &\equiv I_1 + I_2.\end{aligned}$$

First, integrating by parts we obtain

$$I_1 = \mathrm{e}^{A\delta} K_\varepsilon(t-\delta) + \int_0^{t-\delta} A\mathrm{e}^{A(t-s)} K_\varepsilon(s)\,\mathrm{d}s.$$

The semigroup (e^{At}) is analytic, so the operators $A\mathrm{e}^{At}$ are continuous and (1.8) yields

$$\lim_{\varepsilon\to 0+} \mathrm{e}^{A\delta} K_\varepsilon(t-\delta) = 0$$

and

$$\lim_{\varepsilon\to 0+} A\mathrm{e}^{A(t-s)} K_\varepsilon(s) = 0, \qquad s \in [0, t-\delta].$$

Therefore, for any $\delta > 0$ fixed we get $I_1 \longrightarrow 0$ as $\varepsilon \to 0+$ by the dominated convergence theorem. Moreover,

$$|I_2| \leqslant \sup_{0\leqslant t\leqslant T} \left|\mathrm{e}^{At}\right|_{L(H)} L\delta,$$

so I_2 can be made arbitrarily small (uniformly in ε) by choosing δ sufficiently small, and the first statement of Proposition 1.2 follows. The second assertion can be proved in a similar way. $\square$

The following example shows that the assumptions on the convergence of coefficients adopted in Theorem 1.1 are strictly weaker than those used in Proposition 1.2.

Example 2.9. Let $A\colon \mathrm{Dom}(A) \longrightarrow H$ be a selfadjoint operator such that there exists an orthonormal basis $\{f_n\}_{n=1}^{\infty}$ consisting of eigenvectors of the operator A, $Af_i = \lambda_i f_i$, where $0 > \lambda_i \longrightarrow -\infty$. As is well known, A is an infinitesimal generator of an analytic selfadjoint C_0-semigroup (e^{At}) given by

$$\mathrm{e}^{At} = \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n t} f_n \otimes f_n, \qquad t \geqslant 0.$$

Let us define $a_n\colon [0,T] \times H \longrightarrow H$ by

$$a_n(s,x) = f_n, \qquad s \in [0,T],\ x \in H,\ n \geqslant 1.$$

Then

$$\begin{aligned}
\left| \int_0^t \mathrm{e}^{A(t-s)} a_n(s,x)\,\mathrm{d}s \right|_H &= \left| \sum_{k=1}^{\infty} \int_0^t \mathrm{e}^{\lambda_k(t-s)} \bigl\langle f_k, a_n(s,x) \bigr\rangle f_k \,\mathrm{d}s \right| \\
&= \left| \int_0^t \mathrm{e}^{\lambda_n(t-s)} f_n \,\mathrm{d}s \right| = \left| \frac{1}{\lambda_n} \bigl(\mathrm{e}^{\lambda_n t} - 1\bigr) f_n \right| \\
&\leqslant \frac{1}{|\lambda_n|} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

On the other hand,

$$\left| \int_0^t a_n(s,x)\,\mathrm{d}s \right|_H = t$$

for every $n \geqslant 1$.

Further, let us consider a Wiener process w in H of the form $w(t) = \beta(t) f_1$, where β is a standard one-dimensional Brownian motion. The covariance operator of the process w is $W = f_1 \otimes f_1$ (i.e., $Wh = \langle h, f_1 \rangle f_1$ for $h \in H$). Obviously, $W = W^{1/2}$ and $W f_k = \delta_{1k} f_1$ for $k \geqslant 1$. Define operators $b_n(s,x) \in L(H)$ by

$$b_n(s,x)h \equiv b_n h = \sum_{k=1}^{\infty} \langle h, f_k \rangle f_{k+n-1}, \qquad s \in [0,T],\ x, h \in H,\ n \geqslant 1.$$

Noticing that $b_n f_1 = f_n$ for $n \geqslant 1$ and using the positivity of the operator $\mathrm{e}^{A(t-s)} b_n W b_n^* \mathrm{e}^{A(t-s)}$ we obtain

$$\begin{aligned}
\left| \int_0^t \mathrm{e}^{A(t-s)} b_n W b_n^* \mathrm{e}^{A(t-s)} \, \mathrm{d}s \right|_{\mathscr{N}} &= \operatorname{tr} \left(\int_0^t \mathrm{e}^{A(t-s)} b_n W b_n^* \mathrm{e}^{A(t-s)} \, \mathrm{d}s \right) \\
&= \sum_{k=1}^{\infty} \left\langle \left(\int_0^t \mathrm{e}^{A(t-s)} b_n W b_n^* \mathrm{e}^{A(t-s)} \, \mathrm{d}s \right) f_k, f_k \right\rangle \\
&= \int_0^t \sum_{k=1}^{\infty} \left\langle \mathrm{e}^{A(t-s)} b_n W b_n^* \mathrm{e}^{A(t-s)} f_k, f_k \right\rangle \mathrm{d}s \\
&= \int_0^t \sum_{k=1}^{\infty} \left| W^{1/2} b_n^* \mathrm{e}^{A(t-s)} f_k \right|^2 \mathrm{d}s \\
&= \int_0^t \sum_{k=1}^{\infty} \left| W b_n^* \mathrm{e}^{A(t-s)} f_k \right|^2 \mathrm{d}s \\
&= \int_0^t \sum_{k=1}^{\infty} \left| \mathrm{e}^{A(t-s)} b_n W f_k \right|^2 \mathrm{d}s \\
&= \int_0^t \left| \mathrm{e}^{A(t-s)} b_n f_1 \right|^2 \mathrm{d}s = \int_0^t \left| \mathrm{e}^{A(t-s)} f_n \right|^2 \mathrm{d}s \\
&= \int_0^t \left| \mathrm{e}^{\lambda_n (t-s)} f_n \right|^2 \mathrm{d}s \leqslant \frac{1}{2|\lambda_n|} \xrightarrow{n \to \infty} 0.
\end{aligned}$$

At the same time,

$$\begin{aligned}
\left| \int_0^t b_n W b_n^* \, \mathrm{d}s \right|_{\mathscr{N}} &= t \, |b_n W b_n^*|_{\mathscr{N}} = t \operatorname{tr}(b_n W b_n^*) \\
&= t \sum_{j=1}^{\infty} |b_n W f_j|^2 = t |f_n|^2 = t \nrightarrow 0
\end{aligned}$$

as $n \to \infty$.

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