120 (1995) MATHEMATICA BOHEMICA No. 3, 293–298

A CHARACTERIZATION OF CHAOTIC FUNCTIONS WITH ENTROPY ZERO VIA THEIR MAXIMAL SCRAMBLED SETS

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(Received March 22, 1994)

Abstract. In this note we characterize chaotic functions (in the sense of Li and Yorke) with topological entropy zero in terms of the structure of their maximal scrambled sets. In the interim a description of all maximal scrambled sets of these functions is also found.

Keywords: chaotic functions, scrambled sets

MSC 1991: 58F13, 26A18, 54H20

1. Introduction

Let $C(I)$ denote the class of continuous funcions $f\colon I \to I$, where $I \subset \mathbb{R}$ is a compact interval. In order to describe how complicated is the dynamics of these functions the notion of chaos in the sense of Li and Yorke [7], [6] is quite a frequently employed approach.

Definition 1.1. Let $f\colon I \to I$ be a continuous function. Suppose that there exists $S \subset I$ with at least two elements such that for any $x, y \in S$, $x \neq y$, and any periodic point p of f

(1) $\limsup\limits_{n\to\infty} |f^n(x) - f^n(y)| > 0$,

(2) $\liminf\limits_{n\to\infty} |f^n(x) - f^n(y)| = 0$,

(3) $\limsup\limits_{n\to\infty} |f^n(x) - f^n(p)| > 0$.

(Here $f^0 = \mathrm{Id}$, f^n is the n-th iterate of f and a point $p \in I$ is said to be *periodic* – of period r – if there is a positive integer r such that $f^r(p) = p$ and $f^i(p) \neq p$ for

This work has been partially supported by the DGICYT PB91-0575.

any $1 \leqslant i < r$.) Then we say that f is *chaotic (in the sense of Li and Yorke)* and S is called a *scrambled set* of f.

It is well known that when the (topological) entropy $h(f)$ of f is positive (recall that $h(f) \in \mathbb{R}^+ \cup \{0\} \cup \{+\infty\}$, see Alder, Konheim and McAndrew [1] for the definition) then it is chaotic (see for example Janková and Smítal [4]). On the other hand, if $h(f) = 0$ then f can be both chaotic and non-chaotic (cf. Smítal [10]). When f is chaotic and $h(f) = 0$ then it must be necessarily of type 2^∞, that is, it has periodic points of period 2^n for any $n \geqslant 0$ and no other periods (see Šarkovskii [9]).

However, Definition 1.1 has an obvious drawback. Since the notion of a scrambled set is local, it can happen that the knowledge of a particular scrambled set of f does not help very much in getting a global picture of the behaviour of f (see Jiménez López [5] for more on this). So the following questions naturally arise: which are the maximal scrambled sets of a chaotic function and what are the relations among them (by a *maximal scrambled* set of f we mean a scrambled set which is not properly included in any scrambled set of f). In this note we find an answer to these questions for the case $h(f) = 0$ (see Theorem 2.6 in Section 2). This allows us to characterize chaotic functions with entropy zero in terms of their maximal scrambled sets as follows.

Definition 1.2. Let $f \in C(I)$ and $S, R \subset I$.

(1) We say that S and R are *equivalent* if there exists a bijection $\psi\colon S \to R$ such that

$$\lim_{n\to\infty} \big(f^n(x) - f^n(\psi(x))\big) = 0 \qquad \text{for any } x \in S.$$

(2) We say that S and R are *separated* if there exists $\delta > 0$ such that

$$\liminf_{n\to\infty} \big|f^n(x) - f^n(y)\big| \geqslant \delta \qquad \text{for any } x \in S \text{ and } y \in R.$$

Theorem 1.3. *Let $f \in C(I)$ be a chaotic function. Then $h(f) = 0$ if and only if every pair of maximal scrambled sets of f are either equivalent or separated.*

We will prove Theorem 1.3 in Section 3.

2. Describing all maximal scrambled sets of entropy zero functions

In order to get information about the structure of maximal scrambled sets of entropy zero functions the following definition will be useful.

Definition 2.1. Let $f \in C(I)$ and let J be a compact subinterval of I. We say that J is *periodic* – of period r – if $f^r(J) = J$ and $f^i(J) \cap f^j(J) = \emptyset$ for any $0 \leqslant i < j < r$.

Throughout this section we will assume that $f \in C(I)$ is an entropy zero function and denote the set of points from I for which $\omega_f(x)$ is infinite by $W(f)$ (recall that $\omega_f(x)$ is the set of accumulation points of the sequence $(f^n(x))_n$).

The following result is well known (see Smítal [10] and also Fedorenko and Smítal [3]).

Theorem 2.2. *Let $x \in W(f)$. Then there exists a sequence $(J_n)_{n=1}^{\infty}$ of periodic intervals exhibiting for any n the following properties:*

(1) J_n *has period* 2^n,

(2) $J_n \subset J_{n+1}$,

(3) $\omega_f(x) \subset \bigcup_{i=0}^{2^n-1} f^i(J_n)$.

In what follows we will fix for any $x \in W(f)$ a sequence $(J_n)_{n=1}^{\infty}$, which will be denoted by $(J_n(x))_{n=1}^{\infty}$. Also we define

$$C(x) = \bigcap_{n=1}^{\infty} \bigcup_{i=0}^{2^n-1} f^i(J_n(x))$$

and

$$A(x) = \{y \in I\colon \text{ for any } n \text{ there exists } k \text{ such that } f^k(y) \in J_n(x)\}.$$

Obviously, $x \in A(x)$ and $\omega_f(x) \subset C(x)$. Moreover, we have

Proposition 2.3. *Let x, $y \in W(f)$. Then either $C(x) = C(y)$ and $A(x) = A(y)$ or $C(x) \cap C(y) = \emptyset$ and $A(x) \cap A(y) = \emptyset$.*

Proof. This is an immediate consequence of Proposition 2.2 (7) from Preston [8]. □

The above proposition allows us to consider in $W(f)$ the equivalence relation $\sim$ defined by

$$x \sim y \quad \text{if} \quad C(x) = C(y).$$

We call every equivalence class with respect to $\sim$ a *$\sim$-class*. Moreover, for every $\sim$-class E let us fix some $x \in E$ and define $(J_n(E))_{n=1}^{\infty} = (J_n(x))_{n=1}^{\infty}$, $C(E) = C(x)$, $A(E) = A(x)$. Then we have

Lemma 2.4. *Let E and F be different $\sim$-classes. Then there exists $\delta > 0$ such that*

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| \geqslant \delta \qquad \textit{for any } x \in E \textit{ and } y \in F.$$

Proof. Recall that $C(E) \cap C(F) = \emptyset$ while $\omega_f(x) \subset C(E)$ and $\omega_f(y) \subset C(F)$ for any $x \in E$, $y \in F$. Since $C(E)$ and $C(F)$ are closed, the lemma easily follows by taking $\delta = \operatorname{dist}(C(E), C(F))$. □

To deepen the knowledge of every $\sim$-class E we will assign to any $x \in E$ a sequence of nonnegative integers $c(x) \in (\mathbb{N} \cup \{0\})^\infty$, where $c(x)_n$ is the first integer k for which $f^k(x) \in J_n(E)$. Moreover, given $x, y \in E$ we say that

$$x \sim_i y \quad \text{if} \quad c(x)_n \equiv c(y)_n \ (mod\ 2^n) \quad \text{for any } n.$$

We call every equivalence class with respect to $\sim_i$ a $\sim_i$-*class* (of course in a strict sense there are as many "$\sim_i$" equivalence relations as $\sim$-classes are). Now we get

Lemma 2.5. *Let E be a $\sim$-class. Then the following statements hold.*
(1) *If $E_1 \subset E$ is a $\sim_i$-class then*

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0 \qquad \text{for any } x, y \in E_1.$$

(2) *If $E_1, E_2 \subset E$ are different $\sim_i$-classes then there exists $\delta > 0$ such that*

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| \geqslant \delta \qquad \text{for any } x \in E_1 \text{ and } y \in E_2.$$

Proof. (1) Fix $\varepsilon > 0$ and choose an appropriate $f^i(J_k(E))$ such that its length is less than ε. Now, given $x, y \in E_1$ put $k(1) = c(x)_k$, $k(2) = c(y)_k$, let us say $k(1) \geqslant k(2)$. Then clearly $f^{k(1)}(x), f^{k(1)}(y) \in J_k(E)$, from which $f^{k(1)+i+m2^k}(x), f^{k(1)+i+m2^k}(y) \in f^i(J_k(E))$ for any m. This implies (1).

(2) Clearly there exists some k such that $c(x)_k \not\equiv c(y)_k$ (mod 2^k) for any $x \in E_1$, $y \in E_2$. Then it is sufficient to take $\delta = \min\limits_{0 \leqslant i < j < 2^k} \operatorname{dist}(f^i(J_k(E)), f^j(J_k(E)))$. □

Our final step consists in considering for every $\sim_i$-class the equivalence relation $\sim^s$ defined by

$$x \sim^s y \quad \text{if} \quad \lim_{n\to\infty} f^n(x) - f^n(y) = 0,$$

where we call every equivalence class with respect to $\sim^s$ a $\sim^s$-*class*. Now, in the light of Lemmas 2.4 and 2.5 we easily get

Theorem 2.6. *Let $f \in C(I)$, $h(f) = 0$, $S \subset I$. Then S is a maximal scrambled set of f if and only if there exists a $\sim_i$-class E containing more than one $\sim^s$-classes such that $S \subset E$ and S contains exactly one representative of every one of these $\sim^s$-classes.*

3. Proof of Theorem 1.3

We conclude this note by proving Theorem 1.3.

Proof of Theorem 1.3. The "only if" part is a direct consequence of Theorem 2.6 and Lemmas 2.4 and 2.5 (2). So let us prove the "if" part.

Assume that $h(f) > 0$. As is well known (see for example Fedorenko, Šarkovskii and Smítal [2]), there exist then some k and $C \subset I$ such that $f^k(C) = C$ and $g = f^k|_C$ is topologically conjugate to the shift τ on the space of sequences of zeros and ones $\{0,1\}^\infty$, that is, there is an homeomorphism $\varphi\colon C \to \{0,1\}^\infty$ for which $\varphi \circ g = \tau \circ \varphi$. Here recall that $\tau(\alpha)_n = \alpha_{n+1}$ for any n and any $\alpha \in \{0,1\}^\infty$, while $\{0,1\}^\infty$ is equipped with the topology of pointwise convergence (which induces a metric ϱ in this space). Notice also that $p \in C$ is periodic if and only if $\varphi(p)$ is a periodic sequence, that is, there exists r such that $\tau^r(\varphi(p)) = \varphi(p)$.

Now consider the points $\alpha^i \in \{0,1\}^\infty$, $i = 1,2,3$, defined by

$$\begin{aligned}
\alpha^1 &= 0,1,0,0,1,1,0,0,0,1,1,1,0,0,0,0,1,1,1,1,\dots,\\
\alpha^2 &= 1,0,1,1,0,0,1,1,1,0,0,0,1,1,1,1,0,0,0,0,\dots,\\
\alpha^3 &= 0,0,1,1,1,1,0,0,0,0,0,0,1,1,1,1,1,1,1,1,\dots
\end{aligned}$$

and put $x_i = \varphi^{-1}(\alpha^i)$ for any i. Since clearly

$$\liminf_{n\to\infty} \varrho(\tau^n(\alpha^1), \tau^n(\alpha^2)) > 0,$$
$$\limsup_{n\to\infty} \varrho(\tau^n(\alpha^1), \tau^n(\alpha^3)) > 0, \qquad \liminf_{n\to\infty} \varrho(\tau^n(\alpha^1), \tau^n(\alpha^3)) = 0,$$
$$\limsup_{n\to\infty} \varrho(\tau^n(\alpha^2), \tau^n(\alpha^3)) > 0, \qquad \liminf_{n\to\infty} \varrho(\tau^n(\alpha^2), \tau^n(\alpha^3)) = 0$$

and

$$\limsup_{n\to\infty} \varrho(\tau^n(\alpha^i), \tau^n(\beta)) > 0 \qquad \text{for any periodic sequence } \beta \text{ and any } i,$$

we easily get that

$$\liminf_{n\to\infty} |f^n(x_1) - f^n(x_2)| > 0$$

while $\{x_1, x_3\}$ and $\{x_2, x_3\}$ are scrambled sets of f (notice in particular that

$$\liminf_{n\to\infty} |f^n(x_i), f^n(p)| > 0$$

for any periodic point $p \notin C$ and any i because C is closed). Then consider maximal scrambled sets S and R containing respectively $\{x_1, x_3\}$ and $\{x_2, x_3\}$. Obviously S and R are not separated. Moreover, S and R are not equivalent, since otherwise

$$\liminf_{n\to\infty} |f^n(x_2) - f^n(\psi(x_1))| = 0$$

for the corresponding bijection $\psi\colon S \to R$, which is clearly impossible.

This completes the proof. $\square$

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