THE COUNTERPARTS OF SOME CARDINAL FUNCTIONS IN BITOPOLOGICAL SPACES II

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Abstract. In this paper, bitopological counterparts of the cardinal functions Lindelöf number, weak Lindelöf number and spread are introduced and studied. Some basic relations between these functions and the functions in [3] are given.

Keywords: bi-Lindelöf number, weak bi-Lindelöf number, bispread, bi-quasi-uniform weight

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In the preceding paper of this series, counterparts of the functions weight, density and cellularity were defined [3]. Here, *bi-Lindelöf number* is defined, and shown to be equal to the joint Lindelöf number. Following this we define the *weak bi-Lindelöf number*, and consider its relation with bicellularity. *Bidiscreteness* is introduced, the *bispread* of a bitopological space is defined, and the special properties of the various cardinal functions under consideration which hold on p-q metric space are obtained. Considering the *bi-quasi-uniform weight* of Kopperman and Meyer [7], some basic relations are obtained in this class of spaces for the biweight, bicellularity, bi-quasi-uniform weight and weak bi-Lindelöf number. For notation and terminology which is not explained here, we refer to [2], [5] and [6]. As in [3], bitopological counterparts of topological cardinal invariants are denoted by preceding the usual name with b (bw = biweight, etc.). The prefix j denotes the corresponding invariant applied to the joint topology.

1. Bi-Lindelöf number, weak bi-Lindelöf number and bispread

1.1. Definition. Let (X,u,v) be a bitopological space. X is called α *bi-Lindelöf* if every open dual cover has a subcover whose cardinal number is at most α. The cardinal number

$$\mathrm{bL}(X)=\min\{\alpha\colon X \text{ is } \alpha \text{ bi-Lindelöf}\}$$

is called the *bi-Lindelöf number* of (X,u,v).

1.2. Theorem. *For every bitopological space (X,u,v), we have*

$$\mathrm{bL}(X)=\mathrm{jL}(X).$$

Proof. $\mathrm{bL}(X)\leqslant \mathrm{jL}(X)$ is immediate from our observation that with every open dual cover d we may associate the jointly open dual cover $\{U\cap V\colon UdV\}$. Hence, it is enough to show that $\mathrm{jL}(X)\leqslant \mathrm{bL}(X)$. Let $\delta=\{I_\alpha\colon \alpha\in A\}$ be a jointly open cover of X. For each $\alpha\in A$, we can choose pairwise disjoint sets A_α such that $\bigcup\{U_\lambda\cap V_\lambda\colon \lambda\in A_\alpha\}=I_\alpha$ and $(U_\lambda,V_\lambda)\in u\times v$. Hence, $d=\{(U_\lambda,V_\lambda)\colon \lambda\in\bigcup A_\alpha\}$ is an open dual cover of X. Choose a subcover $e=\{(U_\lambda,V_\lambda)\colon \lambda\in B\}$ of d with $|B|\leqslant \mathrm{bL}(X)$, and $C=\{\alpha\colon \exists\lambda\in B(\lambda\in A_\alpha)\}$. It is easy to see that $\bigcup\{I_\alpha\colon \alpha\in C\}=X$ and $|C|\leqslant \mathrm{bL}(X)$. Hence, we obtain $\mathrm{jL}(X)\leqslant \mathrm{bL}(X)$. □

By an open pair we shall mean an ordered pair of sets (G,H) with $G\in u$ and $H\in v$. The following definition generalizes the concept of the weak Lindelöf number [6].

1.3. Definition. Let (X,u,v) be a bitopological space, d an open dual family, that is $d\subseteq u\times v$. Then d is called a *weak open dual cover* of X if given an open pair (G,H) with $G\cap H\neq\emptyset$, there exists $(U,V)\in d$ such that $G\cap V\neq\emptyset$ and $H\cap U\neq\emptyset$. If every open dual cover has a weak subcover whose cardinal number is at most α, then X is called *weak α bi-Lindelöf.* The cardinal number

$$\mathrm{wbL}(X)=\min\{\alpha\colon X \text{ is weak } \alpha \text{ bi-Lindelöf}\}$$

will be called the *weak bi-Lindelöf number* of X.

1.4. Theorem.

(i) $\mathrm{wbL}(X)\leqslant \mathrm{jwL}(X)\leqslant \mathrm{bL}(X)$

(ii) $\mathrm{wbL}(X)\leqslant \mathrm{bc}(X)$.

Proof. (i) Let $d=\{(U_\alpha,V_\alpha)\colon \alpha\in A\}$ be an open dual cover of X. Clearly, $\delta=\{U_\alpha\cap V_\alpha\colon \alpha\in A\}$ is a jointly open cover of X. We choose a weak subcover

$\delta' = \{U_\alpha \cap V_\alpha \colon \alpha \in A'\}$ of δ with $|A'| \leqslant \mathrm{jwL}(X)$. Since $\overline{\bigcup \delta'}^{u \vee v} = X$, for each open pair (G, H) with $G \cap H \neq \emptyset$ there exists $\alpha \in A'$ such that $G \cap H \cap U_\alpha \cap V_\alpha \neq \emptyset$. Hence, $G \cap V_\alpha \neq \emptyset$ and $H \cap U_\alpha \neq \emptyset$. Thus the open dual family $\{(U_\alpha, V_\alpha) \colon \alpha \in A'\}$ is a weak subcover of d whose cardinality is at most $\mathrm{jwL}(X)$. We have shown that $\mathrm{wbL}(X) \leqslant \mathrm{jwL}(X)$. Since $\mathrm{jwL}(X) \leqslant \mathrm{jL}(X)$, by Theorem 1.2 we have $\mathrm{jwL}(X) \leqslant \mathrm{bL}(X)$ as well.

(ii) Let e be an open dual cover of X and $\mathscr{C} = \{(U_\alpha, V_\alpha) \colon \alpha \in A\}$ a maximal bicellular refinement of e (such a refinement exists by Zorn's Lemma). Let us show that $\mathscr{C}$ is a weak subcover of X. Suppose the contrary is true. Then there exists an open pair (G, H) with $G \cap H \neq \emptyset$ such that for each $\alpha \in A$, $U_\alpha \cap H = \emptyset$ or $V_\alpha \cap G = \emptyset$. Take $x \in G \cap H$ and choose $(R, S) \in e$ with $x \in R \cap S$. Let $U = G \cap R$ and $V = H \cap S$. Then $U \cap V \neq \emptyset$ and for each α, $U_\alpha \cap V = \emptyset$ or $V_\alpha \cap U = \emptyset$. Hence, $\mathscr{C}^\star = \mathscr{C} \cup \{(U, V)\}$ is a bicellular family in X, and clearly, $\mathscr{C}^\star \prec e$. Since $(U, V) \notin \mathscr{C}$, this contradicts the maximality of $\mathscr{C}$. Now for each $\alpha \in A$ we choose $(R_\alpha, S_\alpha) \in e$ with $U_\alpha \subseteq R_\alpha, V_\alpha \subseteq S_\alpha$. Then the family $\{(R_\alpha, S_\alpha) \colon \alpha \in A\}$ is a weak subcover of e whose cardinality is at most $\mathrm{bc}(X)$. Thus we have $\mathrm{wbL}(X) \leqslant \mathrm{bc}(X)$. $\square$

1.5. Definition. A bitopological space (X, u, v) is called *bidiscrete* if for each $x \in X$ there exists an open pair (U_x, V_x) with $x \in U_x \cap V_x$ satisfying the condition

$$\forall y \in X, x \neq y \Rightarrow U_x \cap V_y = \emptyset \quad \text{or} \quad U_y \cap V_x = \emptyset \quad .$$

Trivially, every bidiscrete space is jointly discrete. For a discrete topological space, it is well known that $w(X) = |X| = \mathrm{d}(X)$. A similar result is, however, not true for bidiscrete bitopological spaces.

1.6. Example. Let $X = \mathbb{R}$. Consider the topologies $u = \{(-\infty, a] \colon a \in \mathbb{R}\}$ and $v = \{[b, \infty) \colon b \in \mathbb{R}\}$ on $\mathbb{R}$. The space $(\mathbb{R}, u, v)$ is bidiscrete and $\mathrm{qrd}(\mathbb{R}) = \mathrm{rd}(\mathbb{R}) = \mathrm{bd}(\mathbb{R}) < \mathrm{bw}(\mathbb{R}) = \omega_1$.

However, we do have:

1.7. Theorem. *If (X, u, v) is bidiscrete, then*

$$\mathrm{bd}(X) \leqslant |X| \leqslant \mathrm{bw}(X).$$

1.8. Definition. Let (X, u, v) be a bitopological space. The cardinal number

$$\mathrm{bs}(X) = \sup \bigl\{|D| \colon D \text{ is bidiscrete in } (X, u, v)\bigr\}$$

is called the *bispread* of X.

This generalizes the spread $\mathrm{s}(X)$ of a topological space X, see for example [6]. The following fact is evident:

1.9. Theorem.

$$\mathrm{bs}(X) \leqslant \mathrm{js}(X).$$

2. p-q METRIZABLE SPACES

It is well known that in a (pseudo) metric topological space several cardinal invariants coincide. Recall that a bitopological space (X, u, v) is called *p-q metrizable* if there exists a pseudo quasi-metric p such that u is the topology of p and v the topology of its conjugate q.

Also, the extent of a topological space X is defined by

$$\mathrm{e}(X) = \sup\big\{|D| \colon D \text{ is closed and discrete in } X\big\}.$$

2.1. Theorem. *If (X, u, v) is weakly pairwise T_1 and p-q metrizable, then*

$$\mathrm{bw}(X) = \mathrm{jw}(X) = \mathrm{bL}(X) = \mathrm{je}(X) = \mathrm{bs}(X) = \mathrm{bc}(X) = \mathrm{jd}(X).$$

Proof. Clearly, $\mathrm{bw}(X) \geqslant \mathrm{jw}(X) \geqslant \mathrm{bL}(X) \geqslant \mathrm{je}(X)$. If A is a bidiscrete subspace of X, it is also jointly discrete. Furthermore, X is jointly T_2 and perfectly normal. Hence, using a standart topological argument, we easily obtain $\mathrm{je}(X) \geqslant |A|$, that is $\mathrm{je}(X) \geqslant \mathrm{bs}(X)$. Now let $\mathscr{C} = \{(U_\alpha, V_\alpha) \colon \alpha \in B\}$ be a bicellular family in X. For each $\alpha \in B$, choose $x_\alpha \in U_\alpha \cap V_\alpha$. It is easy to see that $D = \{x_\alpha \colon \alpha \in B\}$ is bidiscrete, and $|D| = |\mathscr{C}| \leqslant \mathrm{bs}(X)$. Hence, $\mathrm{bs}(X) \geqslant \mathrm{bc}(X)$. Now we will show that $\mathrm{bc}(X) \geqslant \mathrm{jd}(X)$. Let p be a pseudo quasi metric compatible with (X, u, v) and let q be the conjugate of p. For $i = 1, 2, \ldots$ consider the family

$$\mathscr{G}_i = \left\{B \subset X \colon x,\ y \in B,\ x \neq y \Rightarrow p(x, y) \geqslant \frac{1}{i} \ \text{ or } \ q(x, y) \geqslant \frac{1}{i}\right\}.$$

By using Teichmüller-Tukey Lemma, for each $i = 1, 2, \ldots$ we can find a maximal set $G_i \in \mathscr{G}_i$ such that

$$x,\ y \in G_i,\ x \neq y \Rightarrow p(x, y) \geqslant \frac{1}{i} \ \text{ or } \ q(x, y) \geqslant \frac{1}{i}.$$

It be can easily checked that for each $i = 1, 2, \ldots$,

$$\mathscr{C}_i = \left\{\left(B_p\left(x, \frac{1}{2i}\right), B_q\left(x, \frac{1}{2i}\right)\right) \colon x \in G_i\right\}$$

is a bicellular family in X, and $|G_i| = |\mathscr{C}_i| \leqslant \mathrm{bc}(X)$. Let $G = \bigcup\{G_i : i = 1, 2, \ldots\}$. Clearly, $|\mathscr{C}| \leqslant \mathrm{bc}(X)$. Now we show that G is jointly dense. Suppose the contrary. Let $x \in X \setminus \overline{G}^{u \vee v}$. Then there exists a natural number i_0 such that

$$(p \vee q)(a, G_{i_0}) \geqslant (p \vee q)(a, G) \geqslant \frac{1}{i_0}.$$

Consider the set $H = \{a\} \cup G_{i_0}$. For $x, y \in H$, we have $(p \vee q)(x, y) \geqslant \frac{1}{i_0}$ and so $p(x, y) \geqslant \frac{1}{i_0}$ or $q(x, y) \geqslant \frac{1}{i_0}$. But this contradicts the maximality of G_{i_0}. Thus, G is as desired. Hence, $\mathrm{bc}(X) \geqslant \mathrm{jd}(X)$. (If G_i is empty, then u and v are discrete topologies, and the result is immediate.)

$\mathrm{jd}(X) \geqslant \mathrm{bw}(X)$: Let A be a jointly dense subset of X with $|A| = \mathrm{jd}(X)$. It is easy to see that the family

$$d = \Big\{\big(B_p(y, r),\ B_q(y, r)\big) : y \in A,\ r \in \mathbb{Q}\Big\}$$

is a bibase for X and $|A| \geqslant |d|$, that is $\mathrm{jd}(X) \geqslant \mathrm{bw}(X)$. □

The following example shows that in general $\mathrm{bd}(X)$ cannot be included in the above equalities:

2.2. Example. Consider the set $X = \{(x, y) : x \geqslant 0, y \geqslant 0\} \subset \mathbb{R}^2$. Let u consist of $\emptyset$ and all subsets G of X satisfying

(i) $(x, y) \in G,\ 0 < x' \leqslant x \Rightarrow (x', y) \in G$

(ii) $(x, y) \in G,\ 0 < y \leqslant y' \Rightarrow (x, y') \in G$

(iii) $\exists y > 0$ with $(0, y) \in G$.

Clearly, u is a topology on X, and so is $v = \{G^{-1} : G \in u\}$. The space (X, u, v) is weakly pairwise T_1 and p-q metrizable [2]. The set $A = \{(x, y) : x \geqslant 0,\ y \geqslant 0$ and $x, y \in \mathbb{Q}\}$ is bidense in X, with $\mathrm{bd}(X) = |A| = \omega$. However, $\mathrm{bw}(X) = \mathrm{jd}(X) = \omega_1$. Hence, $\mathrm{bd}(X) < \mathrm{bw}(X)$.

If we remove the condition that (X, u, v) is weakly pairwise T_1, we obtain the following more limited result:

2.3. Theorem. *If (X,u,v) is p-q metrizable, then*

$$\mathrm{bc}(X) = \mathrm{jd}(X) = \mathrm{bw}(X).$$

3. Pairwise completely regular spaces

Let $\{(X_\alpha, u_\alpha, v_\alpha)\}_{\alpha\in A}$ be a family of bitopological spaces. Consider the product bitopological space (X, u, v), where $X = \prod\limits_{\alpha\in A} X_\alpha,\ u = \prod\limits_{\alpha\in A} u_\alpha,\ v = \prod\limits_{\alpha\in A} v_\alpha$.

The following theorem generalizes the well known properties of the weight in topological spaces:

3.1. Theorem.

$$\mathrm{bw}(X) = |A| \sup \big\{ \mathrm{bw}(X_\alpha) \colon \alpha \in A \big\}.$$

3.2. Definition. Let (X, u, v) be a pairwise completely regular space. The cardinal number

$$\mathrm{bq}(X) = \min \big\{ |\delta| \colon \delta \text{ is a base for a quasi-uniformity compatible with } X \big\}$$

is called the *biquasi-uniform weight* of X.

3.3. Theorem. *If (X, u, v) is pairwise completely regular, then*

$$\mathrm{bw}(X) \leqslant \mathrm{bq}(X) \cdot \mathrm{bc}(X).$$

Proof. If (X, u, v) is p-q metrizable, then by Theorem 2.3 the assertion is immediate. Assume X is not p-q metrizable. Consider the family $\mathscr{P}$ of p-q metrics, with $|\mathscr{P}| = \mathrm{bq}(X)$ (the gage of X). If $|\mathscr{P}| \geqslant \mathrm{bw}(X)$, then the proof is complete. Suppose $|\mathscr{P}| < \mathrm{bw}(X)$. Consider the bitopological space X_p determined by $p \in \mathscr{P}$. By Theorem 3.1, we have

$$\mathrm{bw}\left(\prod \{X_p \colon p \in \mathscr{P}\}\right) = |\mathscr{P}| \sup \big\{ \mathrm{bw}(X_p) \colon p \in \mathscr{P} \big\}.$$

It can be checked that $\mathrm{bw}(X) \leqslant \mathrm{bw}(\prod\{X_p \colon p \in \mathscr{P}\})$ (cf.[4]). Hence, $\mathrm{bw}(X) \leqslant \sup\{\mathrm{bw}(X_p) \colon p \in \mathscr{P}\}$. By Theorem 2.3, $\mathrm{bw}(X_p) = \mathrm{bc}(X_p)$ and so $\mathrm{bw}(X) \leqslant \sup\{\mathrm{bc}(X_p) \colon p \in \mathscr{P}\}$. Clearly, $\sup\{\mathrm{bc}(X_p) \colon p \in \mathscr{P}\} \leqslant \mathrm{bc}(X)$. Finally, we obtain $\mathrm{bw}(X) \leqslant \mathrm{bc}(X)$. This completes the proof. □

Now we give a stronger result than Theorem 3.3.

3.4. Theorem. *If (X,u,v) is pairwise completely regular, then*

$$\mathrm{bw}(X) \leqslant \mathrm{bq}(X) \cdot \mathrm{wbL}(X).$$

Proof. Let δ be a covering base [1] of a quasi-uniformity compatible with (X,u,v), and $|\delta| = \mathrm{bq}(X)$. We take an open pair (G,H) with $G \cap H \neq \emptyset$. Let $x \in G \cap H$. Then there exists an open normal dual cover $d \in \delta$ such that $\mathrm{St}(d,x) = \bigcup\{U\colon \exists V (U,V) \in d, x \in V\} \subseteq G$ and $\mathrm{St}(x,d) = \bigcup\{V\colon \exists U (U,V) \in d, x \in U\} \subseteq H$. Let $e \in \delta$, $e \prec \star d$. Choose $(R,S) \in e$ and $x \in R \cap S$. Consider a weak subcover I_e of e with $|I_e| \leqslant \mathrm{wbL}(X)$. There exists $(L,T) \in I_e$ such that $S \cap L \neq \emptyset$ and $R \cap T \neq \emptyset$. Since $e \prec \star d$, there exists an open pair $(U,V) \in d$ such that $\mathrm{St}(e,L) = \bigcup\{R\colon (R,S) \in e, S \cap L \neq \emptyset\} \subseteq U, \mathrm{St}(T,e) = \bigcup\{S\colon (R,S) \in e, R \cap T \neq \emptyset\} \subseteq V$. Clearly, $x \in U \cap V$. Since $x \in \mathrm{St}(e,L) \subseteq U \subseteq \mathrm{St}(d,x) \subseteq G$ and $x \in \mathrm{St}(T,e) \subseteq V \subseteq \mathrm{St}(x,d) \subseteq H$, the family

$$d' = \Big\{ \big(\mathrm{St}(e,L), \mathrm{St}(T,e) \big) \colon e \in \delta, (L,T) \in I_e \Big\}$$

is a bibase for X. Hence, we obtain

$$\mathrm{bw}(X) \leqslant |d'| \leqslant \mathrm{bq}(X) \cdot \mathrm{wbL}(X).$$

$\square$

Remark. Note that Theorem 3.3 can be also obtained as a consequence of Theorems 1.4(ii) and 3.4.

3.5. Theorem. [7] *If (X,u,v) is pairwise completely regular, then*

$$\mathrm{bq}(X) \leqslant \mathrm{bw}(X).$$

As a consequence of Theorems 3.4 and 3.5 we have the following

3.6. Corollary. *If (X,u,v) is pairwise completely regular and* F *is an element of* $\{\mathrm{bc}, \mathrm{jc}, \mathrm{bs}, \mathrm{js}, \mathrm{bL}, \mathrm{wbL}, \mathrm{jwL}\}$, *then*

$$\mathrm{bw}(X) = \mathrm{bq}(X) \cdot \mathrm{F}(X).$$

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