INERTIAL LAW OF QUADRATIC FORMS ON MODULES OVER PLURAL ALGEBRA

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Abstract. Quadratic forms on a free finite-dimensional module are investigated. It is shown that the inertial law can be suitably generalized provided the vector space is replaced by a free finite-dimensional module over a certain linear algebra over $\mathbb{R}$ (*real plural algebra*) introduced in [1].

Keywords: linear algebra, free module, bilinear form, quadratic form, polar basis

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I. INTRODUCTION

Definition I.1. The *real plural algebra of order m* is every linear algebra $\mathbf{A}$ on $\mathbb{R}$ having as a vector space over $\mathbb{R}$ a basis

$$\{1, \eta, \eta^2, \ldots, \eta^{m-1}\}, \quad \text{with } \eta^m = 0.$$

Definition I.2. The *system of projections* $\mathbf{A} \to \mathbb{R}$ is a system of mappings $p_k\colon \mathbf{A}$ onto $\mathbb{R}$, defined for $k = 0, \ldots, m-1$, as follows:

$$\forall \beta \in \mathbf{A}, \quad \beta = \sum_{i=0}^{m-1} b_i \eta^i; \quad p_k(\beta) \overset{\text{def}}{=} b_k.$$

To make the paper selfcontained we present several propositions proved in [1].

Proposition I.3. *$\mathbf{A}$ is a local ring with the maximal ideal $\eta\mathbf{A}$. The ideals $\eta^j\mathbf{A}$, $1 \leqslant j \leqslant m$, are all ideals of $\mathbf{A}$.*

Proposition I.4. *The ring* $\mathbf{A}$ *is isomorphic to the factor ring of polynomials* $\mathbb{R}[x]/(x^m)$.

Agreement I.5. Throughout the paper we denote by $\mathbf{A}$ the $\mathbb{R}$-algebra introduced in this section. The capital $\mathbf{M}$ always denotes the free finite-dimensional module over the algebra $\mathbf{A}$.

Proposition I.6. *Let* $\{E_1, \dots, E_n\}$ *be a (linearly independent) system of generators of a module* $\mathbf{M}$. *If* $U_1, \dots, U_k$ *are linearly independent elements from* $\mathbf{M}$ *then*

(1) $k \leqslant n$,

(2) *After a suitable renumbering of the elements* $E_1, \dots, E_n$, $\{U_1, \dots, U_k, E_{k+1}, \dots, E_n\}$ *will be a (linearly independent) set of generators of* $\mathbf{M}$.

Proposition I.7. *Let* $\mathscr{E} = \{E_1, \dots, E_n\}$ *be a basis of the* $\mathbf{A}$*-module* $\mathbf{M}$. *Let us define a system* $P_0, \dots, P_{m-1}$ *of vector spaces over* $\mathbb{R}$*:*

$$\mathbf{P}_j = [\eta^j E_1, \dots, \eta^j E_n], \quad 0 \leqslant j \leqslant m-1.$$

If we view $\mathbf{M}$ *as an* $\mathbb{R}$*-vector space, then the following statements are valid:*

(1) $\mathbf{M} = \bigoplus\limits_{j=0}^{m-1} \mathbf{P}_j$,

(2) $\forall X \in \mathbf{M}\ \exists!(X_0, \dots, X_{m-1}) \in \mathbf{P}_0^m;\ X = \sum\limits_{j=0}^{m-1} \eta^j X_j$.

Theorem I.8. *If* $\varphi\colon \mathbf{M} \to \mathbf{A}$ *is a linear form such that* $(\operatorname{Im}\varphi \setminus \eta\mathbf{A}) \neq \emptyset$ *then there exists exactly one free* $(n-1)$*-dimensional submodule* $\mathscr{N}$ *of* $\mathbf{M}$ *such that*

$$\mathscr{N} = \operatorname{Ker}\varphi.$$

II. Bilinear forms on modules over the algebra A

The relations between bilinear forms, their projections and bilinear forms $\mathbf{P}_0^2 \to \mathbb{R}$ from Proposition II.5 are similar to those between analogous objects in the case of linear forms described in [1]. Thus the proofs of Propositions II.1–II.6 will be omitted.

Proposition II.1. *Let* $\Phi\colon \mathbf{M}^2 \to \mathbf{A}$ *be a bilinear form. Then there exists exactly one system of bilinear forms* $\Phi_0, \dots, \Phi_{m-1}\colon \mathbf{M}^2 \to \mathbb{R}$ *such that*

$$\Phi = \sum_{j=0}^{m-1} \Phi_j \eta^j.$$

Definition II.2. The bilinear forms $\Phi_0, \ldots, \Phi_{m-1}\colon \mathbf{M}^2 \to \mathbb{R}$ from Proposition II.1 will be called *projections of* Φ (Φ_j is the *j-th projection*).

Proposition II.3. *If $\Phi_0, \ldots, \Phi_{m-1}\colon \mathbf{M}^2 \to \mathbb{R}$ are bilinear forms then the mapping $\Phi = \sum\limits_{j=0}^{m-1} \Phi_j \eta^j$ is a bilinear form $\mathbf{M}^2 \to \mathbf{A}$ if and only if $\forall X, Y \in \mathbf{M}$:*
(1) $\Phi_0(\eta X, Y) = 0$,
(2) $\Phi_k(\eta X, Y) = \Phi_{k-1}(X, Y)$, $1 \leqslant k \leqslant m-1$,
(3) $\Phi_0(X, \eta Y) = 0$,
(4) $\Phi_k(X, \eta Y) = \Phi_{k-1}(X, Y)$, $1 \leqslant k \leqslant m-1$.

Proposition II.4. *Let $\Phi_0, \ldots, \Phi_{m-1}\colon \mathbf{M}^2 \to \mathbb{R}$ be a system of bilinear forms such that $\sum\limits_{j=0}^{m-1} \Phi_j \eta^j$ is a bilinear form $\mathbf{M}^2 \to \mathbf{A}$. Then $\forall X, Y \in \mathbf{M}$, $X = \sum\limits_{j=0}^{m-1} \eta^j X_j$, $Y = \sum\limits_{j=0}^{m-1} \eta^k Y_k$, Y_k, $X_j \in \mathbf{P}_0$ we have*

$$\Phi_k(X, Y) = \sum_{j=0}^{k} \sum_{h=0}^{l} \Phi_{k-j}(X_h, Y_{j-h}), \quad 0 \leqslant k \leqslant m-1.$$

Proposition II.5. *Let $\Phi\colon \mathbf{M}^2 \to \mathbf{A}$ be a bilinear form, let $\Phi_0, \ldots, \Phi_{m-1}$ be a system of its projections. Then there exists exactly one system of bilinear forms $F_0, \ldots, F_{m-1}\colon \mathbf{P}_0^2 \to \mathbb{R}$ such that $\forall X, Y \in \mathbf{M}$, $X = \sum\limits_{j=0}^{m-1} \eta^j X_j$, $Y = \sum\limits_{k=0}^{m-1} \eta^k Y_k$, X_j, $Y_k \in \mathbf{P}_0$ the following relation is true:*

$$(*) \qquad \Phi_k(X, Y) = \sum_{j=0}^{k} \sum_{h=0}^{j} F_{k-j}(X_h, Y_{j-h}), \quad 0 \leqslant k \leqslant m-1.$$

Proposition II.6. *Let $\{F_j\}_{j=0}^{m-1}$ be a system of bilinear forms $\mathbf{P}_0^2 \to \mathbb{R}$ and let $\{\Phi_k\}_{k=0}^{m-1}$ be the system of bilinear forms $\mathbf{M}^2 \to \mathbb{R}$ defined as follows: $\forall X, Y \in \mathbf{M}$, $X = \sum\limits_{j=0}^{m-1} \eta^j X_j$, $Y = \sum\limits_{k=0}^{m-1} \eta^k Y_k$;*

$$(**) \qquad \Phi_k(X, Y) \stackrel{\text{def}}{=} \sum_{j=0}^{k} \sum_{h=0}^{j} F_{k-j}(X_h, Y_{j-h}), \quad 0 \leqslant k \leqslant m-1.$$

Then the mapping $\Phi = \sum\limits_{k=0}^{m-1} \Phi_k \eta^k$ is a bilinear form $\mathbf{M}^2 \to \mathbf{A}$ determined uniquely by the system $\{F_j\}$.

Definition II.7. A bilinear form $\Phi\colon \mathbf{M}^2 \to \mathbf{A}$ is called a *bilinear form of order* k $(0 \leqslant k \leqslant m)$ if

(1) $\forall (X,Y) \in \mathbf{M}^2;\ \Phi(X,Y) \in \eta^k \mathbf{A}$,

(2) $\exists (U,V) \in \mathbf{M}^2;\ \Phi(U,V) \notin \eta^{k+1}\mathbf{A}$.

In the special case $k=0$ the bilinear form is called the *epiform.*

Proposition II.8. *If Φ is a bilinear form of order k then there exists at least one epiform Λ such that*

$$\Phi = \eta^k \Lambda.$$

P r o o f. Let Φ be a bilinear form of order k. Hence we have

$$(*) \qquad \Phi_0 \equiv \Phi_1 \equiv \ldots \equiv \Phi_{k-1} \equiv 0 \wedge \exists (U,V) \in \mathbf{M}^2; \quad \Phi_k(U,V) \neq 0.$$

Let us denote $\Phi^* = \Phi_k + \ldots + \eta^{m-k-1}\Phi_{m-1}$. Then $\Phi = \eta^k \Phi^*$, though generally Φ^* is not a bilinear form $\mathbf{M} \to \mathbf{A}$. According to Proposition II.5 there is a uniquely determined system $\{F_j\}$ of bilinear forms $\mathbf{P}_0^2 \to \mathbb{R}$ fulfilling II.5.$(*)$ for the bilinear form Φ. Since II.8.$(*)$ is true we have from II.5.$(*)$:

$$(**) \qquad F_0 \equiv F_1 \equiv \ldots \equiv F_{k-1} \equiv 0.$$

Let us define a system $\{H_j\}_{j=0}^{m-1}$ of bilinear forms $\mathbf{P}_0^2 \to \mathbb{R}$ as follows. Let

$$(***) \qquad H_0 = F_k, \quad H_1 = F_{k+1}, \ldots, H_{m-k-1} = F_{m-1}$$

and let bilinear forms $H_{m-k}, \ldots, H_{m-1}$ be chosen arbitrarily.

With respect to Proposition II.6 for the system $\{H_0, \ldots, H_{m-k-1}, H_{m-k}, \ldots, H_{m-1}\}$ we have the system $\{\Lambda_j\}$ of bilinear forms $\mathbf{M}^2 \to \mathbb{R}$ given by

$$\Lambda_k(X,Y) = \sum_{j=0}^{k} \sum_{h=0}^{j} H_{k-j}(X_k, Y_{j-h}), \quad 0 \leqslant k \leqslant m-1,$$

for which Λ defined by $\Lambda = \sum\limits_{j=0}^{m-1} \Lambda_j \eta^j$ is a bilinear form $\mathbf{M}^2 \to \mathbf{A}$. For $r \geqslant k$ we get [using $(**)$, $(***)$]: $\forall X,\, Y \in \mathbf{M}$, $X = \sum\limits_{j=0}^{m-1} \eta^j X_j$, $Y = \sum\limits_{k=0}^{m-1} \eta^k Y_k$;

$$\begin{aligned}
\Phi_r(X,Y) &= \sum_{j=0}^{r} \sum_{h=0}^{j} F_{r-j}(X_h, Y_{j-h}) = \sum_{j=0}^{r-k} \sum_{h=0}^{j} F_{r-j}(X_h, Y_{j-h}) \\
&= \sum_{j=0}^{r-k} \sum_{h=0}^{j} H_{(r-k)-j}(X_h, Y_{j-h}) = \Lambda_{r-k}, \quad k \leqslant r \leqslant m-k-1,
\end{aligned}$$

i.e. $\Lambda_0 = \Phi_k, \Lambda_1 = \Phi_{k+1}, \ldots, \Lambda_{m-k-1} = \Phi_{m-1}$. Clearly $\eta^k \Lambda = \Phi$ and since $\exists (U,V) \in \mathbf{M}^2$; $\Phi_k(U,V) = \Lambda_0(U,V) \neq 0$, Λ is an epiform. $\square$

A g r e e m e n t II.9. In what follows ${}_2\Phi$ denotes the quadratic form determined by the symmetric bilinear form Φ.

Proposition II.10. *Let ${}_2\Phi$ be a quadratic form on the module $\mathbf{M}$. Then there exists a polar basis of $\mathbf{M}$ with respect to ${}_2\Phi$.*

P r o o f by induction for $n = \dim \mathbf{M}$.

1. The proposition is clear for $n = 1$.

2. Let Proposition II.10 be true for all $(n-1)$-dimensional $\mathbf{A}$-modules, $n \geqslant 2$.

(a) Let Φ be a symmetric epiform, i.e. $\exists (U,V) \in \mathbf{M}^2$; $\Phi(U,V)$ is a unit. Then there exists $Y \in \mathbf{M}$ such that ${}_2\Phi(Y)$ is a unit. Indeed, in the opposite case, we should have $\Phi(U,V) = \frac{1}{2}[{}_2\Phi(U+V) - {}_2\Phi(U) - {}_2\Phi(V)] \in \eta\mathbf{A}$ for any $(U,V) \in \mathbf{M}^2$, a contradiction. Thus the linear form $\varphi(X) = \Phi(X,Y)$ is an epiform $\mathbf{M} \to \mathbf{A}$. According to Theorem I.8 the kernel of Φ is the free $(n-1)$-dimensional module $\mathscr{N}$, i.e. $\forall W \in \mathbf{M}$; $W \in \mathscr{N} \Leftrightarrow \Phi(W,Y) = 0$. Due to the induction hypothesis we get that $\mathscr{N}$ has a polar basis $\{U_1, \ldots, U_{n-1}\}$ of the quadratic form ${}_2\Phi/\mathscr{N}$.

Since ${}_2\Phi(Y)$ is a unit we may easily see that $\{U_1, \ldots, U_{n-1}, Y\}$ is a linearly independent system.

Putting $U_n = Y$, we obtain a polar basis $\{U_1, \ldots, U_{n-1}, U_n\}$ of $\mathbf{M}$ (according to Proposition I.6).

(b) Let Φ be a bilinear form of order k $(\neq 0)$. According to Proposition II.8 there exists a bilinear epiform Ψ with $\Phi = \eta^k\Psi$. By (a) we can construct a polar basis for the form Ψ, i.e. $[U_1, \ldots, U_n] = \mathbf{M}$ and for $i \neq j$, $j \leqslant n$ we have $\Psi(U_i, U_j) = 0$, hence $\Phi(U_i, U_j) = \eta^k\Psi(U_i, U_j) = 0$. $\square$

Definition II.11. A polar basis $\{U_1, \ldots, U_n\}$ of a quadratic form on the module $\mathbf{M}$ is called the *normal polar basis* if for every i, $1 \leqslant i \leqslant n$, there exists k, $0 \leqslant k \leqslant m$, such that

$$ {}_2\Phi(U_i) = \pm\eta^k. $$

Theorem II.12. *Let a quadratic form ${}_2\Phi$ on the $\mathbf{A}$-module $\mathbf{M}$ be given. Then there exists a normal polar basis of $\mathbf{M}$ with respect to ${}_2\Phi$.*

P r o o f. Let ${}_2\Phi$ be a quadratic form $\mathbf{M} \to \mathbf{A}$ and let $\{U_1, \ldots, U_n\}$ be its polar basis. Putting $\gamma_i = {}_2\Phi(U_i)$, $1 \leqslant i \leqslant n$, we can write every γ_i in the form $\gamma_i = \pm\eta^{k(i)}\varepsilon_i$, where ε_i is a unit for which $p_0(\varepsilon_i) > 0$. By Proposition 1.4 in [1] we have a unit α_i such that $\alpha_i^2 = \varepsilon_i$, $\forall i$, $1 \leqslant i \leqslant n$. Let us put $W_i = \frac{1}{\alpha_i}U_i$ for every i. Then we obtain ${}_2\Phi(W_i) = \pm\eta^{k(i)}$, $\forall i$, $1 \leqslant i \leqslant n$. Evidently, $\Phi(W_i, W_j) = 0$ for $i \neq j$ and the system of vectors $\{W_i\}$ is linearly independent (since α_i are units). $\square$

Definition III.1. Let ${}_2\Phi$ be a quadratic form on $\mathbf{M}$ and let $\mathscr{U} = \{U_1, \dots, U_n\}$ be its normal polar basis. Putting $\gamma_i = {}_2\Phi(U_i)$, $1 \leqslant i \leqslant n$, we define a system of sets as follows:

$$\mathscr{I}_k = \{i \in \mathbb{N}(n)\,;\ \gamma_i = \pm\eta^k\}, \quad 0 \leqslant k \leqslant m.$$

If we denote $\pi_k = \operatorname{card}(\mathscr{I}_k)$, $0 \leqslant k \leqslant m$, then

$$\mathfrak{Ch}({}_2\Phi, \mathscr{U}) = (\pi_0, \dots, \pi_m)$$

is called the *characteristic of the quadratic form ${}_2\Phi$ with respect to the basis $\mathscr{U}$.*

Definition III.2. For any quadratic form ${}_2\Phi$ on $\mathbf{M}$, let us denote by $\mathscr{V}_k^\Phi$ the set $\{Y \in \mathbf{M}\,;\ \eta^k\Phi(X, Y) = 0, \forall X \in \mathbf{M}\}$, $0 \leqslant k \leqslant m$.

The following lemma is evident:

Lemma III.3. *If $\mathscr{U}$ is a basis of $\mathbf{M}$ and ${}_2\Phi$ is a quadratic form, then*

$$\mathscr{V}_k^\Phi = \{Y \in \mathbf{M}\,;\ \eta^k\Phi(U, Y) = 0, \forall U \in \mathscr{U}\}, \quad 0 \leqslant k \leqslant m.$$

Proposition III.4. *Let ${}_2\Phi$ be a quadratic form on $\mathbf{M}$ and let $\mathscr{U}$ be its normal polar basis. Then $\mathscr{V}_k^\Phi$ is a submodule of $\mathbf{M}$ and as an $\mathbb{R}$-vector space it has the dimension*

$$\dim_{\mathbb{R}} \mathscr{V}_k^\Phi = \sum_{j=0}^{m-k-1} (k+j)\pi_j + m \sum_{j=m-k}^{m} \pi_j,$$

where $(\pi_0, \dots, \pi_m) = \mathfrak{Ch}({}_2\Phi, \mathscr{U})$.

P r o o f. $\mathscr{V}_k^\Phi$ is clearly a submodule of $\mathbf{M}$. Let $\mathscr{U} = \{U_1, \dots, U_n\}$ and let us consider a $Y \in \mathbf{M}$, $Y = \sum\limits_{i=1}^{n} \zeta_i U_i$. Putting $\gamma_i = {}_2\Phi(U_i)$, $i \in \mathbb{N}(n)$, we obtain: $Y \in \mathscr{V}_k^\Phi \Leftrightarrow \forall i,\ 1 \leqslant i \leqslant n;\ \eta^k\Phi(U_i, Y) = 0 \Leftrightarrow \forall i,\ 1 \leqslant i \leqslant n;\ \eta^k\gamma_i\zeta_i = 0$. According to Definition III.1 we get that $Y \in \mathscr{V}_k^\Phi$ if and only if the following conditions are valid:

$$\begin{array}{rl}
(0) & i \in \mathscr{I}_0 \Rightarrow \zeta_i = y_{i0}\eta^{m-k} + y_{i1}\eta^{m-k+1} + \dots + y_{ik-1}\eta^{m-1} \\
(1) & i \in \mathscr{I}_1 \Rightarrow \zeta_i = y_{i0}\eta^{m-k-1} + \dots + y_{ik}\eta^{m-1} \\
(j) & i \in \mathscr{I}_j \Rightarrow \zeta_i = y_{i0}\eta^{m-k-j} + \dots + y_{ik+j-1}\eta^{m-1}, \quad 0 \leqslant j \leqslant m-k-1 \\
(m-k-1) & i \in \mathscr{I}_{m-k-1} \Rightarrow \zeta_i = y_{i0}\eta + \dots + y_{im-2}\eta^{m-1} \\
(m-k) & i \in \bigcup\limits_{s=0}^{k} \mathscr{I}_{m-s} \Rightarrow \zeta_i \in \mathbf{A}, \quad \text{where all } y_{ih} \in \mathbb{R}.
\end{array}$$

Let us construct the following system of submodules in $\mathscr{V}_k^\Phi$:

$$\mathscr{V}_{kj}^\Phi = \Big\{ Y \in \mathbf{M};\, Y \in \mathscr{V}_k^\Phi \wedge Y = \sum_{i \in \mathscr{I}_j} \zeta_i U_i \Big\}, \quad 0 \leqslant j \leqslant m.$$

The condition (0) implies that $\bigcup\limits_{i \in \mathscr{I}_0} \{\eta^{m-k} U_i, \ldots, \eta^{m-1} U_i\}$ is an $\mathbb{R}$-basis of $\mathscr{V}_{k0}^\Phi$, therefore $\dim_{\mathbb{R}} \mathscr{V}_{k0}^\Phi = \pi_0 k$. Analogously, conditions (j) imply that $\dim_{\mathbb{R}} \mathscr{V}_{kj}^\Phi = \pi_j(k+j)$, $0 \leqslant j \leqslant m-k-1$, and the condition $(m-k)$ implies that $\dim_{\mathbb{R}} \mathscr{V}_{kj}^\Phi = \pi_j m$, $m-k \leqslant j \leqslant m$. Evidently, $\mathscr{V}_k^\Phi = \bigoplus\limits_{j=0}^{m} \mathscr{V}_{kj}^\Phi$. Thus we have

$$\dim_{\mathbb{R}} \mathscr{V}_k^\Phi = \sum_{j=0}^{m} \dim_{\mathbb{R}} \mathscr{V}_{kj}^\Phi = \sum_{j=0}^{m-k-1} (k+j)\pi_j + m \sum_{j=m-k}^{m} \pi_j.$$

□

Theorem III.5. *Let a quadratic form ${}_2\Phi$ on $\mathbf{M}$ be given. If $\mathscr{U}$, $\mathscr{V}$ are arbitrary normal polar bases of the form ${}_2\Phi$, then*

$$\mathfrak{Ch}({}_2\Phi, \mathscr{U}) = \mathfrak{Ch}({}_2\Phi, \mathscr{V}).$$

Proof. Let $\mathfrak{Ch}({}_2\Phi, \mathscr{U}) = (\pi_0, \ldots, \pi_m)$. Then Proposition III.4 implies

$$\dim_{\mathbb{R}} \mathscr{V}_k^\Phi = \sum_{j=0}^{m-k} \pi_j(k+j) + \sum_{j=m-k+1}^{m} \pi_j m,$$
$$\dim_{\mathbb{R}} \mathscr{V}_{k-1}^\Phi = \sum_{j=0}^{m-k} \big(\pi_j(k+j) - \pi_j\big) + \sum_{j=m-k+1}^{m} \pi_j m.$$

Consequently, we have $\dim_{\mathbb{R}} \mathscr{V}_k^\Phi - \dim_{\mathbb{R}} \mathscr{V}_{k-1}^\Phi = \sum\limits_{j=0}^{m-k} \pi_j$.

Let $\mathfrak{Ch}({}_2\Phi, \mathscr{V}) = (\nu_0, \ldots, \nu_m)$. Then we obtain $\dim_{\mathbb{R}} \mathscr{V}_k^\Phi - \dim_{\mathbb{R}} \mathscr{V}_{k-1}^\Phi = \sum\limits_{n=0}^{m-k} \nu_h$, i.e. $\sum\limits_{j=0}^{m-k} \pi_j = \sum\limits_{h=0}^{m-k} \nu_h$.

Putting $k = m$, $m-1$, …, 0, we get

$$\pi_0 = \nu_0,\ \pi_0 + \pi_1 = \nu_0 + \nu_1,\ \ldots,\ \sum_{j=0}^{m-1} \pi_j + \pi_m = \sum_{h=0}^{m-1} \nu_h + \nu_m,$$

which successively yields $\pi_0 = \nu_0$, $\pi_1 = \nu_1$, …, $\pi_m = \nu_m$. □

Definition III.6. Let ${}_2\Phi$ be a quadratic form on $\mathbf{M}$ and let $\mathscr{U} = \{U_1, \dots, U_n\}$ be its normal polar basis. Putting $\gamma_i = {}_2\Phi(U_i)$, $i \in \mathbb{N}(n)$, we define a system of sets as follows:

$$P_k = \{i \in \mathbb{N}(n)\,;\, \gamma_i = \eta^k\},$$
$$N_k = \{i \in \mathbb{N}(n)\,;\, \gamma_i = -\eta^k\}, \quad 0 \leqslant k \leqslant m-1.$$

If we denote $\mathfrak{p}_k = \operatorname{card} P_k$, $\mathfrak{n}_k = \operatorname{card} N_k$, $0 \leqslant k \leqslant m-1$, then

$$\mathscr{G}({}_2\Phi, \mathscr{U}) = (\mathfrak{p}_0, \dots, \mathfrak{p}_{m-1}, \mathfrak{n}_0, \dots, \mathfrak{n}_{m-1})$$

is called the *plural signature of the quadratic form ${}_2\Phi$ with respect to the basis $\mathscr{U}$.*

Theorem III.7. *Let a quadratic form ${}_2\Phi$ on $\mathbf{M}$ be given. If $\mathscr{U}$, $\mathscr{V}$ are arbitrary normal polar bases of the form ${}_2\Phi$, then*

$$\mathscr{G}({}_2\Phi, \mathscr{U}) = \mathscr{G}({}_2\Phi, \mathscr{V}).$$

P r o o f. Let $\mathscr{U} = \{U_1, \dots, U_m\}$, $X \in \mathbf{M}$, $X = \sum\limits_{i=1}^{n} \xi_i U_i$, arbitrary. Then (according to Definition III.6) we get

$${}_2\Phi(X) = \sum_{i=1}^{n} \gamma_i \xi_i^2 = \sum_{h=0}^{m-1} \left(\sum_{i \in P_h} \xi_i^2 \eta^h - \sum_{i \in N_h} \xi_i^2 \eta^h \right).$$

If $\xi_i = \sum\limits_{j=0}^{m-1} x_{ij} \eta^j$, then

$$\begin{aligned} {}_2\Phi(X) &= \sum_{h=0}^{m-1} \left(\sum_{\substack{j+k=0 \\ i \in P_h}} x_{ij} x_{ik} \eta^{j+k+h} - \sum_{\substack{j+k=0 \\ i \in N_n}} x_{ij} x_{ik} \eta^{j+k+h} \right) \\ &= \sum_{s=0}^{m-1} \left(\sum_{\substack{j+k=s-h \\ 0 \leqslant h \leqslant s \\ i \in P_h}} x_{ij} x_{ik} - \sum_{\substack{j+k=s-h \\ 0 \leqslant h \leqslant s \\ i \in N_h}} x_{ij} x_{ik} \right) \eta^s. \end{aligned}$$

Denoting

$${}_2\Phi_s = \sum_{\substack{j+k=s-h \\ 0 \leqslant h \leqslant s \\ i \in P_h}} x_{ij} x_{ik} - \sum_{\substack{j+k=s-h \\ 0 \leqslant h \leqslant s \\ i \in N_h}} x_{ij} x_{ik}, \quad 0 \leqslant s \leqslant m-1$$

we obtain quadratic forms $\mathbf{M} \to \mathbb{R}$ such that ${}_2\Phi = \sum\limits_{s=0}^{m-1} \Phi_s \eta^s$.

Let us consider $X \in \mathscr{V}^{\Phi}_{m-s}$. This is equivalent to the following relations for $\{x_{ij}\}$ $[X = \sum\limits_{i=1}^{n} \xi_i U_i,\ \xi_i = \sum\limits_{j=0}^{m-1} x_{ij}\eta^j$, see the proof of Proposition III.4, $\mathscr{I}_h = P_h \cup N_h]$:

$$\begin{array}{rl}
(0) & i \in \mathscr{I}_0 \Rightarrow x_{i0} = \ldots = x_{is-1} = 0 \\
(1) & i \in \mathscr{I}_1 \Rightarrow x_{i0} = \ldots = x_{is-2} = 0 \\
(h) & i \in \mathscr{I}_h \Rightarrow x_{i0} = \ldots = x_{is-h-1} = 0, \quad 0 \leqslant h \leqslant s-1; \\
(s-1) & i \in \mathscr{I}_{s-1} \Rightarrow x_{i0} = 0 \\
(s) & i \in \bigcup\limits_{r=s}^{m} \mathscr{I}_r \Rightarrow x_{ij} \quad \text{are arbitrary.}
\end{array}$$

If we put ${}_2F_s = {}_2\Phi_s / \mathscr{V}^{\Phi}_{m-s}$, $0 \leqslant s \leqslant m-1$, then conditions $(0), \ldots, (s-1)$ imply

$$\begin{gathered}
{}_2F_s = \sum_{\substack{h=s\\ i\in P_s}} x_{i0}^2 + \underbrace{\sum_{\substack{j+k=s-h\\ 0\leqslant h<s\\ i\in P_h}} x_{ij}x_{ik}}_{=0} - \sum_{\substack{h=s\\ i\in N_s}} x_{i0}^2 - \underbrace{\sum_{\substack{j+k=s-h\\ 0\leqslant h<s\\ i\in N_h}} x_{ij}x_{ik}}_{=0} \\
\Rightarrow {}_2F_s = \sum_{i\in P_S} x_{i0}^2 - \sum_{i\in N_s} x_{i0}^2, \quad 0 \leqslant s \leqslant m-1.
\end{gathered}$$

Since every ${}_2F_s$ is a real quadratic form on the vector space $\mathscr{V}^{\Phi}_{m-s}$, $(\mathfrak{p}_s, \mathfrak{n}_s)$ is its signature. This implies the invariance of the plural signature of the form Φ. $\square$

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