NOTE ON SIMULTANEOUS SOLUTIONS OF A SYSTEM OF SCHRÖDER'S EQUATIONS

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(Received November 22, 1993)

Abstract. We investigate simultaneous solutions of a system of Schröder's functional equations. Under certain assumptions these solutions are used in transformations of functional-differential equations the initial set of which consists of the initial point only.

Keywords: functional equation, simultaneous solution, differential equation with delays, global transformation

MSC 1991: 39B05, 39B62; 34K05

The aim of this paper is to find a simultaneous solution φ of a system of Schröder's equations

$$\begin{aligned}\varphi(f(x)) &= \lambda_f \varphi(x)\\ \varphi(g(x)) &= \lambda_g \varphi(x)\end{aligned} \qquad \text{on } I, \tag{1}$$

where λ_f, $\lambda_g \notin \{-1, 0, 1\}$ are constants and f, g are given continuous bijections mapping an interval I with endpoints a, b onto itself. For reasons that will be mentioned later we wish to receive some reasonable conditions on f and g which guarantee the existence of an n-times differentiable solution ($n \geqslant 1$) with a positive derivative on I.

Throughout this paper suppose that f^n means the n-th iterate of a function f for positive integers n, f^0 is the identity function and f^n is the $(-n)$-th iterate of the inverse function f^{-1} for negative integers n. Further, we say that $f(x) = O\{v(x)\}$ as $x \to x^*$, $x^* \in I$ if there are positive constants K, ε such that

$$|f(x)| \leqslant K|v(x)| \qquad \text{for } x \in (x^* - \varepsilon, x^* + \varepsilon),$$

where ε is sufficiently small.

Finally, define $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$.

Proposition 1. *Let system* (1) *has a solution with* $\varphi'(x) > 0$ *on* I. *Then each of the functions* f, g *has at most one fixed point in* I *which is necessarily the same for both the functions* f *and* g.

Proof. Suppose on the contrary that there exist fixed points $p, q \in I$, $p \neq q$ of f and g, respectively. Then $\varphi(p) = \varphi(q) = 0$ which contradicts $\varphi'(x) > 0$ on I. $\square$

Remark 1. If bijections f, g have no fixed point in I then they are necessarily increasing ones. Substituting then $\alpha(x) = \log \varphi(x)$, we convert system (1) into a system of Abel's equations

$$\begin{aligned} \alpha(f(x)) &= \alpha(x) + c_f \\ \alpha(g(x)) &= \alpha(x) + c_g \end{aligned} \qquad \text{on } I, \tag{2}$$

where $c_f = \log \lambda_f$, $c_g = \log \lambda_g$ are nonzero constants. Since simultaneous solutions of system (2) were studied and fully described in [5] and [9], in the sequel we deal with the case when the functions f, g have a unique (necessarily the same) fixed point in I.

The conditions which we must impose on f or g to ensure the existence of a solution $\varphi \in C^n(I)$, $\varphi'(x) > 0$ on I are known provided a single equation from system (1) is considered (see [4] or [1]). To recapitulate them we introduce the set

$$\begin{aligned} V_p^n(I) = \{f \in C^n(I);\ & f(I) = I,\ f(p) = p,\ f^2(x) \neq x \text{ for } x \neq p, \\ & f'(x) \neq 0,\ |f'(p)| \neq 1,\ f'(x) = f'(p) + O\{(x-p)^\delta\} \text{ as } x \to p,\ \delta > 0\}, \end{aligned}$$

where $n \geqslant 1$ and $p \in I$. Note that the asymptotic property

$$f'(x) = f'(p) + O\{(x-p)^\delta\} \text{ as } x \to p$$

is satisfied for any $f \in C^n(I)$, $n \geqslant 2$.

Then we have

Theorem 1. *Let* $f \in V_p^n(I)$ *for some* $n \geqslant 1$. *Then Schröder's equation*

$$\varphi(f(x)) = \lambda_f \varphi(x) \qquad \text{on } I,$$

where $\lambda_f = f'(p)$, *has a unique one parameter family of* n*-times differentiable solutions, with a positive (negative) derivative on* I *given by the formula*

$$\varphi(x) = c \lim_{k\to\gamma\infty} \frac{f^{2k}(x) - p}{\lambda_f^{2k}} \qquad \text{on } I,$$

where $\gamma = \operatorname{sgn}(\log \frac{1}{|f'(p)|})$ *and* c *is any positive (negative, respectively) real constant. Moreover, if* $p = a$ ($p = b$, $p \in (a,b)$) *then* φ *is a bijection of* I *onto* $\mathbb{R}^+$ ($\mathbb{R}^-$, $\mathbb{R}$, *respectively).*

Proof. The proof was given in [1]. □

Now consider a function $f \in V_p^n(I)$ for some $n \geqslant 1$ and let φ be a C^n-diffeomorphism on I such that

$$f(x) = \varphi^{-1}(\lambda_f \varphi(x)) \qquad \text{on } I,$$

where $\lambda_f = f'(p)$. Then denote by $\{f_+^u\}$ or $\{f_-^u\}$ the family of functions defined respectively by the relation

$$f_+^u(x) = \varphi^{-1}(|\lambda_f|^u \varphi(x)) \qquad \text{on } I,$$

or

$$f_-^u(x) = \varphi^{-1}(-|\lambda_f|^u \varphi(x)) \qquad \text{on } I,$$

where $u \in \mathbb{R}$. Note that for an increasing f the class $\{f_+^u\}$ is identical with the continuous iteration group fulfilling $f_+^0(x) = x$ and $f_+^1(x) = f(x)$ for $x \in I$. The same is not true for the class $\{f_-^u\}$; in fact we have $f_-^u \circ f_-^v = f_+^{u+v}$. Since the classes $\{f_+^u\}$, $\{f_-^u\}$ involve functions generated by the same bijection φ, we put $\{f(\varphi)\} = \{f_+^u\} \cup \{f_-^u\}$.

Proposition 2. *Consider system* (1), *where* $f, g \in V_p^n(I)$ *for some* $n \geqslant 1$. *Then* (1) *has a solution* $\varphi \in C^n(I)$ *with* $\varphi'(x) > 0$ *on* I *if and only if* $g \in \{f(\varphi)\}$.

Proof. System (1) has a solution φ with the above stated properties if and only if

$$f(x) = \varphi^{-1}(\lambda_f \varphi(x)) \qquad \text{on } I$$

and

$$g(x) = \varphi^{-1}(\lambda_g \varphi(x)) \qquad \text{on } I.$$

Putting $u = \frac{\log|\lambda_g|}{\log|\lambda_f|}$ we can rewrite the last relation as

$$g(x) = \varphi^{-1}(\gamma|\lambda_f|^u \varphi(x)) \qquad \text{on } I,$$

where $\gamma = \operatorname{sgn} \lambda_g$ and vice versa. □

Lemma. *Suppose* $f \in V_p^n(I)$ *for some* $n \geqslant 1$ *and denote* $\Phi_f = \{g \in C^n(I);$ $g(I) = I,\ g \circ f = f \circ g\}$. *Then* $\Phi_f = \{f(\varphi)\}$.

P r o o f. Let $g \in \{f(\varphi)\}$, i.e.

$$g(x) = \varphi^{-1}\big(\gamma|\lambda_f|^u \varphi(x)\big) \qquad \text{on } I$$

for a suitable C^n-diffeomorphism φ on I and $u \in \mathbb{R}$, where $\gamma = \operatorname{sgn} g'(p)$. Then g is obviously an n-times differentiable function mapping I onto itself and

$$f \circ g(x) = \varphi^{-1}\big(\lambda_f \varphi(g(x))\big) = \varphi^{-1}\big(\lambda_f \gamma |\lambda_f|^u \varphi(x)\big) = \varphi^{-1}\big(\gamma|\lambda_f|^u \varphi(f(x))\big) = g \circ f(x)$$

for every $x \in I$, hence $g \in \Phi_f$.

Further, suppose $g \in \Phi_f$. Since there exists $\varphi \in C^n(I)$, $\varphi'(x) > 0$ on I such that the equation

$$\varphi(f(x)) = \lambda_f \varphi(x)$$

is fulfilled for every $x \in I$, we have

$$\varphi(g \circ f(x)) = \varphi(f \circ g(x)) = \lambda_f \varphi(g(x)) \qquad \text{on } I.$$

Thus we have obtained the same Schröder's equation as above with the function $\varphi \circ g$ as a C^n solution having a nonzero derivative on I. Hence, due to Theorem 1

$$\varphi(g(x)) = c\varphi(x) \qquad \text{on } I$$

must hold for a suitable real constant c. Put $u = \frac{\log|c|}{\log|\lambda_f|}$. Then $c = \gamma|\lambda_f|^u$, where $\gamma = \operatorname{sgn} g'(p)$ and

$$g(x) = \varphi^{-1}\big(\gamma|\lambda_f|^u \varphi(x)\big) \qquad \text{on } I,$$

i.e. $g \in \{f(\varphi)\}$. □

Now we can summarize these results in the following statement giving a simple necessary and sufficient condition which guarantees the existence of a solution φ of system (1) with the required properties.

Proposition 3. *Let* $f \in V_p^n(I)$ *for some* $n \geqslant 1$ *and let* $g\colon I \xrightarrow{\text{onto}} I$ *be a* C^n *function. Then system* (1) *has a solution* $\varphi \in C^n(I)$, $\varphi'(x) > 0$ *on* I *if and only if* $f \circ g = g \circ f$ *on* I.

E x a m p l e 1. Consider a system of Schröder's equations

$$\begin{aligned} \varphi(x^r) &= r\varphi(x) \\ \varphi(g(x)) &= \lambda_g \varphi(x), \end{aligned}$$

where $x \in (0,\infty)$, $r \notin \{-1,0,1\}$ is a real constant and g is an infinitely smooth function mapping the interval $(0,\infty)$ onto itself. Then according to Proposition 3 this system has a solution $\varphi \in C^\infty((0,\infty))$, $\varphi'(x) > 0$ on $(0,\infty)$ iff $g(x^r) = (g(x))^r$. The only differentiable solutions of the above equation are functions $g(x) = x^s$, $x \in (0,\infty)$, $s \neq 0$ (see [4], p.121).

Remark 2. The previous statement can be easily generalized to the case when simultaneous solutions of a system of m Schröder's equations ($m \geqslant 2$) are investigated.

Notation. Suppose that f, $g \in V_p^n(I)$ for some $n \geqslant 1$ are commuting functions. Then for every $x \in I$ denote $C(x) = \{f^k \circ g^l(x);\, k,l \in \mathbb{Z}\}$ and let I_x be an interval with endpoints a_x, b_x, where $a_x = \inf C(x)$ and $b_x = \sup C(x)$. The endpoints belong to I_x provided they belong to I as well.

Corollary 1. *Let f, $g \in V_p^n(I)$ for some $n \geqslant 1$ and $f \circ g = g \circ f$. Then the following statements are equivalent:*

(i) *$f^k = g^l$ for suitable nonzero integers k, l;*

(ii) *$(f'(p))^k = (g'(p))^l$ for the same integers k, l as in* (i) *(if they exist);*

(iii) *$C(x)$ is not dense in I_x for any $x \in I$, $x \neq p$.*

Proof. Using Proposition 3 we get the relations

$$\begin{aligned} f^k(x) &= \varphi^{-1}\big((f'(p))^k \varphi(x)\big) \\ g^l(x) &= \varphi^{-1}\big((g'(p))^l \varphi(x)\big) \end{aligned} \qquad \text{on } I,$$

where k, l are integers and φ is a suitable C^n-diffeomorphism on I. This implies the equivalences (i) and (ii).

Further, rewrite the previous relations as

$$\varphi(C(x)) = \big\{\varphi(f^k \circ g^l(x));\ \ k,\, l \in \mathbb{Z}\big\} = \big\{(f'(p))^k (g'(p))^l \varphi(x);\ \ k,\, l \in \mathbb{Z}\big\},$$

where $x \in I$. Now we have to distinguish three cases with respect to the position of p. First, e.g., let p be the left endpoint of I. In this case $I_x = [a,b)$, $\varphi(I_x) = \mathbb{R}^+$ and therefore $C(x)$ is dense in I_x ($x \in (a,b)$) if and only if the set $\{(f'(p))^k (g'(p))^l t;$ $k,\, l \in \mathbb{Z}\}$ is dense in $\mathbb{R}^+$ ($t = \varphi(x) > 0$). This holds iff $(f'(p))^k \neq (g'(p))^l$ for any couple of nonzero integers k, l.

The equivalence of (ii) and (iii) can be similarly proved provided $p \in (a,b)$ or $p = b$. □

Corollary 2. *Let f, $g \in V_p^n(I)$ for some $n \geqslant 1$, $f \circ g = g \circ f$ and $f'(p) = g'(p)$. Then $f = g$.*

Now we make two remarks in which the assumptions imposed on f and g in Proposition 3 are weakened.

Remark 3. Assuming that the above considered C^n bijections f, g are continuous only we may expect only invertible continuous solutions of (1) instead of smooth solutions with a nonzero derivative. As in the previous part we introduce the set $V_p^0(I)$ which consists of all continuous bijections of I onto itself such that $f(p) = p$, $f^2(x) \neq x$ for $x \neq p$ and $\operatorname{sgn}(f^2(x_1) - x_1) = \operatorname{sgn}(x_2 - f^2(x_2))$ for some (then any) $x_1 \in (a, p)$, $x_2 \in (p, b)$. Then we have

Proposition 4. *Let $f \in V_p^0(I)$. Then there exists an invertible continuous solution φ of the equation*

$$(3) \qquad \varphi(f(x)) = \lambda_f \varphi(x) \qquad \text{on } I,$$

λ_f being a suitable real constant, which depends on an arbitrary function.

Proof. Take $x^* \in I$, $x^* \neq p$ and for the sake of simplicity let $x^* > p$ (i.e. $p \neq b$). Assuming that f is an increasing bijection from $V_p^0(I)$ we consider a positive real constant λ_f satisfying $\operatorname{sgn}(\log \lambda_f) = \operatorname{sgn}(f(x^*) - x^*)$ and a continuous function ψ defined on $[x^*, f(x^*)]$ or $[f(x^*), x^*]$ such that $\psi(f(x^*)) = \lambda_f \psi(x^*)$. Then with respect to [3], p.48 there exists a continuous solution φ of (3) fulfilling $\varphi(x) = \psi(x)$ on $[x^*, f(x^*)]$ or $[f(x^*), x^*]$. Obviously, if ψ is invertible then φ is invertible as well.

Provided a decreasing bijection f from $V_p^0(I)$ is considered we investigate the equation

$$\varphi^*(f^2(x)) = \lambda_f^* \varphi^*(x) \qquad \text{on } I.$$

Since f^2 is an increasing bijection from $V_p^0(I)$ we get an invertible continuous solution φ^* of this equation depending on an arbitrary function. Then it is easy to verify that the function

$$\varphi(x) = \begin{cases} \varphi^*(x), & \text{for } x \in I,\ x \leqslant p, \\ \lambda_f \varphi^*(f^{-1}(x)), & \text{for } x \in I,\ x > p, \end{cases}$$

where $\lambda_f = -\sqrt{\lambda_f^*}$, is an invertible continuous function satisfying (3).

The case $p = b$ can be dealt with quite similarly. $\square$

In the sequel we want to find conditions under which system (1) has an invertible continuous solution. Consequently, in accordance with Proposition 3, we take into our considerations commuting bijections f, g from $V_p^0(I)$. However, this property is now only a necessary, but not a sufficient condition for the existence of such a solution. Indeed, consider functions $f(x) = x^2$ and

$$g(x) = (6\sqrt[2^n]{x} - 8)^{2^n} \qquad \text{for } x \in (2^{2^n}, 2^{2^{n+1}}],$$

$n \in \mathbb{Z}$ and put $g(1) = 1$. Then even if $f,\ g \in V_1^0([1,\infty))$ and

$$f \circ g(x) = ((6\sqrt[2^n]{x} - 8)^{2^n})^2 = (6\sqrt[2^{n+1}]{x^2} - 8)^{2^{n+1}} = g \circ f(x)$$

for every $x \in [1,\infty)$, system (1) has no invertible continuous solution on $[1,\infty)$. It is a consequence of the relation $f(2^{2^n}) = g(2^{2^n})$ which is valid for any $n \in \mathbb{Z}$ and which contradicts the fact that if system (1) has an invertible continuous solution then f, g have to fulfil precisely one of the following conditions:

(a) $f^k = g^l$ for suitable nonzero integers $k,\ l$,

(b) $f^k(x) \neq g^l(x)$ for all nonzero integers $k,\ l$ and all $x \in I,\ x \neq p$.

Indeed, if $f^k(x_0) = g^l(x_0)$ for an $x_0 \in I$, $x_0 \neq p$ then we get

$$\varphi^{-1}(\lambda_f^k \varphi(x_0)) = \varphi^{-1}(\lambda_g^l \varphi(x_0)),$$

hence with respect to $\varphi(x_0) \neq 0$,

$$\lambda_f^k = \lambda_g^l.$$

Consequently,

$$f^k(x) = \varphi^{-1}(\lambda_f^k \varphi(x)) = \varphi^{-1}(\lambda_g^l \varphi(x)) = g^l(x)$$

for all $x \in I$, $x \neq p$.

In the sequel we denote by Λ_f the set of all $\lambda_f \in \mathbb{R}$ such that equation (3) has an invertible continuous solution. Then we can formulate the following sufficient condition for (1) to have an invertible continuous solution.

Proposition 5. *Suppose that $f,\ g \in V_p^0(I)$, $f \circ g = g \circ f$ and let hypothesis* (a) *be fulfilled. Then system* (1) *has an invertible continuous solution for any reals $\lambda_f \in \Lambda_f$, $\lambda_g \in \Lambda_g$ such that $\lambda_f^k = \lambda_g^l$. Moreover, this solution depends on an arbitrary function.*

Proof. Assume that at least one of the functions f, g is decreasing. Let, e.g., f have this property. Since the functions f^2, g^2 are increasing bijections of I onto itself satisfying $f^{2k} = g^{2l}$, due to Theorem 3 in [8] there exists an increasing bijection

$h\colon I \xrightarrow{\text{onto}} I$ and integers $r = \frac{l}{(k,l)}$, $s = \frac{k}{(k,l)}$ such that $f^2 = h^r$, $g^2 = h^s$. Obviously $h \in V_p^0(I)$ and due to Proposition 4 the equation

$$\varphi^*(h(x)) = \lambda_h \varphi^*(x) \qquad \text{on } I,$$

where λ_h is a suitable positive real constant, has an invertible continuous solution φ^* depending on an arbitrary function. Then the function φ^* is a solution of the system

$$\begin{aligned} \varphi^*(f^2(x)) &= \lambda_h^r \varphi^*(x), \\ \varphi^*(g^2(x)) &= \lambda_h^s \varphi^*(x) \end{aligned} \qquad \text{on } I$$

as well. Putting $\lambda_f = -\sqrt{\lambda_h^r}$ and $\lambda_g = \gamma\sqrt{\lambda_h^s}$ ($\gamma = 1$ or $\gamma = -1$ for an increasing or decreasing g, respectively) we can see that the function

$$\varphi(x) = \begin{cases} \varphi^*(x), & \text{for } x \in I,\ x \leqslant p, \\ \lambda_f \varphi^*(f^{-1}(x)), & \text{for } x \in I,\ x > p \end{cases}$$

is a solution of system (1) with the required properties.

Provided both f and g are increasing bijections from $V_p^0(I)$ it is enough to consider functions f, g instead of f^2, g^2 in the previous part of the proof and put $\lambda_f = \lambda_h^r$, $\lambda_g = \lambda_h^s$ and $\varphi(x) = \varphi^*(x)$ on I. $\square$

Remark 4. As mentioned above, functions f, g with different fixed points cannot yield a solution φ of (1) with a positive derivative. In such a case we will consider rather a more general system. As an example we show that the system

$$\begin{aligned} \varphi(h_1(x)) &= \lambda_1 \varphi(x), \\ \varphi(h_2(x)) &= \lambda_2 \varphi(x) + \omega \end{aligned} \qquad \text{on } I, \tag{4}$$

where the bijections h_i have a unique fixed point $p_i \in I$ ($i = 1, 2$), $p_1 \neq p_2$, can be, under certain assumptions, fulfilled by a function φ having the required differential properties. Note that by differentiating this system and putting $x = p_i$ we get with respect to $\varphi'(x) > 0$ on I that $\lambda_i = h_i'(p_i)$ $(i = 1, 2)$. If $|h_1'(p_1)| = 1$ then the only continuous solutions of the first equation from system (4) are constant functions. Similarly, the latter equation from (4) has no required solution provided $|h_2'(p_2)| = 1$. That is why the functions h_i must belong to the class $V_{p_i}^n(I)$ $(i = 1, 2)$ if we wish to obtain an n-times differentiable solution of (4) for some $n \geqslant 1$ with a positive derivative on I.

Proposition 6. *Let $h_1 \in V_{p_1}^n(I)$, $h_2 = f \circ g \in V_{p_2}^n(I)$ for some $n \geqslant 1$, where $g\colon I \xrightarrow{\text{onto}} I$ is a C^n function commuting with h_1, and let for every $x \in I$*

$$\lim_{k \to \gamma\infty} \frac{g^{2k}(f(x)) - g^{2k}(x)}{(g'(p_1))^{2k}} = c \neq 0,$$

where $\gamma = \operatorname{sgn}(\log \frac{1}{|g'(p_1)|})$. Then system (4) has a C^n solution with a positive derivative on I if and only if $\lambda_1 = h_1'(p_1)$, $\lambda_2 = h_2'(p_2) = g'(p_1)$ and $\omega = c$.

P r o o f. Using Proposition 3 we can obtain the existence of a function $\varphi \in C^n(I)$, $\varphi'(x) > 0$ on I such that

$$\begin{aligned} \varphi(h_1(x)) &= \lambda_1 \varphi(x) \\ \varphi(g(x)) &= \lambda_2 \varphi(x) \end{aligned} \qquad \text{on } I,$$

where $\lambda_1 = h_1'(p_1)$ and $\lambda_2 = g'(p_1)$. Then according to the formula mentioned in Theorem 1 we have

$$\begin{aligned} c &= \lim_{k\to\gamma\infty} \frac{g^{2k}(f(x)) - g^{2k}(x)}{(g'(p_1))^{2k}} = \lim_{k\to\gamma\infty} \frac{g^{2k}(f(x)) - p_1}{(g'(p_1))^{2k}} - \lim_{k\to\gamma\infty} \frac{g^{2k}(x) - p_1}{(g'(p_1))^{2k}} \\ &= \varphi(f(x)) - \varphi(x), \end{aligned}$$

hence

$$\varphi(h_2(x)) = \varphi(f(g(x))) = \varphi(g(x)) + c = \lambda_2 \varphi(x) + c.$$

□

E x a m p l e 2. We investigate the system

$$\begin{aligned} \varphi(x^r) &= \lambda_1 \varphi(x) \\ \varphi(ax^s) &= \lambda_2 \varphi(x) + \omega \end{aligned} \qquad \text{on } (0,\infty),$$

where $a > 0$, r, $s \notin \{-1, 0, 1\}$. The functions $h_1(x) = x^r$ and $h_2(x) = ax^s$ are C^∞ bijections of $(0,\infty)$ onto itself with unique fixed points $p_1 = 1$ and $p_2 = \sqrt[s-1]{\frac{1}{a}}$, respectively, lying in $(0,\infty)$. The function h_2 can be expressed as the composition $f \circ g$, where $f(x) = ax$ and $g(x) = x^s$. It is easy to check that all the assumptions of Proposition 6 are satisfied. Particularly,

$$\lim_{k\to\gamma\infty} \frac{(ax)^{s^{2k}} - x^{s^{2k}}}{s^{2k}} = \lim_{k\to\gamma\infty} \frac{a^{s^{2k}} - 1}{s^{2k}} = \ln a,$$

where $\gamma = \operatorname{sgn}(\ln |\frac{1}{s}|)$. Then the system considered has a solution $\varphi(x) = \ln x$ provided $\lambda_1 = h_1'(p_1) = r$, $\lambda_2 = h_2'(p_2) = s$ and $\omega = \ln a$.

Now we give an application of the previous results to the theory of pointwise transformations of differential equations with several delays (for similar situations see [5], [6] and [1]).

Consider an equation of the n-th order with delays $\tau_1, \ldots, \tau_m$

$$y^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x) y^{(i)}(x) + \sum_{i=0}^{n-1} \sum_{j=1}^{m} q_{ij}(x) y^{(i)}(\tau_j(x)) = 0 \qquad \text{on } [x_0, b). \tag{5}$$

The most general pointwise transformation converting any equation (5) into a linear homogeneous differential equation of the same order with an independent variable $t \in J$, a dependant variable z and delays $\mu_1, \ldots, \mu_m$ has the form

$$z(t) = g(t) y(h(t)), \tag{6}$$

where h is a C^n-diffeomorphism of $J = h^{-1}([x_0, b))$ onto $[x_0, b)$, $h'(t) > 0$ on J, $g \in C^n(J)$, $g(t) \neq 0$ on J and

$$\tau_j \circ h = h \circ \mu_j \qquad \text{on } J \tag{7}$$

for $j = 1, \ldots, m$ (see [7]). To be more precise, this transformation converts the graph of every solution $y(x)$ of (5) into a graph of a solution $z(t)$ of the transformed equation.

Then using (6) and (7) we can prove

Theorem 2. *Consider equation* (5) *with* $p_i, q_{ij} \in C^0([x_0, b))$, $p_{n-1} \in C^{n-1}([x_0, b))$, $\tau_j \in V^1_{x_0}([x_0, b))$ *for* $n = 1$, $\tau_j \in V^{n+1}_{x_0}([x_0, b))$ *for* $n \geqslant 2$, $\tau_j(x_1) < x_1$ *and* $\tau_j'(x_1) > 0$ *for some (then any)* $x_1 \in (x_0, b)$ $(i = 0, \ldots, n-1,\ j = 1, \ldots, m)$. *If* $\tau_{j_1} \circ \tau_{j_2} = \tau_{j_2} \circ \tau_{j_1}$ *for all* $j_1, j_2 \in \{1, \ldots, m\}$ *then equation* (5) *can be transformed to the equation*

$$z^{(n)}(t) + \sum_{i=0}^{n-2} r_i(t) z^{(i)}(t) + \sum_{i=0}^{n-1} \sum_{j=1}^{m} s_{ij}(t) z^{(i)}(\lambda_j t) = 0 \qquad \text{on } \mathbb{R}^+, \tag{8}$$

where $r_i, s_{ij} \in C^0(\mathbb{R}^+)$ *and* $\lambda_j = \tau_j'(x_0)$ $(i = 0, \ldots, n-1,\ j = 1, \ldots, m)$.

P r o o f. Put $\mu_j(t) = \lambda_j t$, where $\lambda_j = \tau_j'(x_0)$ and $\varphi = h^{-1}$ on J. Thus relation (7) between τ_j and μ_j can be rewritten as

$$\varphi(\tau_j(x)) = \lambda_j \varphi(x) \qquad \text{on } h(J) = [x_0, b),$$

$j = 1, \ldots, m$. Due to Proposition 3 and Remark 2 there exists a simultaneous solution of this system which is (similarly as its inverse h) at least n-times differentiable and has a positive derivative on the definition interval.

Further, it was derived in [6] that putting

$$g(t)=\exp\left\{\frac{1}{n}\int_{x_0}^{h(t)} p_{n-1}(s)\,\mathrm{d}s\right\}(h'(t))^{\frac{1-n}{2}} \qquad \text{on } J,$$

the equation received from (5) by means of transformation (6) has a vanishing coefficient at the $(n-1)$-st derivative of the unknown function $z(t)$. The form of the transformed interval follows from the properties of the solution φ given in Theorem 1; particularly $\varphi(x_0)=0$ and $\lim\limits_{x\to b^-}\varphi(x)=\infty$ provided $\lim\limits_{x\to b^-}\tau_j(x)=b$, $j=1,\dots,m$.

□

R e m a r k 5. The form of equation (8) as well as of equation (5) is not uniquely determined by its space of solutions. The form of (8) was chosen so that it corresponds to the used transformation (6) substituted to the form of equation (5).

Corollary 3. *Consider an equation*

$$y'(x)+p(x)y(x)+q_1(x)y(\tau_1(x))+q_2(x)y(\tau_2(x))=0 \qquad \text{on } [x_0,b), \tag{9}$$

where $p,\ q_1,\ q_2\in C^0([x_0,b))$, $\tau_1,\ \tau_2\in C^2([x_0,b))$ are commuting bijections of the interval $[x_0,b)$ onto itself, $\tau_j(x)<x$ on (x_0,b), $\tau_j'(x)>0$ on $[x_0,b)$ and $\tau_j'(x_0)\neq 1$, $j=1,2$.

Then transformation (6) *with $g(t)=\exp\left\{\int_{x_0}^{h(t)} p(s)\,\mathrm{d}s\right\}$ and $h=\varphi^{-1}$, $\varphi(x)=\lim\limits_{n\to\infty}(\tau_1'(x_0))^{-n}(\tau_1^n(x)-x_0)$, globally converts every solution of* (9) *into a solution of the equation*

$$z'(t)+s_1(t)z(\lambda_1 t)+s_2(t)z(\lambda_2 t)=0 \qquad \text{on } \mathbb{R}^+,$$

where $s_j(t)=\exp\left\{\int_{\tau_j(h(t))}^{h(t)} p(s)\,\mathrm{d}s\right\}q_j(h(t))h'(t)$ on $\mathbb{R}^+$ and $\lambda_j=\tau_j'(x_0)$, $j=1,2$.

P r o o f. Substituting $z(t)=g(t)y(h(t))$ (with the above given g and h) into the form of the transformed equation we get

$$y'(h(t))+\sum_{j=1}^{2}\frac{s_j(t)}{h'(t)}\exp\left\{\int_{h(t)}^{\tau_j(h(t))} p(s)\mathrm{d}s\right\}y(\tau_j(h(t)))=0 \qquad \text{on } \mathbb{R}^+,$$

which compared with (9) gives the form of s_1 and s_2. □

R e m a r k 6. The problem of the transformation of a nonlinear differential equation with above introduced delays τ_j into a differential equation with delays $\mu_j(t)=\lambda_j t$ leads again to finding a simultaneous solution of the system $\varphi(\tau_j(x))=\lambda_j\varphi(x)$. Then the change of the independent variable $t=\varphi(x)$, φ being a sufficiently smooth function with a positive derivative on the definition interval, enables us to carry out such a transformation.

Further, let us remark that the case when delays τ_j intersect the identity function at the initial point is usually refered to as the singular case (see [2]).

E x a m p l e 3. The equation

$$y'(x) + py(x) + q_1 y(x^{\alpha_1}) + q_2 y(x^{\alpha_2}) = 0 \qquad \text{on } [1, \infty),$$

where p, q_1, $q_2 \in \mathbb{R}$ and α_1, $\alpha_2 \in (0,1)$ are constants, has a unique one-parameter family of solutions $y(x)$ defined on $[1, \infty)$ (see, e.g. [2]). These solutions can be converted to the functions $z(t) = \exp\left\{p \int_1^{\exp t} \mathrm{d}s\right\} y(\exp t)$ which form the space of solutions of the equation

$$z'(t) + s_1(t) z(\alpha_1 t) + s_2(t) z(\alpha_2 t) = 0 \qquad \text{on } \mathbb{R}^+,$$

where $s_j(t) = q_j \exp\{t + p \exp t - p \exp(\alpha_j t)\}$, $j = 1, 2$.

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