APPROXIMATE PROPERTIES OF PRINCIPAL SOLUTIONS OF VOLTERRA-TYPE INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE AFTEREFFECT

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Abstract. The integrodifferential system with aftereffect ("heredity" or "prehistory")

$$\mathrm{d}x/\,\mathrm{d}t = Ax + \varepsilon \int_{-\infty}^{t} R(t,s)x(s,\varepsilon)\,\mathrm{d}s,$$

is considered; here ε is a positive small parameter, A is a constant $n \times n$ matrix, $R(t,s)$ is the kernel of this system exponentially decreasing in norm as $t \to \infty$. It is proved, if matrix A and kernel $R(t,s)$ satisfy some restrictions and ε does not exceed some bound ε_*, then the n-dimensional set of the so-called principal two-sided solutions $\tilde{x}(t,\varepsilon)$ approximates in asymptotic sense the infinite-dimensional set of solutions $x(t,\varepsilon)$ corresponding a sufficiently wide class of initial functions. For t growing to infinity an estimate of the difference between $x(t,\varepsilon)$ and $\tilde{x}(t,\varepsilon)$ is obtained.

Keywords: integrodifferential equations, principal solutions, small parameter

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1. Introduction

We have considered in [1]–[3] problems concerning the existence and the methods of construction of a certain class of two-sided solutions of a system

$$\mathrm{d}x/\,\mathrm{d}t = Ax + \varepsilon \int_{-\infty}^{t} R(t,s)x(s,\varepsilon)\,\mathrm{d}s, \tag{1}$$

where A is a constant $n \times n$ matrix, ε is a small (positive) parameter, the integral on the right-hand side is the Riemann integral, $R(t,s)$ is a matrix called the kernel of this system, continuous in t, s for $-\infty < s < t < \infty$ and satisfying the inequality

$$\|R(t,s)\| \leqslant c\mathrm{e}^{-\gamma(t-s)}/(t-s)^{1-\alpha}, \tag{2}$$

where c, γ, α are positive constants and $0 < \alpha \leqslant 1$. In [3] these solutions were called *principal two-sided* (or for simplicity *principal*). According to [1]–[3] these solutions (denoted by $\tilde{x}(t,\varepsilon)$)

1) exist, if ε does not exceed some bound ε_*;
2) are defined uniquely on the entire axis t for every given initial vector $x|_{t=t_0} = x_0$;
3) belong to a set of continuous vector functions

$$U_q = \{\psi(t) \colon \|\psi(t)\|\mathrm{e}^{qt} < \infty,\ t \leqslant 0\},$$

where q is a positive number in the left neighbourhood of γ;

4) satisfy a system of ordinary differential equations

$$\mathrm{d}x/\,\mathrm{d}t = Dx, \tag{3}$$

where $D = D(t,\varepsilon)$ is a $n \times n$ matrix with $D(t,\varepsilon) \to A$ as $\varepsilon \to 0$;

5) tend to the corresponding solutions of the degenerate system

$$\mathrm{d}x^0/\,\mathrm{d}t = Ax^0 \tag{4}$$

as $\varepsilon \to 0$;

6) possess characteristic exponents for $t \to \infty$ and $t \to -\infty$ tending to the characteristic exponents of the corresponding solutions $x^0(t)$ as $\varepsilon \to 0$.

Principal solutions of system (1) form n-dimensional set. They constitute a comparatively narrow class within the infinite-dimensional set of solutions $x(t,\varepsilon)$ of system (1) depending on different initial functions $\varphi(t)$ given on the left semiaxis t. These functions describe the so-called "prehistory" or "heredity" corresponding to the given one-sided solution $x(t,\varepsilon)$ for $t > 0$ and define in general the behaviour of this solution. Systems of form (1) are therefore often called the hereditary systems or the systems with prehistory.

Principal solutions of such systems (existing, as a rule, if ε does not exceed some bound) do not depend on initial functions, i.e. do not depend on "prehistory" and are deprived, so to say, of *heredity* signs. A question arises, if the class of *principal* solutions is representative, i.e. if the behaviour of principal solutions reflects the properties of the infinite-dimensional set of the other solutions of system (1). The analysis allows us to give in a certain sense a positive answer to this question, if the parameter ε does not exceed some bound (in general smaller than the bound securing the existence of principal solutions). Namely, principal solutions allow us for such ε to approximate sufficiently well the other solutions being determined by initial functions of a wide class, and also to describe a series of asymptotical properties of these solutions. Below we carry out a proof for the cases, when the initial functions are piecewise continuous and bounded as $t \to -\infty$ and the kernel $R(t,s)$ satisfies a simpler (in comparison to (2)) and more restrictive estimate

$$\|R(t,s)\| \leqslant c\,\mathrm{e}^{-\gamma(t-s)}, \quad t-s \geqslant 0. \tag{5}$$

2. Conditions for existence of principal solutions and their estimate

A necessary condition for the existence of *principal* solutions of system (1) is, as follows from [1]–[3], the following restriction on the eigenvalues $\lambda_1, \ldots, \lambda_n$ of matrix A:

$$\min_j \operatorname{Re} \lambda_j > -\gamma. \tag{6}$$

Hence, the following inequality holds for the matrix e^{At}:

$$\|\mathrm{e}^{At}\| \leqslant m\,\mathrm{e}^{-\mu t}, \quad t \leqslant 0, \tag{7}$$

where m, μ are positive constants satisfying the conditions $m \geqslant 1$, $\mu < \gamma$. Using one of the methods of construction of the *principal* solution with an initial vector (for the sake of simplicity we put $t_0 = 0$)

$$x|_{t=0} = x_0$$

we will construct this solution with the help of successive approximations

$$x_1(t),\ x_2(t,\varepsilon),\ x_3(t,\varepsilon),\ \ldots$$

defined as solutions of the following systems of ordinary differential equations:

$$\begin{aligned} \mathrm{d}x_1/\,\mathrm{d}t &= Ax_1, \\ \mathrm{d}x_{k+1}/\,\mathrm{d}t &= Ax_{k+1} + \varepsilon \int_{-\infty}^{t} R(t,s)\, x_k(s,\varepsilon)\,\mathrm{d}s, \end{aligned} \tag{8}$$

$$k = 1, 2, 3, \ldots$$

with initial values

$$x_1|_{t=0} = x_2|_{t=0} = \cdots = x_0. \tag{9}$$

We obtain

$$
\begin{aligned}
x_1(t) &= \mathrm{e}^{At} x_0,\\
x_2(t,\varepsilon) &= \varepsilon \int_0^t \mathrm{e}^{A(t-\theta)} \int_{-\infty}^{\theta} R(\theta,s)\, x_1(s)\, \mathrm{d}s\, \mathrm{d}\theta + \mathrm{e}^{At} x_0,\\
x_3(t,\varepsilon) &= \varepsilon \int_0^t \mathrm{e}^{A(t-\theta)} \int_{-\infty}^{\theta} R(\theta,s)\, x_2(s,\varepsilon)\, \mathrm{d}s\, \mathrm{d}\theta + \mathrm{e}^{At} x_0,
\end{aligned} \tag{10}
$$

etc.

We can also consider $x_1, x_2, x_3, \ldots$ as successive approximations to solution of the following integral system:

$$
x(t,\varepsilon) = \varepsilon \int_0^t \mathrm{e}^{A(t-\theta)} \int_{-\infty}^{\theta} R(\theta,s)\, x(s,\varepsilon)\, \mathrm{d}s\, \mathrm{d}\theta + \mathrm{e}^{At} x_0, \tag{11}
$$

which we interprete as an operator system

$$
x = \varepsilon L x + \mathrm{e}^{At} x_0, \tag{12}
$$

where L is the operator corresponding to the integral on the right-hand side of (11).

The operator L has been analysed in [3] in the case of the estimate (2) for $R(t,s)$. The following estimate for ε securing uniform convergence of the sequence (10) and at the same time the existence of *principal* solutions of system (1) has been obtained:

$$
\varepsilon < \varepsilon_* = \frac{\alpha^{\alpha}(\gamma-\mu)^{\alpha+1}}{cm\Gamma(\alpha)(1+\alpha)^{1+\alpha}},
$$

where $\Gamma(\alpha)$ is Euler's Gamma-function.[1] In the case of the simpler estimate (5) for $R(t,s)$, i.e. when $\alpha = 1$, we have for ε the estimate

$$
\varepsilon < \varepsilon_* = \frac{(\gamma-\mu)^2}{4cm}. \tag{13}
$$

As we have mentioned above, every *principal* solution $\tilde{x}(t,\varepsilon)$ belongs to the class U_q, hence it satisfies for $t \leqslant 0$ the estimate

$$
\|\tilde{x}(t,\varepsilon)\| \leqslant N\, \mathrm{e}^{-qt}, \tag{14}
$$

where N, q are some positive constants and, if ε is sufficiently small, then it is possible according to results in [3] to put a constant q sufficiently near to constant μ characterising $\exp(-At)$. Here we will derive more a concrete estimate for $\tilde{x}(t,\varepsilon)$.

[1] There is a misprint in (33) of paper [3]: $(\gamma+\mu)$ instead of $(\gamma-\mu)$.

The principal solution $\tilde{x}(t,\varepsilon)$ corresponding to an initial vector $x|_{t=0} = x_0$ can be written in the form

$$\tilde{x}(t,\varepsilon) = \tilde{X}(t,\varepsilon)x_0, \tag{15}$$

where $\tilde{X}(t,\varepsilon)$ is the principal fundamental two-sided (or, for the sake of simplicity, principal) matrix of system (1) normalised for $t = 0$. This matrix satisfies the equation coinciding with (1) or (11), if we substitute $\tilde{X}$ for x. For the sequence $\tilde{X}_1(t), \tilde{X}_2(t,\varepsilon), \ldots$ we obtain formulae coinciding with (10) if we substitute $\tilde{X}_1, \tilde{X}_2, \ldots$ for $x_1, x_2, \ldots$ and put $x_0 = 1$. These approximations converge (at least if the condition (13) is satisied) to the principal fundamental matrix $\tilde{X}(t,\varepsilon)$. The following theorem is true.

Theorem 1. *If parameter ε satisfies condition* (13), *then the principal fundamental matrix $\tilde{X}(t,\varepsilon)$ of system* (1) *normalised for $t = 0$ satisfies for $t \leqslant 0$ the estimate*

$$\|\tilde{X}(t,\varepsilon)\| \leqslant m\,\mathrm{e}^{q_1 t}, \tag{16}$$

where

$$q_1 = -\mu - \frac{(\gamma-\mu)}{2}\left(1 - \sqrt{1 - \frac{4\varepsilon mc}{(\gamma-\mu)^2}}\right) \tag{17}$$

and m, γ, c are the constants in the estimates (5), (7) *for matrices* $R(t,s)$, $\exp(At)$.

P r o o f. Consider the scalar integro-differential equation

$$\mathrm{d}u/\,\mathrm{d}t = -\mu u - mc\varepsilon \int_{-\infty}^{t} \mathrm{e}^{-\gamma(t-s)}\, u(s,\varepsilon)\,\mathrm{d}s. \tag{18}$$

We will seek its principal solution under the initial condition $u|_{t=0} = m$ with the help of the following successive approximation:

$$\begin{aligned} u_1(t) &= m\mathrm{e}^{-\mu t},\\ u_2(t,\varepsilon) &= mc\varepsilon \int_t^0 \mathrm{e}^{-\mu(t-\theta)} \int_{-\infty}^{\theta} \mathrm{e}^{-\gamma(\theta-s)}\, u_1(s)\,\mathrm{d}s\,\mathrm{d}\theta + m\mathrm{e}^{-\mu t},\\ u_{k+1}(t,\varepsilon) &= mc\varepsilon \int_t^0 \mathrm{e}^{-\mu(t-\theta)} \int_{-\infty}^{\theta} \mathrm{e}^{-\gamma(\theta-s)}\, u_k(s,\varepsilon)\,\mathrm{d}s\,\mathrm{d}\theta + m\mathrm{e}^{-\mu t},\\ &\qquad k = 2,3,4,\ldots \end{aligned} \tag{19}$$

The equation (18) and the approximations (19) represent respectively the particular cases of system (1) and approximations (10). Therefore, the results proved in [3] allow to conclude that the sequence $u_1, u_2, u_3, \dots$ converges to the principal solution $\tilde{u}(t,\varepsilon)$ of equation (18) provided parameter ε satisfies the estimate (13).

The comparison of (19) and (10) shows owing to estimates (5), (7) that approximations (19) are majorizing for $t \leqslant 0$ with respect to the sequence

$$\|\tilde{X}_1(t)\|,\ \|\tilde{X}_2(t,\varepsilon)\|,\ \dots$$

Hence, we obtain the inequality

$$\|\tilde{X}(t,\varepsilon)\| \leqslant \tilde{u}(t,\varepsilon), \quad t \leqslant 0. \tag{20}$$

At the same time, we can obtain the principal solution $\tilde{u}(t,\varepsilon)$ of Eq. (18) directly (see [1], [2]) in the form

$$\tilde{u}(t,\varepsilon) = m\,\mathrm{e}^{qt}, \tag{21}$$

where m is the initial value of $\tilde{u}(t,\varepsilon)$ for $t=0$ and $q = q(\varepsilon)$ is the root of the algebraic equation

$$q = -\mu - \frac{\varepsilon mc}{\gamma + q}, \tag{22}$$

namely, the root tending to $-\mu$ as $\varepsilon \to 0$.

The equation (22) has two roots; the first root q_1 is expressed by (17) and the second equals

$$q_2 = -\gamma + \frac{\gamma-\mu}{2}\left(1 - \sqrt{1 - \frac{4\varepsilon mc}{(\gamma-\mu)^2}}\right). \tag{23}$$

The root q_2 tends to $-\gamma$ as $\varepsilon \to 0$. Therefore, just the root q_1 corresponds to the principal solution $\tilde{u}(t,\varepsilon)$. Thus, the inequality (20) may be reduced to the inequality (16). Theorem is proved. □

3. Asymptotic properties of principal solutions

Let us be given a so called one-sided solution $x(t,\varepsilon)$ of system (1) with an initial function $\varphi(t)$ on the left semiaxis t, which is piecewise continuous, bounded as $t \to -\infty$ and satisfies the estimate

$$\|\varphi(t)\| \leqslant b, \quad t \leqslant 0, \tag{24}$$

where b is a fixed constant.

Let $\tilde{x}(t,\varepsilon)$ be a principal solution of system (1) with an arbitrary initial vector x_0. We can write such a solution in the form

$$\tilde{x}(t,\varepsilon) = \tilde{X}(t,\varepsilon)x_0, \tag{25}$$

where $\tilde{X}(t,\varepsilon)$ is the principal fundamental matrix of system (1) normalised for $t = 0$. Consider the difference

$$y = \tilde{x} - x \tag{26}$$

and the corresponding equation for this difference:

$$\mathrm{d}y/\mathrm{d}t = Ay + \varepsilon \int_0^t R(t,s)y(s,\varepsilon)\,\mathrm{d}s + \varepsilon F_1(t,\varepsilon)x_0 + \varepsilon F_2(t), \tag{27}$$

where the matrix $F_1(t,\varepsilon)$ and the vector $F_2(t)$ are expressed by formulae

$$F_1(t,\varepsilon) = \int_{-\infty}^0 R(t,s)\tilde{X}(s,\varepsilon)\,\mathrm{d}s, \tag{28}$$

$$F_2(t) = \int_{-\infty}^0 R(t,s)\varphi(s)\,\mathrm{d}s, \tag{29}$$

and the vector x_0 is arbitrary.

Since the initial function $\varphi(t)$ is given and the fundamental matrix $\tilde{X}(t,\varepsilon)$ is the same for all principal solutions of system (1), we can treat functions F_1 and F_2 as known. Thus the system (27) is a linear nonhomogeneous Volterra-type system without aftereffect. It is known [4] that such a system has a unique solution for every initial vector given for $t = 0$. Our principal solution (25) of system (1) depends on an arbitrary (for the moment) vector x_0. Taking it in view, we will seek the solution $y(t,\varepsilon)$ tending to zero as $t\to\infty$ with the help of the following successive approximations considered for $t\geqslant 0$:

$$y_1(t,\varepsilon)=\varepsilon\int_\infty^t \mathrm{e}^{A(t-\theta)}\bigl[F_1(\theta,\varepsilon)x_0+F_2(\theta)\bigr]\,\mathrm{d}\theta, \tag{30}$$

$$y_{k+1}(t,\varepsilon)=\varepsilon\int_\infty^t \mathrm{e}^{A(t-\theta)}\int_0^\theta R(\theta,s)y_k(s,\varepsilon)\,\mathrm{d}s\,\mathrm{d}\theta+y_1(t,\varepsilon), \tag{31}$$

$$k=1,2,3,\dots$$

If the sequence $y_1,y_2,\dots$ converges (uniformly in t), then its limit represents a solution of system (27) for $t\geqslant 0$.

For the purpose of constructing the majorizing sequence with respect to the sequence $y_1,y_2,\dots$ we will derive some estimates. Namely, in accordance with (5), (16), (24), (25) and (28), (29) we have the estimates

$$\begin{aligned}\|F_1(t,\varepsilon)\|\leqslant\int_{-\infty}^0\|R(t,s)\|\cdot\|\tilde X(s,\varepsilon)\|\,\mathrm{d}s\leqslant mc\int_{-\infty}^0 \mathrm{e}^{-\gamma(t-s)}\mathrm{e}^{q_1 s}\,\mathrm{d}s\\ =M_1\mathrm{e}^{-\gamma t},\end{aligned} \tag{32}$$

$$\begin{aligned}\|F_2(t)\|\leqslant\int_{-\infty}^0\|R(t,s)\|\cdot\|\varphi(s)\|\,\mathrm{d}s\leqslant c\int_{-\infty}^0 b\mathrm{e}^{-\gamma(t-s)}\,\mathrm{d}s\\ =M_2\mathrm{e}^{-\gamma t},\end{aligned} \tag{33}$$

where $t\geqslant 0$ and

$$M_1=\frac{mc}{\gamma+q_1},\qquad M_2=\frac{cb}{\gamma}. \tag{34}$$

The following estimates are also needed:

$$\left\|\int_\infty^t \mathrm{e}^{A(t-\theta)}F_1(\theta,\varepsilon)\,\mathrm{d}\theta\right\|\leqslant m\,M_1\int_t^\infty \mathrm{e}^{-\mu(t-\theta)}\mathrm{e}^{-\gamma\theta}\,\mathrm{d}\theta=\frac{m\,M_1}{\gamma-\mu}\mathrm{e}^{-\gamma t}, \tag{35}$$

$$\left\|\int_\infty^t \mathrm{e}^{A(t-\theta)}F_2(\theta)\,\mathrm{d}\theta\right\|\leqslant m\,M_2\int_t^\infty \mathrm{e}^{-\mu(t-\theta)}\mathrm{e}^{-\gamma\theta}\,\mathrm{d}\theta=\frac{m\,M_2}{\gamma-\mu}\mathrm{e}^{-\gamma t}. \tag{36}$$

We write every member of the sequence (30), (31) in the form of a sum (separating an arbitrary vector x_0):

$$y_j=Y_jx_0+\bar y_j,\quad 1,2,3,\dots, \tag{37}$$

where Y_j is a matrix, $\bar{y}_j$ is a vector. The sequence $y_1, y_2, \dots$ can be considered as a linear combination of two sequences $\{Y_j\}$ and $\{\bar{y}_j\}$ defined by the formulae

$$\begin{aligned} &Y_1(t,\varepsilon) = \varepsilon \int_\infty^t \mathrm{e}^{A(t-\theta)} F_1(\theta,\varepsilon)\,\mathrm{d}\theta, \\ (38)\qquad &Y_{k+1}(t,\varepsilon) = \varepsilon \int_\infty^t \mathrm{e}^{A(t-\theta)} \int_0^\theta R(\theta,s) Y_k(s,\varepsilon)\,\mathrm{d}s\,\mathrm{d}\theta + Y_1(t,\varepsilon), \\ &\bar{y}_1(t,\varepsilon) = \varepsilon \int_\infty^t \mathrm{e}^{A(t-\theta)} F_2(\theta)\,\mathrm{d}\theta, \\ (39)\qquad &\bar{y}_{k+1}(t,\varepsilon) = \varepsilon \int_\infty^t \mathrm{e}^{A(t-\theta)} \int_0^\theta R(\theta,s) \bar{y}_k(s,\varepsilon)\,\mathrm{d}s\,\mathrm{d}\theta + \bar{y}_1(t), \end{aligned}$$

$$k = 1, 2, 3, \dots$$

For $Y_1(t,\varepsilon)$ and $\bar{y}_1(t,\varepsilon)$ we have in accordance with (35), (36) the estimates

$$\|Y_1(t,\varepsilon)\| \leqslant \varepsilon \overline{M}_1 \mathrm{e}^{-\gamma t}, \qquad \|\bar{y}_1(t,\varepsilon)\| \leqslant \varepsilon \overline{M}_2 \mathrm{e}^{-\gamma t}, \tag{40}$$

where

$$\overline{M}_1 = \frac{m\,M_1}{\gamma-\mu}, \qquad \overline{M}_2 = \frac{m\,M_2}{\gamma-\mu}. \tag{41}$$

The sequences $\{Y_j\}$ and $\{\bar{y}_j\}$ have the same structure, therefore we will analyse only the first of them. For this purpose we construct the scalar sequence

$$\begin{aligned} &v_1(t,\varepsilon) = K_1 \mathrm{e}^{-\gamma t}, \\ (42)\qquad &v_{k+1}(t,\varepsilon) = mc\varepsilon \int_t^\infty \mathrm{e}^{-\mu(t-\theta)} \int_0^\theta \mathrm{e}^{-\gamma(\theta-s)} v_k(s,\varepsilon)\,\mathrm{d}s\,\mathrm{d}\theta + v_1(t,\varepsilon), \end{aligned}$$

$$k = 1, 2, 3, \dots,$$

where $K_1 = \varepsilon \overline{M}_1$.

This sequence is majorizing with respect to the sequence $Y_j(t,\varepsilon)$ in accordance with the estimates (40), (5), (7). We will show that the following lemma for the sequence $\{v_j(t,\varepsilon)\}$ is true.

Lemma. *If a positive parameter ε satisfies the estimate* (13), *then the sequence* (42) *converges to the function*

$$v_*(t,\varepsilon) = K_1 \frac{\gamma-\mu}{q_1+\gamma} \mathrm{e}^{q_2 t}, \tag{43}$$

where constants q_1 and q_2 are expressed by (17), (23), *and this function for* $t \geqslant 0$ *is a solution of the equation*

$$\mathrm{d}v/\,\mathrm{d}t = -\mu v - \varepsilon mc \int_0^t \mathrm{e}^{-\gamma(t-s)} v(s,\varepsilon) - (\gamma-\mu) K_1 \mathrm{e}^{-\gamma t}. \tag{44}$$

Proof. Consider on the right t semiaxis a class of continuous functions decreasing as $t \to \infty$ not slower than $\exp(qt)$, where the negative number q satisfies the inequality $-\gamma < q < -\mu$. For every function $\psi(t)$ of this class we will take the quantity

$$\max_{t \geqslant 0} |\psi(t) \mathrm{e}^{-qt}|$$

as its norm $\|\psi\|_q$; thus

$$\|\psi(t,\varepsilon)\|_q = \max_{t \geqslant 0} |\psi(t,\varepsilon) \mathrm{e}^{-qt}| \leqslant N, \tag{45}$$

where N is a constant.

The sequence (42) represents the successive approximations to the solution of the integral equation

$$v(t,\varepsilon) = m\, c\varepsilon \int_t^\infty \mathrm{e}^{-\mu(t-\theta)} \int_0^\theta \mathrm{e}^{-\gamma(\theta-s)} v(s,\varepsilon)\, \mathrm{d}s\, \mathrm{d}\theta + K_1 \mathrm{e}^{-\gamma t} \tag{46}$$

or (in operator form)

$$v = Lv + K_1 \mathrm{e}^{-\gamma t}, \tag{47}$$

where L is the integral operator corresponding to the integral on the right-hand side of (46). For the norm $\|Lv\|_q$ we obtain

$$\begin{aligned}
\|Lv(t,\varepsilon)\|_q &= \max_{t \geqslant 0} |Lv(t,\varepsilon) \mathrm{e}^{-qt}| \\
&= \max_{t \geqslant 0} \left[mc\varepsilon \int_t^\infty \mathrm{e}^{-\mu(t-\theta)} \int_0^\theta \mathrm{e}^{-\gamma(\theta-s)} \mathrm{e}^{qs} v(s,\varepsilon) \mathrm{e}^{-qs}\, \mathrm{d}s\, \mathrm{d}\theta \cdot \mathrm{e}^{-qt} \right] \\
&= \max_{t \geqslant 0} \left[\frac{\varepsilon mc}{\gamma+q} \left(\frac{1}{-q-\mu} \mathrm{e}^{qt} - \frac{1}{\gamma-\mu} \mathrm{e}^{-\gamma t} \right) \mathrm{e}^{-qt} \right] \|v(t,\varepsilon)\|_q \\
&\leqslant \frac{\varepsilon mc}{(\gamma+q)(-q-\mu)} \|v(t,\varepsilon)\|_q.
\end{aligned}$$

Thus

$$\|v(t,\varepsilon)\|_q \leqslant \varepsilon \varrho \|v(t,\varepsilon)\|_q, \tag{48}$$

where

$$\varrho = \frac{mc}{(\gamma + q)(-q - \mu)}. \tag{49}$$

The relation $\varepsilon\varrho = 1$ represents an algebraic equation in q, and this equation coincides with the equation (22). Its roots $q_1(\varepsilon)$, $q_2(\varepsilon)$ are defined by the formulae (17), (23), and

$$-\gamma < q_2(\varepsilon) < q_1(\varepsilon) < -\mu$$

and $q_1(\varepsilon) \to -\mu, \quad q_2(\varepsilon) \to -\gamma$ as $\varepsilon \to 0$.

Further analysis of this equation shows that the inequality $\varepsilon\varrho < 1$ holds for all values of q in the interval (q_2, q_1), provided ε satisfies the estimate (13). Thus this estimate provides a contraction of operator L and the convergence of sequence (42) to a solution $v_*(t,\varepsilon)$ of the integral equation (46). After differentiating (46) we come to the conclusion that the function $v_*(t,\varepsilon)$ also satisfies the integrodifferential equation (44).

It would be possible to estimate the function $v_*(t,\varepsilon)$ with the help of expressions (42) for its successive approximations. However, we can obtain a more accurate estimate after defining $v_*(t,\varepsilon)$ directly as a finite solution of equation (44). The equations of such form (linear nonhomogeneous) are solvable in finite form (see [4]).

Namely, the general solution of the equation (44) can be written in the form

$$v(t,\varepsilon,C) = Cv^0(t,\varepsilon) + \bar{v}(t,\varepsilon), \tag{50}$$

where $v^0(t,\varepsilon)$ is the solution of the corresponding homogeneous equation (for $K = 0$) with the initial value $v|_{t=0} = 1$, C is an arbitrary constant and

$$\bar{v}(t,\varepsilon) = -(\gamma - \mu)K_1 \int_0^t v^0(t-s,\varepsilon)\,\mathrm{e}^{-\gamma s}\,\mathrm{d}s. \tag{51}$$

The function $v^0(t,\varepsilon)$ has the form

$$v^0(t,\varepsilon) = C_1\mathrm{e}^{q_1 t} + C_2\mathrm{e}^{q_2 t},$$

where the constants q_1, q_2 are the above mentioned roots of the algebraic equation (22) and C_1, C_2 are some constants depending on γ, q_1, q_2. As a result of some calculations we obtain

$$v(t,\varepsilon,C) = C\left[\frac{\gamma + q_1}{q_1 - q_2}\mathrm{e}^{q_1 t} - \frac{\gamma + q_2}{q_1 - q_2}\mathrm{e}^{q_2 t}\right] + K_1\frac{\gamma - \mu}{q_1 - q_2}\left(\mathrm{e}^{q_2 t} - \mathrm{e}^{q_1 t}\right). \tag{52}$$

It is necessary now to choose such a constant C that we could obtain from (52) just the solution $v_*(t,\varepsilon)$.

It is seen from formulae (42) that all approximations $v_{k+1}(t,\varepsilon), k = 1, 2, 3, \ldots$ have the form

$$(P_k(t,\varepsilon) + K_1)\,\mathrm{e}^{-\gamma t},$$

where $P_k(t,\varepsilon)$ are polynomials in t of degree k tending to zero as $\varepsilon \to 0$. Therefore all $v_{k+1}(t,\varepsilon)$ tend to zero as $t \to \infty$ and possess the same characteristic exponent $-\gamma$. Their limit $v_*(t,\varepsilon)$ tends also to zero as $t \to \infty$ and its characteristic exponent is the nearer to $-\gamma$, the smaller ε is.

Since $q_1(\varepsilon) \to -\mu$ and $q_2(\varepsilon) \to -\gamma$ as $\varepsilon \to 0$, we obtain the required solution $v_*(t,\varepsilon)$ of equation (44) from the general solution (52) choosing the constant C just so that all members with the exponent $\exp(q_1 t)$ vanish. We put

$$C = C_* = K_1 \frac{\gamma - \mu}{\gamma + q_1}. \tag{53}$$

Then we obtain

$$v_*(t,\varepsilon) = v(t,\varepsilon,C_*) = K_1 \frac{\gamma - \mu}{\gamma + q_1} \mathrm{e}^{q_2 t}. \tag{54}$$

The lemma is proved. $\square$

This lemma testifies to the convergence of the sequence (38) on the right semiaxis t to a matrix function $Y_*(t,\varepsilon)$, and also to the validity of the estimation

$$\|Y_*(t,\varepsilon)\| \leqslant K_1 \frac{\gamma - \mu}{\gamma + q_1} \mathrm{e}^{q_2 t}, \tag{55}$$

if ε satisfies (13).

The same analysis may be carried out for the sequence (39). We obtain the same result with regard to the convergence of this sequence to the vector function $\bar{y}_*(t,\varepsilon)$, and derive the same estimate for $t \geqslant 0$:

$$\|\bar{y}_*(t,\varepsilon)\| \leqslant K_2 \frac{\gamma - \mu}{\gamma + q_1} \mathrm{e}^{q_2 t}, \tag{56}$$

where $K_2 = \varepsilon \overline{M}_2$. We can rewrite these estimates with the help of the expressions (34), (41) for M_1, M_2, $\overline{M}_1$, $\overline{M}_2$, and we obtain

$$\|Y_*(t,\varepsilon)\| \leqslant \varepsilon \frac{m^2 c}{(\gamma + q_1)^2} \mathrm{e}^{q_2 t}, \tag{57}$$

$$\|\bar{y}_*(t,\varepsilon)\| \leqslant \varepsilon \frac{mbc}{\gamma(\gamma + q_1)} \mathrm{e}^{q_2 t}. \tag{58}$$

Let us return now to the solution $y(t,\varepsilon)$ of system (27) and to the sequence (30), (31) used for finding this solution. According to (37) we can consider the sequence (30), (31) as a linear combination of the sequences (38) and (39). The results obtained above with regard to these sequences lead us to the conclusion that the sequence (30), (31) converges, if ε satisfies (13), to a solution $y_*(t,\varepsilon)$ and this solution can be written in the form

$$y_*(t,\varepsilon) = Y_*(t,\varepsilon)x_0 + \bar{y}_*(t,\varepsilon), \tag{59}$$

where the matrix $Y_*(t,\varepsilon)$ and the vector $\bar{y}_*(t,\varepsilon)$ satisfy the estimates (57), (58) and x_0 is an arbitrary (at the moment) vector.

We can now prove the following theorem.

Theorem 2. *Let us be given a one-sided solution $x(t,\varepsilon)$ of system* (1) *with an initial function $\varphi(t)$ satisfying for $t \leqslant 0$ the estimate* (24). *Then, if ε satisfies the estimate* (13) *and, additionally, the inequality*

$$\frac{\varepsilon m^2 c}{(\gamma + q_1)^2} < 1, \tag{60}$$

where the constants γ, m, c, q_1 are defined by (5), (7), (17), *then there exists such a principal solution $\tilde{x}_*(t,\varepsilon)$ of system* (1) *that*

$$\|\tilde{x}_*(t,\varepsilon) - x(t,\varepsilon)\| \leqslant \varepsilon M \mathrm{e}^{q_2 t}, \tag{61}$$

where the constant q_2 is expressed by (23) *and M is a constant depending on γ, m, c, q_1 and also on the constant b of inequality* (24).

Remark The inequality (60) may be written with the help of the algebraic equation (22) in the form

$$\frac{m(1-u)}{1+u} < 1, \tag{62}$$

where

$$u = \left(1 - \frac{4\varepsilon mc}{(\gamma-\mu)^2}\right)^{1/2}.$$

Proof. The function $y_*(t,\varepsilon)$ expressed by (59) is equal to the difference (26) between the principal solution $\tilde{x}(t,\varepsilon)$ with the initial vector $\tilde{x}|_{t=0} = x_0$ and the one-sided solution $x(t,\varepsilon)$ with the initial function $\varphi(t)$. Up to now we have considered the vector x_0 as arbitrary. But, if we put in (59) $t = 0$, then this relation turns into a system of linear algebraic equations in matrix form with respect to the vector x_0:

$$x_0 - \varphi(0) = Y_*(0,\varepsilon)x_0 + \bar{y}_*(0,\varepsilon). \tag{63}$$

The condition (60) provides the estimate

$$\|Y_*(0,\varepsilon)\| < 1.$$

Hence, the inverse matrix

$$\left[E - Y_*(0,\varepsilon)\right]^{-1}$$

exists, where E is the unit matrix, and the equation (63) has the following solution for x_0:

$$x_0 = x_0^* = \left[E - Y_*(0,\varepsilon)\right]^{-1}\left[\varphi(0) + \bar{y}_*(0,\varepsilon)\right]. \tag{64}$$

Just this initial vector corresponds to the principal solution $\tilde{x}_*(t,\varepsilon)$, for which the difference

$$\tilde{x}_*(t,\varepsilon) - x(t,\varepsilon),$$

denoted by $y_*(t,\varepsilon)$, satisfies the estimate resulting from (57)–(59). Namely, we obtain for $t \geqslant 0$ the inequality

$$\|x_0^*(t,\varepsilon) - x(t,\varepsilon)\| \leqslant \frac{\varepsilon mc}{\gamma + q_1}\left[\frac{m}{\gamma + q_1}\|x_0^*\| + \frac{b}{\gamma}\right]\mathrm{e}^{q_2 t}, \tag{65}$$

where $\|x_0^*\|$ may be estimated according to (64). This inequality corresponds to (61). Theorem is proved. □

E x a m p l e. Consider as an illustration the scalar equation

$$\ddot{x} = \varepsilon \int_{-\infty}^{t} \mathrm{e}^{-\gamma(t-s)} x(s,\varepsilon)\,\mathrm{d}s. \tag{66}$$

Its principal solution under the initial conditions $x|_{t=0} = x_0, \dot{x}_{t=0} = \dot{x}_0$ equals (see [1], [2])

$$\tilde{x}(t,\varepsilon,x_0,\dot{x}_0) = C_1\mathrm{e}^{\lambda_1 t} + C_2\mathrm{e}^{\lambda_2 t}, \tag{67}$$

where

$$C_1 = \frac{\dot{x}_0 - \lambda_2 x_0}{\lambda_1 - \lambda_2}, \qquad C_2 = \frac{\dot{x}_0 - \lambda_1 x_0}{\lambda_2 - \lambda_1} \tag{68}$$

and $\lambda_1(\varepsilon), \lambda_2(\varepsilon)$ are the roots of the algebraic equation

$$\lambda^2 = \frac{\varepsilon}{\gamma + \lambda}, \tag{69}$$

tending to zero as $\varepsilon \to 0$. This solution exists provided (see [1], [2])

$$\varepsilon < \frac{4}{27}\gamma^3. \tag{70}$$

Consider the one-sided solution $x(t, \varepsilon)$ corresponding to the initial function

$$\varphi(t) \equiv 0, \quad t < 0, \quad \varphi(0) = a_0, \quad \dot{\varphi}(0) = a_1. \tag{71}$$

This solution satisfies a Volterra-type equation without aftereffect

$$\ddot{x} = \varepsilon \int_0^t \mathrm{e}^{-\gamma(t-s)} x(s, \varepsilon)\,\mathrm{d}s \tag{72}$$

and the initial conditions

$$x|_{t=0} = a_0, \;\; \dot{x}|_{t=0} = a_1.$$

We obtain this solution in the finite form:

$$x(t, \varepsilon, a_0, a_1) = \bar{c}_1 \left[\mathrm{e}^{\lambda_1 t} - m_1 \mathrm{e}^{\lambda_3 t}\right] + \bar{c}_2 \left[\mathrm{e}^{\lambda_2 t} - m_2 \mathrm{e}^{\lambda_3 t}\right], \tag{73}$$

where constants $\bar{c}_1$, $\bar{c}_2$ are expressed by a_0, a_1 and constants m_1, m_2 are equal to

$$m_1 = \frac{\gamma + \lambda_3}{\gamma + \lambda_1}, \qquad m_2 = \frac{\gamma + \lambda_3}{\gamma + \lambda_2},$$

where $\lambda_3 = \lambda_3(\varepsilon)$ is the third root of equation (69). The roots λ_1, λ_2, λ_3 of this equation satisfy the inequalities

$$-\gamma < \lambda_3 < -\frac{2}{3}\gamma, \quad -\frac{2}{3}\gamma < \lambda_2 < 0, \quad 0 < \lambda_1 < \frac{1}{3}\gamma$$

for all ε restricted by the condition (70), and $\gamma_3 \to -\gamma$ as $\varepsilon \to 0$. For the function

$$y(t, \varepsilon, x_0, \dot{x}_0) = \tilde{x}(t, \varepsilon, x_0, \dot{x}_0) - x(t, \varepsilon, a_0, a_1)$$

we have the integrodifferential equation

$$\ddot{y} = \varepsilon \int_0^t \mathrm{e}^{-\gamma(t-s)}\, y(s, \varepsilon)\,\mathrm{d}s + \varepsilon \int_{-\infty}^0 \mathrm{e}^{-\gamma(t-s)}\, \tilde{x}(s, \varepsilon, x_0, \dot{x}_0)\,\mathrm{d}s, \tag{74}$$

where $\tilde{x}(s, \varepsilon, x_0, \dot{x}_0)$ is expressed by (67). Its solution with a characteristic exponent for $t \to \infty$ tending to $-\gamma$ as $\varepsilon \to 0$ is following:

$$y_*(t, \varepsilon, x_0, \dot{x}_0) = \frac{\varepsilon}{\lambda_3^2}\left[\frac{C_1}{\gamma + \lambda_1} + \frac{C_2}{\gamma + \lambda_2}\right] \mathrm{e}^{\lambda_3 t}, \tag{75}$$

where C_1, C_2 are expressed by (68). We have considered up to now initial values x_0, $\dot{x}_0$ and at the same time constants C_2, C_2 as arbitrary. However, if we put $t = 0$, then we obtain the relations

$$y_*|_{t=0} = \tilde{x}|_{t=0} - x|_{t=0}, \qquad \dot{y}_*|_{t=0} = \dot{\tilde{x}}|_{t=0} - \dot{x}|_{t=0}$$

or

$$\begin{aligned} \frac{\varepsilon}{\lambda_3^2}\left[\frac{C_1}{\gamma + \lambda_1} + \frac{C_2}{\gamma + \lambda_2}\right] &= C_1 + C_2 - a_0, \\ \frac{\varepsilon}{\lambda_3}\left[\frac{C_1}{\gamma + \lambda_1} + \frac{C_2}{\gamma + \lambda_2}\right] &= \lambda_1\, C_1 + \lambda_2\, C_2 - a_1. \end{aligned} \tag{76}$$

These relations represent a linear algebraic system for C_1 and C_2. The calculations show that the determinant of this system does not equal zero, if $0 < \varepsilon < \frac{4}{27}\gamma^3$. Just then we can express C_1, C_2 and also $x_0, \dot{x}_0$ with the help of (68) in terms of a_0, a_1. Thus we shall find the initial values for the principal solution

$$\tilde{x}(t, \varepsilon, x_0, \dot{x}_0)$$

approximating the one-sided solution (73) according to Theorem 2.

If $\varepsilon = \frac{4}{27}\gamma^3$, then the above mentioned determinant is equal to zero and it is not possible to find the required C_1, C_2 for arbitrary a_0, a_1. Thus, the principal solutions of Eq. (66) possess the asymptotic properties in the sense of Theorem 2 for the same range of values of ε, where these solutions exist. The additional restriction on ε corresponding to inequality (60) is not needed. Nevertheless, there are also cases of other kind, when additional restriction on ε is essential, i.e. the principal solutions do not possess the just mentioned asymptotic properties for all values of ε securing the existence of these solutions. For exemple, in the case of the equation

$$\ddot{x} + 2x = \varepsilon \int_{-\infty}^{t} \mathrm{e}^{-\gamma(t-s)} x(s, \varepsilon)\, \mathrm{d}s, \quad \gamma = 0.5$$

the following result can be obtained (we omit calculations): the principal solutions exist if $\varepsilon < 2.25$, but possess the above mentioned asymptotic properties if $\varepsilon < \frac{2}{3} + \frac{1}{108}$.

Conclusion. The results obtained testify to the fact that the infinite dimensional set of one-sided solutions $x(t,\varepsilon)$ of system (1) corresponding to piecewise continuous and bounded for $t \to \infty$ initial functions is equivalent in asymptotical sense to the finite-dimensional set of *principal* solutions $\tilde{x}(t,\varepsilon)$ of this system, if the parameter ε does not exceed a certain bound and the matrix A and the kernel $R(t,s)$ satisfy some conditions. Every solution $x(t,\varepsilon)$ approaches sufficiently quickly some principal solution $\tilde{x}(t,\varepsilon)$ as t grows and thus is well approximated by this solution beginning from some moment t. Hence, such asymptotic properties as stability, boundedness, oscillation are inherent to one-sided solutions, if these properties are valid for the principal solutions. However, the latter solutions constitute a finite-dimensional set and satisfy some system of ordinary differential equations of the form (3). Thus some essential structural features of the set of solutions of system (1) (finite dimensionality, boundedness etc.) remain in asymptotical sense or completely the same as is typical for the solutions of a system without aftereffect, and these features do not depend on *prehistory, heredity.* If equations (1.1) describe a dynamical system, then we propose the following interpretation of the principal solutions. Namely, we assume that principal solutions describe the natural behaviour of this system corresponding to its inherent relations and properties. *Prehistory* not corresponding to such natural behaviour leads to perturbations of such behaviour but only for a short time. Thereafter the influence of *prehistory* is fading and the dynamical system returns to its natural evolution described just by the *principal* solutions.

However, we want to underline that if the parameter ε exceeds the bound securing the validity of the above mentioned asymptotic properties of *principal* solutions, then *prehistory* or *heredity* has full influence upon the properties of the solutions on the right t semiaxis. Therefore it is of practical importance to determine this bound of values of ε in the cases of given concrete equations.

Remark. An analogous theorem can be also proved in the case of a more complicated estimate (2) for the kernel $R(t,s)$, and of a set of one-sided solutions corresponding to initial functions $\varphi(t)$ restricted for $t \leqslant 0$ by the inequality

$$\|\varphi(t)\| \leqslant b\,\|\varphi(0)\|\mathrm{e}^{-\beta t}, \quad t < 0,\ 0 < \beta < \gamma.$$

It may be expected that the similar results concerning the existence of principal solutions and their asymptotic properties are to some degree valid in the case of system (1) with a variable matrix $P(t)$ (instead of a constant matrix A), if all solution of the homogeneous equation

$$\mathrm{d}x/\,\mathrm{d}t = P(t)x$$

are exponentially bounded.

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