# $\alpha$-COMPACT FUZZY TOPOLOGICAL SPACES

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*Abstract.* The purpose of this paper is to introduce and discuss the concept of $\alpha$-compactness for fuzzy topological spaces.

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## 1. Introduction

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh in his classical paper [7]. Subsequently several authors have applied various basic concepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. The concept of $\alpha$-compactness for topological spaces has been discussed in [5]. The purpose of this paper is to introduce and study $\alpha$-compactness for fuzzy topological spaces, thus filling a gap in the existing literature on fuzzy topological spaces.

## 2. Definitions

Throughout this paper $(X, T)$ will denote a fuzzy topological space. If $A$ is a fuzzy set in a fuzzy topological space then the closure and interior of $A$ will be as usual denoted by $\operatorname{Cl} A$ and $\operatorname{Int} A$, respectively. We now introduce the following definitions.

**Definition 2.1.** Let $A$ be a fuzzy subset of a fuzzy topological space $X$. $A$ is said to be fuzzy $\alpha$-open if $A \subset \operatorname{Int}\operatorname{Cl}\operatorname{Int} A$. The set of all fuzzy $\alpha$-open subsets of $X$ will be denoted by $F\alpha(X)$.

**Definition 2.2.** In a fuzzy topological space $(X, T)$, a family $\upsilon$ of fuzzy subsets of $X$ is called an $\alpha$-covering of $X$ iff $\upsilon$ covers $X$ and $\upsilon \subset F\alpha(X)$.

**Definition 2.3.** A fuzzy topological space $(X, T)$ is said to be $\alpha$-compact if every $\alpha$-open cover of $X$ has a finite subcover.

**Definition 2.4.** Let $(X, T)$ and $(Y, S)$ be fuzzy topological spaces. A mapping $f\colon X \to Y$ is called fuzzy $\alpha$-continuous if the inverse image of each fuzzy open set in $Y$ is fuzzy $\alpha$-open in $X$.

**Definition 2.5.** A mapping $f\colon X \to Y$ is said to be fuzzy $\alpha$-irresolute if the inverse image of every fuzzy $\alpha$-open set in $Y$ is fuzzy $\alpha$-open in $X$.

**Definition 2.6.** Let $(X, T)$ and $(Y, S)$ be fuzzy topological spaces and let $T_\xi$ be a fuzzy topology on $X$ which has $F\alpha(X)$ as a subbase. A mapping $f\colon X \to Y$ is called fuzzy $\xi$-continuous if $f\colon (X, T_\xi) \to (Y, S)$ is fuzzy continuous; $f$ is said to be fuzzy $\xi'$-continuous if $f\colon (X, T_\xi) \to (Y, S_\xi)$ is fuzzy continuous.

## 3. Results

**Theorem 3.1.** *Let $(X, T)$ and $(Y, S)$ be fuzzy topological spaces and let $T_\xi$ be a fuzzy topology on $X$ which has $F\alpha(X)$ as a subbase. If $f\colon (X, T) \to (Y, S)$ is fuzzy $\alpha$-continuous, then $f$ is fuzzy $\xi$-continuous.*

Proof. Let $f$ be fuzzy $\alpha$-continuous and let $V \in S$. Then $f^{-1}(V) \in F\alpha(X)$ and so $f^{-1}(V) \in T_\xi$. Thus $f$ is fuzzy $\xi$-continuous and this completes the proof. $\square$

**Theorem 3.2.** *Let $(X, T)$ and $(Y, S)$ be fuzzy topological spaces. Let $T_\xi$ and $S_\xi$ be respectively the fuzzy topologies on $X$ and $Y$ which have $F\alpha(X)$ and $F\alpha(Y)$ as subbases. If $f\colon (X, T) \to (Y, S)$ is fuzzy $\alpha$-irresolute then $f$ is fuzzy $\xi'$-continuous.*

Proof. Let $f$ be fuzzy $\alpha$-irresolute and let $V \in S_\xi$. Then

$$V = \bigcup_i \Big( \bigcap_{i=1}^{n} S_{i_{n_i}} \Big) \quad \text{where } S_{i_{n_i}} \in F\alpha(Y, S),$$

and

$$f^{-1}(V) = f^{-1}\Big( \bigcup_i \Big( \bigcap_{i=1}^{n} S_{i_{n_i}} \Big) \Big) = \bigcup_i \Big( \bigcap_{i=1}^{n} f^{-1}(S_{i_{n_i}}) \Big).$$

Since $f$ is fuzzy $\alpha$-irresolute, $f^{-1}(S_{i_{n_i}}) \in F\alpha(X, T)$. This implies that $f^{-1}(V) \in T_\xi$ and thus $f$ is fuzzy $\xi'$-continuous. $\square$

**Theorem 3.3.** *A fuzzy topological space $X$ is $\alpha$-compact if and only if every family of fuzzy $\alpha$-closed subsets of $X$ with finite intersection property has non-empty intersection.*

Proof. Evident. $\square$

**Theorem 3.4.** *Let $(X,T)$ be a fuzzy topological space and $T_\xi$ a fuzzy topology on $X$ which has $F\alpha(X)$ as a subbase. Then $(X,T)$ is $\alpha$-compact if and only if $(X,T_\xi)$ is compact.*

Proof. Let $(X,T_\xi)$ be compact. Then, since $F\alpha(X) \subset T_\xi$, it follows that $(X,T)$ is $\alpha$-compact. $\square$

**Theorem 3.5.** *Let $(X,T)$ be a fuzzy topological space which is $\alpha$-compact. Then each $T_\xi$-closed fuzzy set in $X$ is $\alpha$-compact.*

Proof. Let $U$ be any $T_\xi$-closed fuzzy set in $X$. Let $\{V_{\beta_i}\colon \beta_i \in I\}$ be a $T_\xi$-open cover of $U$. Since $X - U$ is $T_\xi$-open, $\{V_{\beta_i}\colon \beta_i \in I\} \cup (X - U)$ is a $T_\xi$-open cover of $X$. Since $X$ is $T_\xi$-compact, by Theorem 3.4 there exists a finite subset $I_0 \subset I$ such that

$$X = \bigcup\{V_{\beta_i}\colon \beta_i \in I_0\} \cup (X - U).$$

This implies that

$$U \subset \bigcup\{V_{\beta_i}\colon \beta_i \in I_0\}.$$

Hence $U$ is $\alpha$-compact relative to $X$ and this completes the proof. $\square$

**Theorem 3.6.** *Let a fuzzy topological space $(X,T)$ be $\alpha$-compact. Then every family of $T_\xi$-closed fuzzy subsets of $X$ with finite intersection property has non-empty intersection.*

Proof. Let $X$ be $\alpha$-compact. Let $U = \{B_{\beta_i}\colon \beta_i \in I\}$ be any family of $T_\xi$-closed fuzzy subsets of $X$ with finite intersection property. Suppose $\bigcap\{B_{\beta_i}\colon \beta_i \in I\} = \emptyset$.

Then $\{X - B_{\beta_i}\colon \beta_i \in I\}$ is a $T_\xi$-open cover of $X$. Hence it must contain a finite subcover $\{X - B_{\beta_{i_j}}\colon j = 1,2,\ldots,n\}$ for $X$. This implies that $\bigcap\{B_{\beta_{i_j}}\colon j = 1,2,\ldots,n\} = \emptyset$ and contradicts the assumption that $U$ has finite intersection property. $\square$

**Theorem 3.7.** *Let $X$ and $Y$ be fuzzy topological spaces and let $f\colon X \to Y$ be fuzzy $\xi'$-continuous. If a fuzzy subset $G$ of $X$ is $\alpha$-compact relative to $X$, then $f(G)$ is $\alpha$-compact relative to $Y$.*

Proof. Let $\{V_{\beta_i}\colon \beta_i \in I\}$ be a cover of $f(G)$ by $S_\xi$-open fuzzy sets in $Y$. Then $\{f^{-1}(V_{\beta_i})\colon \beta_i \in I\}$ is a cover of $G$ by $T_\xi$-open fuzzy sets in $X$. $G$ is $\alpha$-compact relative to $X$. Hence by Theorem 3.4, $G$ is $T_\xi$-compact. So there exists a finite subset $I_0 \subset I$ such that

$$G \subset \bigcup\{f^{-1}(V_{\beta_i})\colon \beta_i \in I_0\}$$

and so

$$f(G) \subset \{V_{\beta_i} : \beta_i \in I_0\}.$$

Hence $f(G)$ is $T_\xi$-compact relative to $Y$. Thus $f(G)$ is $\alpha$-compact relative to $Y$ and this completes the proof. □

**Corollary 3.8.** *If $f\colon (X,T) \to (Y,S)$ is a fuzzy $\xi'$-continuous surjective function and $X$ is $\alpha$-compact, then $Y$ is $\alpha$-compact.*

**Corollary 3.9.** *If $f\colon (X,T) \to (Y,S)$ is a fuzzy $\alpha$-irresolute surjective function and $X$ is $\alpha$-compact then $Y$ is $\alpha$-compact.*

**Theorem 3.10.** *Let $A$ and $B$ be fuzzy subsets of a fuzzy topological space $X$ such that $A$ is $\alpha$-compact relative to $X$ and $B$ is $T_\xi$-closed in $X$. Then $A \cap B$ is $\alpha$-compact relative to $X$.*

Proof. Let $\{V_{\beta_i} : \beta_i \in I\}$ be a cover of $A \cap B$ by $T_\xi$-open fuzzy subsets of $X$. Since $X - B$ is a $T_\xi$-open fuzzy set,

$$\{V_{\beta_i} : \beta_i \in I\} \cup (X - B)$$

is a cover of $A$. $A$ is $\alpha$-compact and thus $T_\xi$-compact relative to $X$. Hence there exists a finite subset $I_0 \subset I$ such that

$$A \subset \bigcup\{V_{\beta_i} : \beta_i \in I_0\} \cup (X - B).$$

Therefore

$$A \cap B \subset \bigcup\{V_{\beta_i} : \beta_i \in I_0\}.$$

Hence $A \cap B$ is $T_\xi$-compact. Therefore $A \cap B$ is $\alpha$-compact and this completes the proof. □

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