MEDIAN PROPERTIES OF GRAPHS WITH SMALL DIAMETERS

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Abstract. Two numerical invariants $\Delta(G)$ and $\Gamma(G)$ of a graph, related to the concept of median, are studied.

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In [1] the numerical invariants $\Delta(G)$ and $\Gamma(G)$ of a finite undirected graph were studied. Here we will study them in the case of graphs whose diameter is at most 2.

Let G be a finite connected undirected graph without loops and multiple edges. If v is a vertex of G, then the valence $\Delta_G(v)$ of v in G is the sum of distances between v and all other vertices of G. The minimum of $\Delta_G(v)$ taken over all vertices v of G is denoted by $\Delta(G)$. Every vertex m of G for which $\Delta_G(m) = \Delta(G)$ holds is called a median of G.

A pairing P in G is a partition of the vertex set $V(G)$ of G into disjoint pairs, leaving at most one vertex unpaired (when $n = |V(G)|$ is odd). The symbol $\Gamma_G(P)$ denotes the sum of distances between two vertices belonging to the same pair of P. The maximum of $\Gamma_G(P)$ taken over all pairings P in G is denoted by $\Gamma(G)$.

In [1] it is proved that for a tree G always $\Delta(G) = \Gamma(G)$ and for a graph G in general $\Gamma(G) \leqslant \Delta(G) \leqslant 2\Gamma(G)$. In this paper we will consider finite graphs with a diameter at most 2. The number of vertices of a graph will be denoted by n. By $\overline{\beta}$ we denote the edge independence number $\beta(\overline{G})$ of the complement $\overline{G}$ of G. The maximum degree of a vertex in G will be denoted by D to avoid the confusion with the symbol Δ defined above.

We start with three lemmas.

Lemma 1. *Let G be graph with n vertices and with the diameter at most 2, let D be the maximum degree of a vertex of G. Then*

$$\Delta(G) = 2n - D - 2$$

and medians of G are exactly all vertices of degree D.

Proof. Let v be a vertex of G of degree r. Then there are r vertices having distance 1 and $n-r-1$ vertices having distance 2 from v. Thus $\Delta_G(v) = r + 2(n - r - 1) = 2n - r - 2$. This value attains its minimum if r is maximum, i.e. if $r = D$. This implies the assertion. □

Lemma 2. *Let G be a graph with n vertices and with the diameter at most 2, let $\overline{\beta}$ be the edge independence number of its complement $\overline{G}$. Then*

$$\Gamma(G) = \left\lfloor \frac{1}{2} n \right\rfloor + \overline{\beta}.$$

Proof. Let P be a pairing of G in which exactly b pairs are nonadjacent; in G these pairs form an independent set of edges and thus $b \leqslant \overline{\beta}$. These pairs have distance 2, while the remaining $\lfloor \frac{1}{2}n \rfloor - b$ pairs have distance 1. Thus $\Gamma_G(P) = 2b + \lfloor \frac{1}{2}n \rfloor - b = \lfloor \frac{1}{2}n \rfloor + b$. This value attains its maximum if b is maximum, i.e. if $b = \overline{\beta}$. □

Lemma 3. *Let G be a graph with n vertices and with the diameter at most 2, let D be the maximum degree of a vertex in G, let $\overline{\beta}$ be the edge independence number of its complement $\overline{G}$. If $\overline{\beta} \geqslant 1$, then $D \geqslant n - 2\overline{\beta}$.*

Proof. There exists a set of $\overline{\beta}$ independent edges in $\overline{G}$; let M be the set of end vertices of these edges. The set $V(G) - M$ induces a complete subgraph of G; otherwise there would be at least $\overline{\beta}+1$ independent edges in $\overline{G}$, which is not possible. Each vertex of $V(G) - M$ has degree $n - 2\overline{\beta} - 1$ in this complete subgraph. As G is connected and $M \neq \emptyset$, there exists at least one edge joining a vertex of $V(G) - M$ with a vertex of M; then this vertex of $V(G) - M$ has degree at least $n - 2\overline{\beta}$ in G and thus $D \geqslant n - 2\overline{\beta}$. □

Now we shall characterize the graphs (among graphs with a diameter at most 2) for which the extremal cases $\Delta(G) = \Gamma(G)$ and $\Delta(G) = 2\Gamma(G)$ occur.

Theorem 1. *Let G be a graph with $n \geqslant 3$ vertices and with the diameter at most 2. Then $\Delta(G) = 2\Gamma(G)$ if and only if n is odd and G is a complete graph with n vertices.*

P r o o f. Let $\Delta(G) = 2\Gamma(G)$. According to Lemmas 1 and 2 this means $2n - D - 2 = 2\left(\lfloor \frac{1}{2}n \rfloor + \overline{\beta}\right)$. If n is even, this implies $D + 2\overline{\beta} = n - 2$. If $D \leqslant n - 2$, then G is not a complete graph. The complement G contains at least one edge and thus $\overline{\beta} \geqslant 1$. According to Lemma 3 then $D + 2\overline{\beta} \geqslant n$, which is a contradiction. If $D = n - 1$, then G is a complete graph and $\Delta(G) = n - 1$, $\Gamma(G) = \frac{1}{2}n$, therefore $\Delta(G) \neq 2\Gamma(G)$. If n is odd, then $D + 2\overline{\beta} = n - 1$. If $D \leqslant n - 2$, then again $\overline{\beta} \geqslant 1$ and $D + 2\overline{\beta} \geqslant n$, which is a contradiction. Therefore the only possibility is $D = n - 1$ and n odd. Then G is a complete graph with n vertices, $\Delta(G) = n - 1$, $\Gamma(G) = \frac{1}{2}(n - 1)$ and the assertion is true. $\square$

Now for every $n \geqslant 3$ we define a graph H_n and its spanning tree T_n. If n is odd, then the vertices of H_n are u_i, v_i for $i = 1, \ldots, \frac{1}{2}(n - 1)$ and w. For each $i = 1, \ldots, \frac{1}{2}(n - 1)$ the pair $\{u_i, v_i\}$ is non-adjacent. All other pairs of different vertices are adjacent. The tree T_n is the star with the center w which is a spanning tree of H_n.

If n is even, then the vertices of H_n are u_i, v_i for $i = 1, 2, \ldots, \frac{1}{2}n$. For each $i = 2, \ldots, \frac{1}{2}n$ the pair $\{u_i, v_i\}$ is non-adjacent. All other pairs of different vertices are adjacent. The tree T_n is the star with the center u_1 which is a spanning tree of H_n.

For n even we also define another spanning tree T_n^* of H_n. The tree T_n^* has the edges u_1u_i, u_1v_i for $i = 2, \ldots, \frac{1}{2}n$ and the edge v_1v_2.

Theorem 2. *Let G be a graph with $n \geqslant 3$ vertices and with the diameter at most 2. Then $\Delta(G) = \Gamma(G)$ if and only if G is isomorphic to a spanning subgraph of H_n which contains the spanning tree T_n in the case of n odd and the spanning tree T_n or T_n^* in the case of n even.*

P r o o f. Let $\Delta(G) = \Gamma(G)$. According to Lemmas 1 and 2 this is $2n - D - 2 = \lfloor \frac{1}{2}n \rfloor + \overline{\beta}$. If n is even, this implies $D + \overline{\beta} = \frac{3}{2}n - 2$. If $D = n - 1$, then $\overline{\beta} = \frac{1}{2}n - 1$. There exists a set B of $\frac{1}{2}n - 1$ independent edges in $\overline{G}$. Further, there exists a vertex u_1 of degree $n - 1$ in G, i.e. adjacent to all other vertices of G. Evidently it is incident with no edge of B in $\overline{G}$. The other vertex which is incident with no edge of B will be denoted by v_1. The edges of B will be denoted by e_i for $i = 2, \ldots, \frac{1}{2}n$ and the end vertices of each e_i will be denoted by u_i, v_i. Hence u_i, v_i are non-adjacent in G for $i = 2, \ldots, \frac{1}{2}n$ and G is a spanning subgraph of H_n. As v_1 has degree $n - 1$, the tree T_n is a spanning tree of G. If $D = n - 2$, then $\overline{\beta} = \frac{1}{2}n$. There exists a set B of $\frac{1}{2}n$ independent edges in $\overline{G}$. We will denote them by e_i for $i = 1, \ldots, \frac{1}{2}n$ and the end vertices of each e_i will be denoted by u_i, v_i. There exists a vertex of degree $n - 2$; without loss of generality let it be u_1. As G is connected and v_1 is not adjacent to u_1, it is adjacent to some other vertex; without loss of generality let it be adjacent to v_2. We see that G is a spanning subgraph of H_n and T_n^* is its spanning tree. The inequality $D < n-2$ would imply $\overline{\beta} > \frac{1}{2}n$, which is impossible.

Now let n be odd. Then $D + \overline{\beta} = \frac{1}{2}(n-1)$. If $D = n-1$, then $\overline{\beta} = \frac{1}{2}(n-1)$. There exists a set B of $\frac{1}{2}(n-1)$ independent edges in $\overline{G}$. We denote them by e_i for $i = 1, \ldots, \frac{1}{2}(n-1)$ and the end vertices of each e_i will be denoted by u_i, v_i. There exists a vertex of degree $n-1$; it is incident with no edge of B in $\overline{G}$ and thus it is the remaining vertex w. Again G is a spanning subgraph of H_n and T_n is its spanning tree. The inequality $D < n-1$ would imply $\overline{\beta} > \frac{1}{2}(n-1)$, which is impossible.

Now let G be a spanning subgraph of H_n and let T_n be its spanning tree. If n is odd, then $\overline{G}$ contains $\frac{1}{2}(n-1)$ independent edges u_iv_i and thus $\overline{\beta} = \frac{1}{2}(n-1)$; it cannot be greater. Further, T_n contains a vertex w of degree $n-1$ and so does G; we have $D = n-1$. This implies $\Delta(G) = \Gamma(G)$. If n is even, then $\overline{G}$ contains $\frac{1}{2}n - 1$ independent edges u_iv_i for $i = 2, \ldots, \frac{1}{2}n$. As v_1 has degree $n-1$, no edge of G is incident with it and therefore $\frac{1}{2}n$ independent edges in G cannot exist and $\overline{\beta}(G) = \frac{1}{2}n - 1$. The tree T_n contains a vertex v_1 of degree $n-1$. So does G; we have $D = n-1$. This implies $\Delta(G) = \Gamma(G)$.

Finally, let n be even, let G be a spanning subgraph of H_n and suppose that T_n^* is a spanning tree of G, while T_n is not. Then u_1, v_1 are non-adjacent in G. The graph G contains $\frac{1}{2}n$ independent edges u_iv_i for $i = 1, \ldots, \frac{1}{2}n$ and thus $\overline{\beta} = \frac{1}{2}n$. No vertex has degree greater than $n-2$ in G. The tree T_n^* contains a vertex v_1 of degree $n-2$ and so does G; we have $D = n-2$. This implies $\Delta(G) = \Gamma(G)$. $\square$

In [1] the authors suggest the problem to characterize the graphs G for which the ratio between $\Delta(G)$ and $\Gamma(G)$ is equal to a given number α such that $1 \leqslant \alpha \leqslant 2$. We will not solve this problem; we will only state an existence theorem.

By K_n we denote the complete graph with n vertices and by $\overline{K}_n$ its complement, i.e., the graph with n vertices and no edges. The Zykov sum $G_1 \oplus G_2$ of two disjoint graphs G_1, G_2 is the graph obtained by joining each vertex of G_1 with each vertex of G_2 by an edge. A saturated vertex of a graph is a vertex which is adjacent to all the others.

First we prove a lemma.

Lemma 4. *Let n be a positive integer such that $n \geqslant 3$, let b be an integer such that $0 \leqslant b \leqslant \frac{1}{2}(n-1)$. Then there exists a graph G with n vertices, with a saturated vertex and such that $\beta(\overline{G}) = b$.*

Proof. For $b = 0$ this graph is K_n. For $0 < b \leqslant \frac{1}{2}(n-1)$ it is the Zykov sum $K_{n-2b} \oplus \overline{K}_{2b}$ or $K_{n-2b-1} \oplus \overline{K}_{2b+1}$. $\square$

Now we prove a theorem.

Theorem 3. *Let α be a rational number, $1 \leqslant \alpha \leqslant 2$. Then there exists a graph G with a saturated vertex and such that $\Delta(G)/\Gamma(G) = \alpha$.*

P r o o f. As α is rational, it can be expressed as p/q, where p, q are positive integers. From various possibilities of this expression we choose one such that $p \geqslant 2$ and in the case of $\alpha = 1$ we choose $p = q$ to be even. We put $n = p + 1$. In the case of p odd we put $b = q - \frac{1}{2}(p+1)$, in the case of p even we put $b = q - \frac{1}{2}p$. According to Lemma 4 there exists a graph G with n vertices, with a saturated vertex and such that $\beta(\overline{G}) = b$, which implies $\Gamma(G) = \lfloor \frac{1}{2}n \rfloor + b = q$. As G has a saturated vertex, $\Delta(G) = n - 1 = p$. This implies the assertion. □

References

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