A COMBINATORIAL APPROACH TO THE KNOWN PROJECTIVE PLANES OF ORDER NINE

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Abstract. A combinatorial characterization of finite projective planes using strongly canonical forms of incidence matrices is presented. The corresponding constructions are applied to known projective planes of order 9. As a result a new description of the Hughes plane of order nine is obtained.

Keywords: finite projective plane, ternary ring, incidence matrix, system of orthogonal Latin squares, Hall plane of order 9, Hughes plane of order 9

MSC 1991: 51E15

§1. Strongly canonical forms of incidence matrices and of systems of mutually orthogonal Latin squares corresponding to a finite projective plane of order $n = p^r$

Let $\mathbf{A}$ be a finite affine plane of order n. Using the symbols $0, 1, \ldots, n-1$ as coordinates let us represent points of $\mathbf{A}$ as couples (i, j) with

$$i, j \in \mathbf{n} = \{0, 1, ..., n-1\}$$

in such a way that

$$\{\{(i,j) \mid j \in \mathbf{n}\} \mid i \in \mathbf{n}\}; \quad \{\{(i,j) \mid i \in \mathbf{n}\} \mid j \in \mathbf{n}\}$$

are the starting pencils of horizontal or vertical lines, respectively (see Fig. 1). The remaining $n-1$ pencils of parallel lines called cross lines determine $n-1$ mutually orthogonal Latin squares of order n with entries from $\mathbf{n}$.

$$
\begin{array}{c|ccccc|cl}
\multicolumn{1}{c}{} & 0 & 1 & \dots & n-1 & \multicolumn{1}{c}{} & \to & x \\
\cline{2-6}
0 & (0,0) & (1,0) & \dots & (n-1,0) & & \dots & r_1 \\
1 & (0,1) & (1,1) & \dots & (n-1,1) & & \dots & r_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \text{rows (horizontal lines)} \\
n-1 & (0,n-1) & (1,n-1) & \dots & (n-1,n-1) & & \dots & r_n \\
\cline{2-6}
\multicolumn{1}{c}{\downarrow} & \vdots & \vdots & \dots & \vdots & \multicolumn{1}{c}{} & & \\
\multicolumn{1}{c}{y} & c_1 & c_2 & & c_n & \multicolumn{1}{c}{} & & \\
\multicolumn{1}{c}{} & \multicolumn{4}{c}{\text{columns} \quad \text{(vertical lines)}} & \multicolumn{1}{c}{} & &
\end{array}
$$

Fig. 1

Let $n = p^r$ be a power of a prime, and let $0, 1, \dots, p-1$ be elements of the Galois field GF(p). The vectors

$$
\begin{array}{llll}
\underline{0} = (0,0,\dots,0) & \underline{p} = (0,1,0,\dots,0) & \dots & \underline{p^2} = (0,0,1,0,\dots,0) \\
\underline{1} = (1,0,\dots,0) & \underline{p+1} = (1,1,0,\dots,0) & \dots & \underline{p^2+1} = (1,0,1,0,\dots,0) \\
\underline{2} = (2,0,\dots,0) & \vdots & & \vdots \qquad (i+1)\text{-th place} \\
 & & & \qquad\qquad\qquad\qquad \downarrow \\
 & \underline{2p-1} = (p-1,1,0,\dots,0) & \dots & \underline{p^i} = (0,\dots,0,1,0,\dots,0) \\
\vdots \quad \vdots \quad \vdots & \vdots & & \vdots \\
\underline{p-1} = (p-1,0,\dots,0) & \underline{p^2-1} = (p-1,p-1,0,\dots,0) & \dots & \underline{p^r-1} = (p-1,p-1,\dots,p-1)
\end{array}
$$

will be used as new "p-adic" symbols for coordinates instead of the initial symbols $0, 1, \dots, n-1$. The Latin squares can be assumed to be in column standard form having the same first column $(0, 1, \dots, n-1)^T$. Further, suppose that our Latin squares are ordered using the members standing in the first row r_1 and in the second column c_2. More precisely, the j-th square L_j has the entry j in the $(1,0)$-cell for $j \in \{1, 2, \dots, n-1\}$:

$$
L_j = \begin{array}{cccc}
0 & \boxed{\mathrm{j}} & \dots & a^j_{0,n-1} \\
1 & a^j_{11} & \dots & a^j_{1,n-1} \\
2 & a^j_{21} & \dots & a^j_{2,n-1} \\
\vdots & \vdots & & \vdots \\
n-1 & a^j_{n-1,1} & \dots & a^j_{n-1,n-1},
\end{array}
$$

where we already write j instead of $\underline{j}$. Here j can be regarded as the slope of lines of the j-th pencil. Now, the usual coordinatization of $\mathbf{A}$ with help of the associated ternary ring $(\mathbf{n}, T)$ is as follows: $T(u, x, y) = v$ if and only if the entry in the (x, y)-cell of L_u is v.

Let $(\mathbf{n}, T)$ be a ternary ring, and let $+, \cdot$ be the corresponding T-induced binary operations defined on the set $\mathbf{n}$ by

$$x + y = T(1, x, y) \quad \text{for all } x, y \in \mathbf{n},$$
$$u \cdot x = T(u, x, 0) \quad \text{for all } u, x \in \mathbf{n}.$$

Then a ternary ring $(\mathbf{n}, T)$ is called *linear* if and only if $T(u, x, y) = u \cdot x + y$ for all $x, y, u \in \mathbf{n}$.

Moreover, if $(\mathbf{n}, T)$ is linear and $(\mathbf{n}, +)$ is a not necessarily commutative group, then $(\mathbf{n}, T)$ is called a *Cartesian group*. We shall restrict ourselves to Cartesian groups of a power-prime order p^r with commutative addition. Let us remark that Cartesian groups of finite order with non-commutative addition are unknown up to now. Furthermore, if the left or the right distributive law (for multiplication from left, respectively from right over addition) is satisfied, then the Cartesian group becomes a left or right quasifield, respectively. A quasifield with associative multiplication is called a *nearfield*. A left and right (simultaneously) quasifield is called a *semifield*. An associative semifield is of course a field.

We return to a general affine plane $\mathbf{A}$ of power-prime order $n = p^r$, and let $L_1, \ldots, L_{n-1}$ be its Latin squares in column standard ordering, i.e. with the same first column $(0, 1, 2, \ldots, n-1)^T$, with the slope j in the $(1, 0)$-cell of L_j and with the first row $(0, 1, \ldots, n-1)$ in L_1. It is well-known (cf. Theorem 8.4.3, pp. 283–284 of [7], or Theorem 5.9, pp. 123–124 of [1]) that the ternary ring $(\mathbf{n}, T)$ is linear if and only if the set of columns of L_j is the same as the set of columns of L_1, i.e. if the n-tuple of columns of L_j differs only in another ordering from the n-tuple of columns of L_1 for all $j \in \{2, 3, \ldots, n-1\}$. Notice that L_1 is the Cayley table of the induced addition $+$.

In what follows we introduce the incidence matrix of order $N = n^2 + n + 1$ of the projective plane $\mathbf{P} = \overline{\mathbf{A}}$ extending the given affine plane $\mathbf{A}$. The ordering of points and lines of $\mathbf{P}$ will be the same as in Fig. 2 and 3. Moreover, let $1, 2, \ldots, n^2$ be the notation of all of the proper points of $\mathbf{P}$, i.e. all of the points of $\mathbf{A}$; let $(0), (1), \ldots, (n-1), (\infty)$ be the notation of improper points where $0, 1, \ldots, n-1, \infty$ are the corresponding slopes. Similarly let $\langle\infty\rangle$ be the notation of the improper line, let $c_1, c_2, \ldots, c_n$ be the notation of vertical lines with the slope ∞, let $r_1, r_2, \ldots, r_n$ be the notation of horizontal lines with the slope 0 and let $l_0^{(j)}, l_1^{(j)}, \ldots, l_{n-1}^{(j)}$ be the notation of lines with the slope j, for all $j \in 1, 2, \ldots, n-1$.

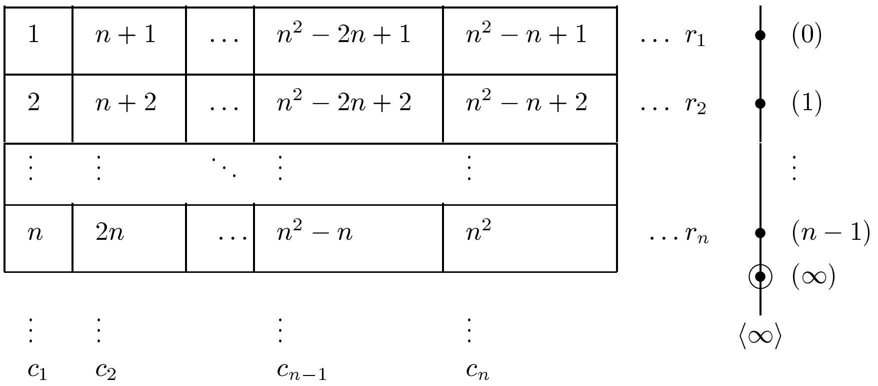

Fig. 2

The incidence matrix in Fig. 3 corresponds to the canonical form of Paige and Wexler (cf. [4] or [7], §8.5). The submatrices $P_{j,k}$, $j,k \in 2, \ldots, n$, are permutation matrices of order n, which means that every row and every column of the matrix contains exactly one unit. Moreover, the incidence matrix cannot contain submatrices of the form $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ and all main diagonal elements of $P_{j,k}$ are necessarily zeros because two distinct points lie simultaneusly on just one line, and two distinct lines intersect at just one point. The submatrix which arises by neglecting the first $n+1$ rows and the first $n+1$ columns will be called the kernel of the incidence matrix which is evidently of order $n^2 = p^{2r}$. We will say that the incidence matrix is of *strongly canonical form*, if every matrix $P_{j,k}$ can be composed by permutation matrices p_{j_1,k_1} of order p^s, where $s < r$. The corresponding $(n-1)$-tuple of Latin squares of order n will be said to be *strongly canonical*, too.

The submatrix of order $n^2 - n$ on the last $n^2 - n$ rows and last $n^2 - n$ columns of the matrix from Fig. 3 will be denoted in the sequel as its reduced kernel.

If $\mathbf{P}_1 = (P, L, I)$, $\mathbf{P}_2 = (P', L', I')$ are projective planes as triples consisting of point sets, line sets, and incidence relations, then a *duality* of $\mathbf{P}_1$ onto $\mathbf{P}_2$ is a couple of bijective mappings $\varphi\colon P \to L'$, $\psi\colon L \to P'$, such that xIy whenever $\psi(y)I'\varphi(x)$ for all $x \in P$ and all $y \in L$. Duality of the projective plane $\mathbf{P}$ onto itself is called an *autoduality* of $\mathbf{P}$. If $\mathbf{P} = (P, L, I)$ is a projective plane, then $\mathbf{P}^* = (L, P, I^*)$ such that $xIy \Leftrightarrow yI^*x$ for all $x \in P$ and all $y \in L$ is the dual plane of $\mathbf{P}$. If M is an incidence matrix of $\mathbf{P}$ with regard to arbitrary orderings of points and lines, then M^T is an incidence matrix of $\mathbf{P}^*$, where lines operate as new points and points as new lines by preserved orderings. If M is an incidence matrix of $\mathbf{P}$, then there exists an *isotopy* of M onto an incidence matrix M^T if and only if $\mathbf{P}$ is autodual. Here,

points lines	$(\infty)\,(0)\,(1)\ldots(n-1)$	$1\ 2\ \ldots\ n$	$n+1\ \ n+2\ldots\ \ 2n$	$\ldots$	$n^2-n+1\ \ n^2-n+2\ldots\ \ n^2$
$\langle\infty\rangle$	$1\ \ 1\ \ 1\ \ \ldots\ 1$				
c_1	1	$1\ 1\ \ldots\ 1$			
c_2	1		$1\ \ \ 1\ \ \ \ldots\ 1$		
$\vdots$	$\vdots$			$\vdots$	
c_n	1				$1\ \ \ 1\ \ \ \ldots\ 1$
r_1	1	1	1		1
r_2	1	1	1		1
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	$\ddots$
r_n	1	1	1		1
$l_0^{(1)}$	1	1			
$l_1^{(1)}$	1	1			
$\vdots$	$\vdots$	$\ddots$	$P_{2,2}$		$P_{2,n}$
$l_{n-1}^{(1)}$	1	1			
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$l_0^{(n-1)}$	1	1			
$l_1^{(n-1)}$	1	1			
$\vdots$	$\vdots$	$\ddots$	$P_{n,2}$		$P_{n,n}$
$l_{n-1}^{(n-1)}$	1	1			

Fig. 3

the isotopy is meant in the sense of Footnote (1) on p. 168 of [7]. The preceding assertion holds especially for a strongly canonical incidence matrix M of $\mathbf{P}$.

§2. A survey of the known projective planes of order nine

In the sequel, projective planes mentioned above will be described using strongly canonical systems of mutually orthogonal Latin squares and strongly canonical incidence matrices.

Let $n = 3^2 = 9$. The labelling set will be $S = \{0, 1, \ldots, 8\}$. Further, we put $S_0 = \{0, 1, 2\}$ and use the same addition and multiplication in S_0 as in $GF(3)$.

Let

$$i = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad j = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \quad k = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

be the permutation matrices of order 3 which form a cyclic group under matrix multiplication. Obviously, $i^T = i$, $j^T = k$, $k^T = j$. Considering the permutation matrices of order 9 defined by

$$
\begin{array}{lll}
I_0 = \begin{pmatrix} i & & \\ & i & \\ & & i \end{pmatrix} & J_0 = \begin{pmatrix} j & & \\ & j & \\ & & j \end{pmatrix} & K_0 = \begin{pmatrix} k & & \\ & k & \\ & & k \end{pmatrix} \\
I_1 = \begin{pmatrix} & & i \\ i & & \\ & i & \end{pmatrix} & J_1 = \begin{pmatrix} & & j \\ j & & \\ & j & \end{pmatrix} & K_1 = \begin{pmatrix} & & k \\ k & & \\ & k & \end{pmatrix} \\
I_2 = \begin{pmatrix} & i & \\ & & i \\ i & & \end{pmatrix} & J_2 = \begin{pmatrix} & j & \\ & & j \\ j & & \end{pmatrix} & K_2 = \begin{pmatrix} & k & \\ & & k \\ k & & \end{pmatrix}
\end{array}
$$

we easily obtain the following equalities:

$$J_0^T = K_0, \quad I_1^T = I_2, \quad J_1^T = K_2, \quad K_1^T = J_2$$

Now, let us investigate the Latin squares L_1 and L_2 defined by

$$
L_1 = \begin{array}{c}
012\ 345\ 678 \\
120\ 453\ 786 \\
201\ 534\ 867 \\
345\ 678\ 012 \\
453\ 786\ 120 \\
534\ 867\ 201 \\
678\ 012\ 345 \\
786\ 120\ 453 \\
867\ 201\ 534
\end{array}
\qquad
L_2 = \begin{array}{c}
021\ 687\ 354 \\
102\ 768\ 435 \\
210\ 876\ 543 \\
354\ 021\ 687 \\
435\ 102\ 768 \\
543\ 210\ 876 \\
687\ 354\ 021 \\
768\ 435\ 102 \\
876\ 543\ 210
\end{array}
$$

Fig. 4

and use them as starting members of strongly canonical systems of Latin squares for all known projective planes of order 9. As L_2 has zero diagonal, the diagonals of all remaining squares of strongly canonical systems of these projective planes are permutations of S with just one fixed label, namely 0, above on the left. Herein L_1 is the addition table of the elementary 3-group of order 9.

1. The Desarguesian plane of order 9 is built up over the field $GF(9)$. There are three possibilities of strongly canonical systems of Latin squares for this plane. One of them is the system ${}^1\mathbf{L}^a = \{L_1, L_2, {}^1L_3^a, \ldots, {}^1L_8^a\}$, where

$$
{}^1L_3^a = \begin{array}{l}
0\boxed{\mathbf{3}}6\ 258\ 147\\
1\mathbf{4}7\ 036\ 258\\
25\mathbf{8}\ 147\ 036\\
360\ \mathbf{5}82\ 471\\
471\ 3\mathbf{6}0\ 582\\
582\ 47\mathbf{1}\ 360\\
603\ 825\ \mathbf{7}14\\
714\ 603\ 8\mathbf{2}5\\
825\ 714\ 60\mathbf{3}
\end{array}
\quad
{}^1L_4^a = \begin{array}{l}
0\boxed{\mathbf{4}}8\ 561\ 723\\
1\mathbf{5}6\ 372\ 804\\
23\mathbf{7}\ 480\ 615\\
372\ \mathbf{8}04\ 156\\
480\ 6\mathbf{1}5\ 237\\
561\ 72\mathbf{3}\ 048\\
615\ 237\ \mathbf{4}80\\
723\ 048\ 5\mathbf{6}1\\
804\ 156\ 37\mathbf{2}
\end{array}
\quad
{}^1L_5^a = \begin{array}{l}
0\boxed{\mathbf{5}}7\ 813\ 462\\
1\mathbf{3}8\ 624\ 570\\
24\mathbf{6}\ 705\ 381\\
381\ \mathbf{2}46\ 705\\
462\ 0\mathbf{5}7\ 813\\
570\ 13\mathbf{8}\ 624\\
624\ 570\ \mathbf{1}38\\
705\ 381\ 2\mathbf{4}6\\
813\ 462\ 05\mathbf{7}
\end{array}
$$

$$
{}^1L_6^a \ldots \quad 0\boxed{6}3\ 174\ 285 \qquad {}^1L_7^a \ldots \quad 0\boxed{7}5\ 426\ 831 \qquad {}^1L_8^a \ldots \quad 0\boxed{8}4\ 732\ 516
$$

Fig. 5

The first row of L_j^a, $j \in \{3, 4, \ldots, 8\}$ coincides with the j-th row of the multiplication table a of $GF(9)$:

a	b	c
12 345 678	12 345 678	12 345 678
21 687 354	21 687 354	21 687 354
36 258 147	36 714 582	36 471 825
48 561 723	48 156 237	48 723 561
57 813 462	57 462 813	57 138 246
63 174 285	63 528 741	63 852 417
75 426 831	75 831 426	75 264 183
84 732 516	84 273 165	84 516 732

Fig. 6

The columns of all Latin squares under consideration coincide with columns of L_1 even though they appear in a different order. So it is sufficient to register only the first rows of the squares. Notice that the diagonals of the squares ${}^1L_3^a$, ${}^1L_4^a$, or ${}^1L_5^a$ coincide with the first row of the squares ${}^1L_4^a$, ${}^1L_5^a$, or ${}^1L_3^a$, respectively. Similarly, the diagonals of the squares ${}^1L_6^a$, ${}^1L_7^a$, or ${}^1L_8^a$ coincide with the first rows of the squares ${}^1L_7^a$, ${}^1L_8^a$, or ${}^1L_6^a$, respectively.

The reduced kernel of the corresponding incidence matrix written in block form is ${}^1\widehat{M}^a$ (Fig. 7).

$$
{}^1\widehat{M}^a = \begin{array}{cccccccc}
J_0 & K_0 & I_1 & J_1 & K_1 & I_2 & J_2 & K_2 \\
K_0 & J_0 & I_2 & K_2 & J_2 & I_1 & K_1 & J_1 \\
I_1 & I_2 & K_0 & K_1 & K_2 & J_0 & J_1 & J_2 \\
J_1 & K_2 & K_1 & I_2 & J_0 & J_2 & K_0 & I_1 \\
K_1 & J_2 & K_2 & J_0 & I_1 & J_1 & I_2 & K_0 \\
I_2 & I_1 & J_0 & J_2 & J_1 & K_0 & K_2 & K_1 \\
J_2 & K_1 & J_1 & K_0 & I_2 & K_2 & I_1 & J_0 \\
K_2 & J_1 & J_2 & I_1 & K_0 & K_1 & J_0 & I_2
\end{array}
$$

Fig. 7

As the incidence matrix of the Desarguesian plane is symmetric, the plane is necessarily autodual. Strongly canonical systems of Latin squares

$$
{}^1\mathbf{L}^b = \{L_1, L_2, L_3^b, \dots, L_8^b\} \quad \text{and} \quad {}^1\mathbf{L}^c = \{L_1, L_2, L_3^c, \dots, L_8^c\}
$$

with similar properties, can be obtained from Tables b and c of Fig. 6 (see [9], pp. 687–8).

2. The Hall plane of order 9 is closely related to quasifields R, S, T of [5], Appendix II, pp. 273–274; and [9], pp. 689–90.

We will present here the multiplication tables of these right quasifields (the first one is a nearfield)

.	.	.
12 345 678	12 345 678	12 345 678
21 687 354	21 687 354	21 687 354
36 274 185	36 418 527	36 751 842
48 526 731	48 751 263	48 163 527
57 832 416	57 163 842	57 418 263
63 158 247	63 824 715	63 572 481
75 461 823	75 236 481	75 824 136
84 713 562	84 572 136	84 236 715
System R	*System S*	*System T*

Fig. 8

As is well-known, the projective planes over these quasifields are isomorphic. This plane is the Hall plane of order 9. The multiplication table of R expresses quaternion group (where $a^4 = 1$, $a^2 = 2$, $ab = -ba$ for all a, b different from 1, 2). All three quasifields lead to strongly canonical systems of Latin squares. We restrict ourselves to the first quasifield. The corresponding strongly canonical system of Latin squares is ${}^2\mathbf{L}^a = \{L_1, L_2, {}^2L_3^a, {}^2L_4^a, \ldots, {}^2L_8^a\}$, where

$$
{}^2L_3^a = \begin{array}{l}
0\boxed{3}6\ 258\ 147\\
1\mathbf{4}7\ 036\ 258\\
25\mathbf{8}\ 147\ 036\\
360\ \mathbf{5}82\ 471\\
471\ 3\mathbf{6}0\ 582\\
582\ 47\mathbf{1}\ 360\\
603\ 825\ \mathbf{7}14\\
714\ 603\ 8\mathbf{2}5\\
825\ 714\ 60\mathbf{3}
\end{array}
\quad
{}^2L_4^a = \begin{array}{l}
0\boxed{4}8\ 723\ 561\\
1\mathbf{5}6\ 804\ 372\\
23\mathbf{7}\ 615\ 480\\
372\ \mathbf{1}56\ 804\\
480\ 2\mathbf{3}7\ 615\\
561\ 04\mathbf{8}\ 723\\
615\ 480\ \mathbf{2}37\\
723\ 561\ 0\mathbf{4}8\\
804\ 372\ 15\mathbf{6}
\end{array}
\quad
{}^2L_5^a = \begin{array}{l}
0\boxed{5}7\ 462\ 813\\
1\mathbf{3}8\ 570\ 624\\
24\mathbf{6}\ 381\ 705\\
381\ \mathbf{7}05\ 246\\
462\ 8\mathbf{1}3\ 057\\
570\ 62\mathbf{4}\ 138\\
624\ 138\ \mathbf{5}70\\
705\ 246\ 3\mathbf{8}1\\
813\ 057\ 46\mathbf{2}
\end{array}
$$

$$
{}^2L_6^a \ldots \quad 0\boxed{6}3\ 174\ 285 \qquad {}^2L_7^a \ldots \quad 0\boxed{7}5\ 831\ 426 \qquad {}^2L_8^a \ldots \quad 0\boxed{8}4\ 516\ 732
$$

Fig. 9

The first row of ${}^2L_j^a$, $j \in \{3, 4, \ldots, 8\}$, coincides with the j-th column of the multiplication table of R, i.e. with the j-th row of the system R^T. Notice that the diagonals of ${}^2L_3^a$, ${}^2L_4^a$ and ${}^2L_5^a$ coincide with the fourth column of the system T, the fifth column of T, and the third column of T, respectively. The triple (${}^2L_6^a$, ${}^2L_7^a$, ${}^2L_8^a$) has the same property.

As in the preceding considerations, the squares ${}^2L_3^a, \ldots, {}^2L_8^a$ have columns which form the same column set as L_1, only the orders in which the columns of L_1 occur in the subsequent squares are different. This follows again from the linearity of the ternary ring which is the left nearfield R^T in the sense of Hughes. Starting with the left quasifield T^T, or S^T, we would analogously get a strongly canonical system of Latin squares ${}^2\mathbf{L}^b$, or ${}^2\mathbf{L}^c$, respectively. From the diagonals of Latin squares of the system one deduces the rows of the multiplication table of the quasifield S^T, or R^T, respectively.

The reduced kernel of the corresponding strongly canonical incidence matrix of the Hall plane of order 9 in block notation is ${}^2\widehat{M}^a$, where

$$
{}^2\widehat{M}^a = \begin{matrix} J_0 & K_0 & I_1 & J_1 & K_1 & I_2 & J_2 & K_2 \\ K_0 & J_0 & I_2 & K_2 & J_2 & I_1 & K_1 & J_1 \\ I_1 & I_2 & K_0 & K_1 & K_2 & J_0 & J_1 & J_2 \\ J_1 & K_2 & J_2 & K_0 & I_1 & K_1 & I_2 & J_0 \\ K_1 & J_2 & J_1 & I_2 & K_0 & K_2 & J_0 & I_1 \\ I_2 & I_1 & J_0 & J_2 & J_1 & K_0 & K_2 & K_1 \\ J_2 & K_1 & K_2 & I_1 & J_0 & J_1 & K_0 & I_2 \\ K_2 & J_1 & K_1 & J_0 & I_2 & J_2 & I_1 & K_0 \end{matrix}
\qquad
{}^2\widehat{M}^{aT} = \begin{matrix} K_0 & J_0 & I_2 & K_2 & J_2 & I_1 & K_1 & J_1 \\ J_0 & K_0 & I_1 & J_1 & K_1 & I_2 & J_2 & K_2 \\ I_2 & I_1 & J_0 & K_1 & K_2 & K_0 & J_1 & J_2 \\ K_2 & J_1 & J_2 & J_0 & I_1 & K_1 & I_2 & K_0 \\ J_2 & K_1 & J_1 & I_2 & J_0 & K_2 & K_0 & I_1 \\ I_1 & I_2 & K_0 & J_2 & J_1 & J_0 & K_2 & K_1 \\ K_1 & J_2 & K_2 & I_1 & K_0 & J_1 & J_0 & I_2 \\ J_1 & K_2 & K_1 & K_0 & I_2 & J_2 & I_1 & J_0 \end{matrix}
$$

Fig. 10

In ${}^2\widehat{M}^{aT}$ the successive changing of elements of rows occurs in accordance with the quaternion group R. The matrix ${}^2\widehat{M}^{aT}$ is not isotopic to ${}^2\widehat{M}^a$.

3. By reordering of rows and by subsequent reordering of columns of the block matrix ${}^2\widehat{M}^{aT}$ we obtain a new block matrix ${}^3\widehat{M}^a$, which we will call the reduced kernel of the dual Hall plane of order 9. The block matrix ${}^3\widehat{M}^a$ is

$$
{}^3\widehat{M}^a = \begin{matrix} J_0 & K_0 & I_1 & J_1 & K_1 & I_2 & J_2 & K_2 \\ K_0 & J_0 & I_2 & K_2 & J_2 & I_1 & K_1 & J_1 \\ I_1 & I_2 & K_0 & J_2 & J_1 & J_0 & K_2 & K_1 \\ J_1 & K_2 & K_1 & K_0 & I_2 & J_2 & I_1 & J_0 \\ K_1 & J_2 & K_2 & I_1 & K_0 & J_1 & J_0 & I_2 \\ I_2 & I_1 & J_0 & K_1 & K_2 & K_0 & J_1 & J_2 \\ J_2 & K_1 & J_1 & I_2 & J_0 & K_2 & K_0 & I_1 \\ K_2 & J_1 & J_2 & J_0 & I_1 & K_1 & I_2 & K_0 \end{matrix}
$$

Fig. 11

The corresponding strongly canonical system of Latin squares of the dual Hall plane is

$$
{}^3\mathbf{L}^a = \{L_1, L_2, {}^3L_3^a, {}^3L_4^a, \ldots, {}^3L_8^a\},
$$

where

$$
{}^3L_3^a = \begin{array}{l}
0\boxed{\mathbf{3}}6\ 274\ 185\\
1\mathbf{4}7\ 085\ 263\\
25\mathbf{8}\ 163\ 074\\
360\ \mathbf{5}17\ 428\\
471\ 3\mathbf{2}8\ 506\\
582\ 40\mathbf{6}\ 317\\
603\ 841\ \mathbf{7}52\\
714\ 652\ 8\mathbf{3}0\\
025\ 730\ 64\mathbf{1}
\end{array}
\qquad
{}^3L_4^a = \begin{array}{l}
0\boxed{\mathbf{4}}8\ 526\ 731\\
1\mathbf{5}6\ 307\ 842\\
23\mathbf{7}\ 418\ 650\\
372\ \mathbf{8}50\ 164\\
480\ 6\mathbf{3}1\ 275\\
561\ 74\mathbf{2}\ 083\\
615\ 283\ \mathbf{4}07\\
723\ 064\ 5\mathbf{1}8\\
804\ 175\ 32\mathbf{6}
\end{array}
\qquad
{}^3L_5^a = \begin{array}{l}
0\boxed{\mathbf{5}}7\ 832\ 416\\
1\mathbf{3}8\ 640\ 527\\
24\mathbf{6}\ 751\ 308\\
381\ \mathbf{2}65\ 740\\
462\ 0\mathbf{7}3\ 851\\
570\ 18\mathbf{4}\ 632\\
624\ 508\ \mathbf{1}73\\
705\ 316\ 2\mathbf{8}4\\
813\ 427\ 06\mathbf{5}
\end{array}
$$

$$
{}^3L_6^a \ldots\quad 0\boxed{6}3\ 158\ 247 \qquad {}^3L_7^a \ldots\quad 0\boxed{7}5\ 462\ 823 \qquad {}^3L_8^a \ldots\quad 0\boxed{8}4\ 713\ 562
$$

Fig. 12

The columns of the squares ${}^3L_j^a$, $j \in \{3, 4, \ldots 8\}$ must be taken from L_1 and their labelling is given by their "leading" elements in the first row. The first row of ${}^3L_j^a$ coincides with the j-th row of the system R. The triples $({}^3L_3^a, {}^3L_4^a, {}^3L_5^a)$ and $({}^3L_6^a, {}^3L_7^a, {}^3L_8^a)$ have the same property as the triples in the Desarguesian case.

4. We come to the Hughes plane of order 9. We shall start from the Desarguesian plane of order 3 understood as the plane over $GF(3) = (S_0, +, \cdot)$ with $S_0 = \{0, 1, 2, \}$. This plane can be described also as a perfect difference set, for example $\{0, 1, 3, 9\}$ (mod 13) (cf. [3], pp. 52–54). We denote it by π_0 and its points by $A_0, A_1, \ldots, A_{12}$.

Further, we take the right nearfield R of order 9 with elements $0, 1, \ldots, 8$ and use homogeneous coordinates (x, y, z) over R (with factor of homogeneity from the right) for points of the projective plane π containing π_0.

We shall proclaim the set $\{A_0, A_1, A_3, A_9, B_0, C_0, D_0, E_0, F_0, G_0\}$ with coordinates according to Fig. 13 to be the improper line of the plane π.

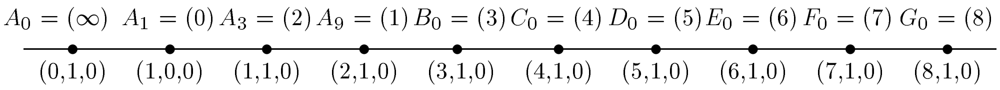

Fig. 13

We shall use the Singer matrix (cf. [3], pp. 293–295)

$$
M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
$$

over $GF(3)$ as the matrix of a collineation (denoted in the following also by M) of π_0. The period of this collineation is 13 and the orbit of A_0 under the collineation subgroup $\langle M\rangle$ generated by M is formed by the points $A_1 = MA_o = M(0,1,0)^T$, $A_2 = M^2(0,1,0)^T, \dots, A_{12} = M^{12}(0,1,0)^T$ with respect to π_0. However we can extend the action of $\langle M\rangle$ to the remaining points $B_0, C_0, \dots, G_0$ of the ideal line so that we get $6\cdot 13 = 78$ points $B_j = M^j B_0, \dots, G_j = M^j G_0$, $j \in \{0,1,2,\dots,12\}$. We obtain the following remarkable dislocation of 81 proper points

	0	1	2	3	4	5	6	7	8	$\to x$
0	A_2	A_4	A_{10}	E_1	G_1	F_1	B_1	D_1	C_1	
1	A_8	A_5	A_6	B_5	C_5	D_5	E_5	F_5	G_5	
2	A_{12}	A_7	A_{11}	F_{11}	E_{11}	G_{11}	C_{11}	B_{11}	D_{11}	
3	C_{12}	C_4	D_{10}	B_2	E_3	E_7	E_6	D_8	B_9	$\dots\ r_3$
4	D_{12}	D_4	B_{10}	G_7	C_2	G_3	B_8	C_9	G_6	$\dots\ r_4$
5	B_{12}	B_4	C_{10}	F_3	F_7	D_2	D_9	F_6	C_8	$\dots\ r_5$
6	F_{12}	F_4	G_{10}	B_6	G_8	E_9	E_2	B_3	B_7	$\dots\ r_6$
7	G_{12}	G_4	E_{10}	E_8	F_9	D_6	D_7	F_2	D_3	$\dots\ r_7$
8	E_{12}	E_4	F_{10}	G_9	C_6	F_8	C_3	C_7	G_2	$\dots\ r_8$
$\downarrow$				$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
y				c_3	c_4	c_5	c_6	c_7	c_8	

Fig. 14

The left above array of this scheme expresses the affine subplane of order 3. The ideal points of this subplane are A_0, A_1, A_3, A_9. We will speak about *primary points* $A_0, A_1, \dots, A_{12}$, whereas the points $B_j, C_j, \dots, G_j$, $j \in \{0,1,2,\dots,12\}$ of the rest will be called *secondary points.* Any two distinct primary points are joined by a unique line also called *primary.* Primary lines can be understood either as lines of π_0 or as extended lines with points A_i, A_{i+1}, A_{i+3}, A_{i+9}, B_i, C_i, D_i, E_i, F_i, G_i for $i \in \{0,1,2,\dots,12\}$ taken modulo 13. Further, we form point sets called *secondary lines*: firstly the *vertical* ones:

$$\begin{array}{ll} c_3 = A_0B_2B_5B_6E_1E_8F_3F_{11}G_7G_9 & c_6 = A_0E_2E_5E_6B_1B_8C_3C_{11}D_7D_9 \\ c_4 = A_0C_2C_5C_6G_1G_8E_3E_{11}F_7F_9 & c_7 = A_0F_2F_5F_6D_1D_8B_3B_{11}C_7C_9 \\ c_5 = A_0D_2D_5D_6F_1F_8G_3G_{11}E_7E_9 & c_8 = A_0G_2G_5G_6C_1C_8D_3D_{11}B_7B_9 \end{array} \tag{4.1}$$

secondly the *horizontal* ones: $r_3, r_4, \ldots, r_8$ with the ideal point A_1, where one obtains r_3 from c_6 and r_6 from c_3 by adding 1 to the indices of all points and similarly for the couples r_4, c_8; r_8, c_4 and r_5, c_7; r_7, c_5, and thirdly the *cross* ones: from (4.2) for $i \in \{2, 3, \ldots, 12\}$.

$$
\begin{array}{l}
A_i\ B_{2+i}\ B_{5+i}\ B_{6+i}\ E_{1+i}\ E_{8+i}\ F_{3+i}\ F_{11+i}\ G_{7+i}\ G_{9+i} \\
A_i\ C_{2+i}\ C_{5+i}\ C_{6+i}\ G_{1+i}\ G_{8+i}\ E_{3+i}\ E_{11+i}\ F_{7+i}\ F_{9+i} \\
A_i\ D_{2+i}\ D_{5+i}\ D_{6+i}\ F_{1+i}\ F_{8+i}\ G_{3+i}\ G_{11+i}\ E_{7+i}\ E_{9+i} \\
A_i\ E_{2+i}\ E_{5+i}\ E_{6+i}\ B_{1+i}\ B_{8+i}\ C_{3+i}\ C_{11+i}\ D_{7+i}\ D_{9+i} \\
A_i\ F_{2+i}\ F_{5+i}\ F_{6+i}\ D_{1+i}\ D_{8+i}\ B_{3+i}\ B_{11+i}\ C_{7+i}\ C_{9+i} \\
A_i\ G_{2+i}\ G_{5+i}\ G_{6+i}\ C_{1+i}\ C_{8+i}\ D_{3+i}\ D_{11+i}\ B_{7+i}\ B_{9+i}
\end{array}
\tag{4.2}
$$

There exist just $13 \cdot 6 = 78$ secondary lines and together with 13 primary lines they form a complete line set of a projective plane π called the *Hughes plane* (and known already in 1907 to Veblen and Wedderburn, cf. [8], pp. 383–4). We shall present here a strongly canonical system of Latin squares of π. The first two are L_1 and L_2 again (Fig. 4) whereas the remaining ones must be written in detail:

$$
{}^4L_3^a = \begin{array}{l}
0\boxed{\mathbf{3}}6\ 258\ 147 \\
1\mathbf{4}7\ 036\ 258 \\
25\mathbf{8}\ 147\ 036 \\[4pt]
360\ \mathbf{7}14\ 825 \\
471\ 8\mathbf{2}5\ 603 \\
582\ 60\mathbf{3}\ 714 \\[4pt]
603\ 471\ \mathbf{5}82 \\
714\ 582\ 3\mathbf{6}0 \\
825\ 360\ 47\mathbf{1}
\end{array}
\qquad
{}^4L_4^a = \begin{array}{l}
0\boxed{\mathbf{4}}8\ 723\ 561 \\
1\mathbf{5}6\ 804\ 372 \\
23\mathbf{7}\ 615\ 480 \\[4pt]
372\ \mathbf{4}80\ 156 \\
480\ 5\mathbf{6}1\ 237 \\
561\ 37\mathbf{2}\ 048 \\[4pt]
615\ 237\ \mathbf{8}04 \\
723\ 048\ 6\mathbf{1}5 \\
804\ 156\ 72\mathbf{3}
\end{array}
\qquad
{}^4L_5^a = \begin{array}{l}
0\boxed{\mathbf{5}}7\ 462\ 813 \\
1\mathbf{3}8\ 570\ 624 \\
24\mathbf{6}\ 381\ 705 \\[4pt]
381\ \mathbf{2}46\ 570 \\
462\ 0\mathbf{5}7\ 381 \\
570\ 13\mathbf{8}\ 462 \\[4pt]
624\ 705\ \mathbf{1}38 \\
705\ 813\ 2\mathbf{4}6 \\
813\ 624\ 05\mathbf{7}
\end{array}
$$

$$
{}^4L_6^a = \begin{array}{l}
0\boxed{\mathbf{6}}3\ 174\ 285 \\
1\mathbf{7}4\ 285\ 063 \\
28\mathbf{5}\ 063\ 174 \\[4pt]
306\ \mathbf{8}52\ 741 \\
417\ 6\mathbf{3}0\ 852 \\
528\ 74\mathbf{1}\ 630 \\[4pt]
630\ 528\ \mathbf{4}17 \\
741\ 306\ 5\mathbf{2}8 \\
852\ 417\ 30\mathbf{6}
\end{array}
\qquad
{}^4L_7^a = \begin{array}{l}
0\boxed{\mathbf{7}}5\ 831\ 426 \\
1\mathbf{8}3\ 642\ 507 \\
26\mathbf{4}\ 750\ 318 \\[4pt]
318\ \mathbf{5}07\ 264 \\
426\ 3\mathbf{1}8\ 075 \\
507\ 42\mathbf{6}\ 183 \\[4pt]
642\ 183\ \mathbf{7}50 \\
750\ 264\ 8\mathbf{3}1 \\
831\ 075\ 64\mathbf{2}
\end{array}
\qquad
{}^4L_8^a = \begin{array}{l}
0\boxed{\mathbf{8}}4\ 516\ 732 \\
1\mathbf{6}5\ 327\ 840 \\
27\mathbf{3}\ 408\ 651 \\[4pt]
327\ \mathbf{1}65\ 408 \\
408\ 2\mathbf{7}3\ 516 \\
516\ 08\mathbf{4}\ 327 \\[4pt]
651\ 840\ \mathbf{2}73 \\
732\ 651\ 0\mathbf{8}4 \\
840\ 732\ 16\mathbf{5}
\end{array}
$$

Fig. 15

The columns of the multiplication table of R enter as the first rows of Latin squares ${}^4L_3^a, {}^4L_4^a, \ldots, {}^4L_8^a$. In the additive group $(S, +, 0)$, where $S = \{0, 1, 2, 3, \ldots, 8\}$ and the addition $+$ is defined by L_1, there are subgroups $(\{0, 1, 2\}, +)$, $(\{0, 3, 6\}, +)$, $(\{0, 4, 8\}, +)$, $(\{0, 5, 7\}, +)$. It is obvious that the Latin squares with nonzero slopes of the same subgroup have up to the order the same columns. The cosets of $(S, +)$ modulo $(S_0, +)$ are $S_0 = \{0, 1, 2\}$, $S_1 = \{3, 4, 5\}$, $S_2 = \{6, 7, 8\}$ and the Latin squares belonging to the slopes of S_1 and/or of S_2 have the following properties:

a) The diagonal of the first square coincides with the first row of the second square, the diagonal of the second square coincides with the first row of the third square and finally, the diagonal of the third square coincides with the first row of the first square again.

b) Every column of an arbitrary square of the system ${}^4\mathbf{L}^a = \{L_1, L_2, {}^4L_3^a, \ldots, {}^4L_8^a\}$ can be divided into three parts such that in each of them there are even permutations of the same coset. Thus it is possible to investigate only Latin 3×9-rectangles formed by the first, fourth and seventh row of each of the squares. This means that the corresponding ternary ring of π is "piecewise linear" (it is well-known that the ternary ring of the Hughes plane π cannot be linear, cf. [1], pp. 199–200). So,the eight Latin squares of the system ${}^4\mathbf{L}^a$ can be divided into four couples such that every couple differs only in the ordering of columns (the set of columns is the same for both squares of the couple) and this ordering is prescribed by the first row of any square.

A modification of a strongly canonical system of Latin squares of the Hughes plane π is presented in [2], p. 293. The squares are normalized with respect to rows, but they are not ordered with respect to their slopes.

The reduced kernel of the corresponding strongly canonical incidence matrix of π is

$$
{}^4\widehat{M}^a = \begin{array}{cccccccc}
J_0 & K_0 & I_1 & J_1 & K_1 & I_2 & J_2 & K_2 \\
K_0 & J_0 & I_2 & K_2 & J_2 & I_1 & K_1 & J_1 \\
I_1 & I_2 & a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\
J_1 & K_2 & a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\
K_1 & J_2 & a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\
I_2 & I_1 & d_1 & f_1 & e_1 & a_1 & c_1 & b_1 \\
J_2 & K_1 & d_3 & f_3 & e_3 & a_3 & c_3 & b_3 \\
K_2 & J_1 & d_2 & f_2 & e_2 & a_2 & c_2 & b_2
\end{array}
$$

Fig. 16

where $J_0, K_0, I_1, \ldots, K_2$ are matrices introduced in Section 1, whereas further 18 permutation matrices of order 9 are as follows

(4.3)
$$\begin{cases}
a_1 = \begin{pmatrix} k & & \\ & & j \\ & j & \end{pmatrix}, \; b_1 = \begin{pmatrix} & j & \\ k & & \\ & & j \end{pmatrix}, \; c_1 = \begin{pmatrix} & & j \\ & j & \\ k & & \end{pmatrix}, \; d_1 = \begin{pmatrix} j & & \\ & & k \\ & k & \end{pmatrix}, \; e_1 = \begin{pmatrix} & k & \\ j & & \\ & & k \end{pmatrix}, \; f_1 = \begin{pmatrix} & & k \\ & k & \\ j & & \end{pmatrix}, \\
a_2 = \begin{pmatrix} & & k \\ & j & \\ j & & \end{pmatrix}, \; b_2 = \begin{pmatrix} k & & \\ & & i \\ & k & \end{pmatrix}, \; c_2 = \begin{pmatrix} & i & \\ i & & \\ & & j \end{pmatrix}, \; d_2 = \begin{pmatrix} & & j \\ k & & \\ & k & \end{pmatrix}, \; e_2 = \begin{pmatrix} & & i \\ & k & \\ i & & \end{pmatrix}, \; f_2 = \begin{pmatrix} j & & \\ & & j \\ & i & \end{pmatrix}, \\
a_3 = \begin{pmatrix} & k & \\ j & & \\ & & j \end{pmatrix}, \; b_3 = \begin{pmatrix} & & i \\ & j & \\ i & & \end{pmatrix}, \; c_3 = \begin{pmatrix} k & & \\ & & k \\ & i & \end{pmatrix}, \; d_3 = \begin{pmatrix} & & j \\ & k & \\ k & & \end{pmatrix}, \; e_3 = \begin{pmatrix} j & & \\ & & i \\ & j & \end{pmatrix}, \; f_3 = \begin{pmatrix} & i & \\ i & & \\ & & k \end{pmatrix}.
\end{cases}$$

As $j^T = k$ and $k^T = j$, it is easily seen that

$$\begin{array}{lll}
d_1 = a_1^T & e_1 = a_3^T & f_1 = a_2^T \\
d_2 = b_1^T & e_2 = b_3^T & f_2 = b_2^T \\
d_3 = c_1^T & e_3 = c_3^T & f_3 = c_2^T
\end{array}$$

From these relations we reconstruct the matrix ${}^4\widehat{M}^{aT}$:

$${}^4\widehat{M}^{aT} = \begin{array}{cccccccc}
K_0 & J_0 & I_2 & K_2 & J_2 & I_1 & K_1 & J_1 \\
J_0 & K_0 & I_1 & J_1 & K_1 & I_2 & J_2 & K_2 \\
I_2 & I_1 & d_1 & f_1 & e_1 & a_1 & c_1 & b_1 \\
K_2 & J_1 & d_2 & f_2 & e_2 & a_2 & c_2 & b_2 \\
J_2 & K_1 & d_3 & f_3 & e_3 & a_3 & c_3 & b_3 \\
I_1 & I_2 & a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\
K_1 & J_2 & a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\
J_1 & K_2 & a_2 & b_2 & c_2 & d_2 & e_2 & f_2
\end{array}$$

Fig. 17

which is isotopic to ${}^4\widehat{M}^a$ (as is seen by interchanging the rows $1 \leftrightarrow 2$, $3 \leftrightarrow 6$, $4 \leftrightarrow 8$, $5 \leftrightarrow 7$). So, we have an easy verification of the well-known fact that the Hughes plane π is autodual (cf. [3], pp. 80–81).

Let us investigate the strong canonical form of the incidence matrix with the reduced kernel of a similar structure as in the matrix ${}^4\widehat{M}^a$, i.e. having the first two rows and columns with the same elements as in the matrix ${}^4\widehat{M}^a$ whereas the inner kernels are different. Combinatorially it is possible to deduce two possibilities for new matrices

$$
{}^4\widehat{M}^b = \begin{matrix}
J_0 & K_0 & I_1 & J_1 & K_1 & I_2 & J_2 & K_2 \\
K_0 & J_0 & I_2 & K_2 & J_2 & I_1 & K_1 & J_1 \\
I_1 & I_2 & u_1 & v_1 & w_1 & x_1 & y_1 & z_1 \\
J_1 & K_2 & u_2 & v_2 & w_2 & x_2 & y_2 & z_2 \\
K_1 & J_2 & u_3 & v_3 & w_3 & x_3 & y_3 & z_3 \\
I_2 & I_1 & x_1 & z_1 & y_1 & u_1 & w_1 & v_1 \\
J_2 & K_1 & x_3 & z_3 & y_3 & u_3 & w_3 & v_3 \\
K_2 & J_1 & x_2 & z_2 & y_2 & u_2 & w_2 & v_2
\end{matrix}
$$

$$
{}^4\widehat{M}^c = \begin{matrix}
J_0 & K_0 & I_1 & J_1 & K_1 & I_2 & J_2 & K_2 \\
K_0 & J_0 & I_2 & K_2 & J_2 & I_1 & K_1 & J_1 \\
I_1 & I_2 & i_1 & j_1 & k_1 & l_1 & m_1 & n_1 \\
J_1 & K_2 & i_2 & j_2 & k_2 & l_2 & m_2 & n_2 \\
K_1 & J_2 & i_3 & j_3 & k_3 & l_3 & m_3 & n_3 \\
I_2 & I_1 & l_1 & n_1 & m_1 & i_1 & k_1 & j_1 \\
J_2 & K_1 & l_3 & n_3 & m_3 & i_3 & k_3 & j_3 \\
K_2 & J_1 & l_2 & n_2 & m_2 & i_2 & k_2 & j_2
\end{matrix}
$$

Fig. 18

where $J_0, K_0, I_1, \ldots, K_2$ are the permutation matrices known from the preceding §2. The inner kernel of each of the new incidence matrices contains 36 permutation matrices such that only 18 of them are distinct. These matrices are as follows:

(4.4)
$$
\left\{
\begin{array}{l}
u_1 = \begin{pmatrix} & j & \\ j & & \\ & & k \end{pmatrix},\ v_1 = \begin{pmatrix} & & k \\ & j & \\ j & & \end{pmatrix},\ w_1 = \begin{pmatrix} j & & \\ & & k \\ & j & \end{pmatrix},\ x_1 = \begin{pmatrix} & k & \\ k & & \\ & & j \end{pmatrix},\ y_1 = \begin{pmatrix} & & j \\ & k & \\ k & & \end{pmatrix},\ z_1 = \begin{pmatrix} k & & \\ & & j \\ & k & \end{pmatrix}, \\
u_2 = \begin{pmatrix} j & & \\ & & j \\ & k & \end{pmatrix},\ v_2 = \begin{pmatrix} & i & \\ k & & \\ & & k \end{pmatrix},\ w_2 = \begin{pmatrix} & & i \\ & j & \\ i & & \end{pmatrix},\ x_2 = \begin{pmatrix} & & k \\ & k & \\ j & & \end{pmatrix},\ y_2 = \begin{pmatrix} k & & \\ & & i \\ & i & \end{pmatrix},\ z_2 = \begin{pmatrix} & j & \\ i & & \\ & & i \end{pmatrix}, \\
u_3 = \begin{pmatrix} & & j \\ & j & \\ k & & \end{pmatrix},\ v_3 = \begin{pmatrix} j & & \\ & & i \\ & i & \end{pmatrix},\ w_3 = \begin{pmatrix} & k & \\ i & & \\ & & k \end{pmatrix},\ x_3 = \begin{pmatrix} k & & \\ & & k \\ & j & \end{pmatrix},\ y_3 = \begin{pmatrix} & i & \\ j & & \\ & & j \end{pmatrix},\ z_3 = \begin{pmatrix} & & i \\ & k & \\ i & & \end{pmatrix},
\end{array}
\right.
$$

(4.5)

$$\left\{\begin{array}{l}
i_1=\begin{pmatrix} & & j\\ & k & \\ j & & \end{pmatrix},\ j_1=\begin{pmatrix} j & & \\ & & j\\ & k & \end{pmatrix},\ k_1=\begin{pmatrix} & k & \\ j & & \\ & & j\end{pmatrix},\ l_1=\begin{pmatrix} & & k\\ & j & \\ k & & \end{pmatrix},\ m_1=\begin{pmatrix} k & & \\ & & k\\ & j & \end{pmatrix},\ n_1=\begin{pmatrix} & j & \\ k & & \\ & & k\end{pmatrix},\\
i_2=\begin{pmatrix} & j & \\ k & & \\ & & j\end{pmatrix},\ j_2=\begin{pmatrix} & & k\\ & k & \\ i & & \end{pmatrix},\ k_2=\begin{pmatrix} j & & \\ & & i\\ & i & \end{pmatrix},\ l_2=\begin{pmatrix} k & & \\ & & j\\ & k & \end{pmatrix},\ m_2=\begin{pmatrix} & i & \\ i & & \\ & & k\end{pmatrix},\ n_2=\begin{pmatrix} & & i\\ & j & \\ j & & \end{pmatrix},\\
i_3=\begin{pmatrix} j & & \\ & & k\\ & j & \end{pmatrix},\ j_3=\begin{pmatrix} & i & \\ i & & \\ & & j\end{pmatrix},\ k_3=\begin{pmatrix} & & i\\ & k & \\ k & & \end{pmatrix},\ l_3=\begin{pmatrix} & k & \\ j & & \\ & & k\end{pmatrix},\ m_3=\begin{pmatrix} & & j\\ & j & \\ i & & \end{pmatrix},\ n_3=\begin{pmatrix} k & & \\ & & i\\ & i & \end{pmatrix}.
\end{array}\right.$$

For the matrices of type (4.4) or (4.5) we deduce

$$\begin{array}{lllllll}
x_1 = u_1^T & y_1 = u_3^T & z_1 = u_2^T & \qquad & l_1 = i_1^T & m_1 = i_3^T & n_1 = i_2^T\\
x_2 = v_1^T & y_2 = v_3^T & z_2 = v_2^T & & l_2 = j_1^T & m_2 = j_3^T & n_2 = j_2^T\\
x_3 = w_1^T & y_3 = w_3^T & z_3 = w_2^T & & l_3 = k_1^T & m_3 = k_3^T & n_3 = k_2^T
\end{array}$$

Using the last relations we obtain transposed matrices

$${}^4\widehat{M}^{bT} = \begin{array}{cccccccc}
K_0 & J_0 & I_2 & K_2 & J_2 & I_1 & K_1 & J_1\\
J_0 & K_0 & I_1 & J_1 & K_1 & I_2 & J_2 & K_2\\
I_2 & I_1 & x_1 & z_1 & y_1 & u_1 & w_1 & v_1\\
K_2 & J_1 & x_2 & z_2 & y_2 & u_2 & w_2 & v_2\\
J_2 & K_1 & x_3 & z_3 & y_3 & u_3 & w_3 & v_3\\
I_1 & I_2 & u_1 & v_1 & w_1 & x_1 & y_1 & z_1\\
K_1 & J_2 & u_3 & v_3 & w_3 & x_3 & y_3 & z_3\\
J_1 & K_2 & u_2 & v_2 & w_2 & x_2 & y_2 & z_2
\end{array}$$

$${}^4\widehat{M}^{cT} = \begin{array}{cccccccc}
K_0 & J_0 & I_2 & K_2 & J_2 & I_1 & K_1 & J_1\\
J_0 & K_0 & I_1 & J_1 & K_1 & I_2 & J_2 & K_2\\
I_2 & I_1 & l_1 & n_1 & m_1 & i_1 & j_1 & k_1\\
K_2 & J_1 & l_2 & n_2 & m_2 & i_2 & j_2 & k_2\\
J_2 & K_1 & l_3 & n_3 & m_3 & i_3 & j_3 & k_3\\
I_1 & I_2 & i_1 & j_1 & k_1 & l_1 & m_1 & n_1\\
K_1 & J_2 & i_3 & j_3 & k_3 & l_3 & m_3 & n_3\\
J_1 & K_2 & i_2 & j_2 & k_2 & l_2 & m_2 & n_2
\end{array}$$

Fig. 19

Comparing the columns of both matrices with the original ones we see that ${}^4\widehat{M}^{bT}, {}^4\widehat{M}^{b}$ and ${}^4\widehat{M}^{cT}, {}^4\widehat{M}^{c}$ are isotopic pairs so that we obtain a similar result as for the Hughes plane: each of the above incidence matrices belongs to a projective plane which is autodual. We shall show that this is only another form of the Hughes plane. From the incidence matrices under investigation we reconstruct the strongly canonical complete systems of mutually orthogonal Latin squares of order 9. The couple of the first and the second row of the reduced kernel ${}^4\widehat{M}^b$ or ${}^4\widehat{M}^c$ lead to the known Latin squares L_1 and L_2 (see Fig. 4). Further, we have:

$$
{}^4L_3^b = \begin{array}{l}
0\boxed{\mathbf{3}}6\ 471\ 582 \\
1\mathbf{4}7\ 582\ 360 \\
25\mathbf{8}\ 360\ 471 \\
360\ \mathbf{1}47\ 258 \\
471\ 2\mathbf{5}8\ 036 \\
582\ 03\mathbf{6}\ 147 \\
603\ 825\ \mathbf{7}14 \\
714\ 603\ 8\mathbf{2}5 \\
825\ 714\ 60\mathbf{3},
\end{array}
\quad
{}^4L_4^b = \begin{array}{l}
0\boxed{\mathbf{4}}8\ 156\ 723 \\
1\mathbf{5}6\ 237\ 804 \\
23\mathbf{7}\ 048\ 615 \\
372\ \mathbf{8}04\ 561 \\
480\ 6\mathbf{1}5\ 372 \\
561\ 72\mathbf{3}\ 480 \\
615\ 480\ \mathbf{2}37 \\
723\ 561\ 0\mathbf{4}8 \\
804\ 372\ 15\mathbf{6},
\end{array}
\quad
{}^4L_5^b = \begin{array}{l}
0\boxed{\mathbf{5}}7\ 813\ 246 \\
1\mathbf{3}8\ 624\ 057 \\
24\mathbf{6}\ 705\ 138 \\
381\ \mathbf{4}62\ 705 \\
462\ 5\mathbf{7}0\ 813 \\
570\ 38\mathbf{1}\ 624 \\
624\ 138\ \mathbf{5}70 \\
705\ 246\ 3\mathbf{8}1 \\
813\ 057\ 46\mathbf{2},
\end{array}
$$

$$
{}^4L_6^b = \begin{array}{l}
0\boxed{\mathbf{6}}3\ 528\ 417 \\
1\mathbf{7}4\ 306\ 528 \\
28\mathbf{5}\ 417\ 306 \\
306\ \mathbf{2}85\ 174 \\
417\ 0\mathbf{6}3\ 285 \\
528\ 17\mathbf{4}\ 063 \\
630\ 741\ \mathbf{8}52 \\
741\ 852\ 6\mathbf{3}0 \\
852\ 630\ 74\mathbf{1},
\end{array}
\quad
{}^4L_7^b = \begin{array}{l}
0\boxed{\mathbf{7}}5\ 264\ 831 \\
1\mathbf{8}3\ 075\ 642 \\
26\mathbf{4}\ 183\ 750 \\
318\ \mathbf{7}50\ 426 \\
426\ 8\mathbf{3}1\ 507 \\
507\ 64\mathbf{2}\ 318 \\
642\ 507\ \mathbf{1}83 \\
750\ 318\ 2\mathbf{6}4 \\
831\ 426\ 07\mathbf{5},
\end{array}
\quad
{}^4L_8^b = \begin{array}{l}
0\boxed{\mathbf{8}}4\ 732\ 165 \\
1\mathbf{6}5\ 840\ 273 \\
27\mathbf{3}\ 651\ 084 \\
327\ \mathbf{5}16\ 840 \\
408\ 3\mathbf{2}7\ 651 \\
516\ 40\mathbf{8}\ 732 \\
651\ 273\ \mathbf{4}08 \\
732\ 084\ 5\mathbf{1}6 \\
840\ 165\ 32\mathbf{7},
\end{array}
$$

$$
{}^4L_3^c = \begin{array}{l}
0\boxed{\mathbf{3}}6\ 714\ 825 \\
1\mathbf{4}7\ 825\ 603 \\
25\mathbf{8}\ 603\ 714 \\
360\ \mathbf{5}82\ 471 \\
471\ 3\mathbf{6}0\ 582 \\
582\ 47\mathbf{1}\ 360 \\
603\ 147\ \mathbf{2}58 \\
714\ 258\ 0\mathbf{3}6 \\
825\ 036\ 14\mathbf{7},
\end{array}
\quad
{}^4L_4^c = \begin{array}{l}
0\boxed{\mathbf{4}}8\ 561\ 237 \\
1\mathbf{5}6\ 372\ 048 \\
23\mathbf{7}\ 480\ 156 \\
372\ \mathbf{1}56\ 804 \\
480\ 2\mathbf{3}7\ 615 \\
561\ 04\mathbf{8}\ 723 \\
615\ 723\ \mathbf{4}80 \\
723\ 804\ 5\mathbf{6}1 \\
804\ 615\ 37\mathbf{2},
\end{array}
\quad
{}^4L_5^c = \begin{array}{l}
0\boxed{\mathbf{5}}7\ 138\ 462 \\
1\mathbf{3}8\ 246\ 570 \\
24\mathbf{6}\ 057\ 381 \\
381\ \mathbf{7}05\ 246 \\
462\ 8\mathbf{1}3\ 057 \\
570\ 62\mathbf{4}\ 138 \\
624\ 570\ \mathbf{8}13 \\
705\ 381\ 6\mathbf{2}4 \\
813\ 462\ 70\mathbf{5},
\end{array}
$$

$$
{}^4L^c_6 = \begin{array}{l}
0\boxed{\mathbf{6}}3\ 852\ 741\\
1\mathbf{7}4\ 630\ 852\\
28\mathbf{5}\ 741\ 630\\
306\ \mathbf{4}17\ 528\\
417\ 5\mathbf{2}8\ 306\\
528\ 30\mathbf{6}\ 417\\
630\ 285\ \mathbf{1}74\\
741\ 063\ 2\mathbf{8}5\\
852\ 174\ 06\mathbf{3},
\end{array}
\quad
{}^4L^c_7 = \begin{array}{l}
0\boxed{\mathbf{7}}5\ 426\ 183\\
1\mathbf{8}3\ 507\ 264\\
26\mathbf{4}\ 318\ 075\\
318\ \mathbf{2}64\ 750\\
426\ 0\mathbf{7}5\ 831\\
507\ 18\mathbf{3}\ 642\\
642\ 831\ \mathbf{5}07\\
750\ 642\ 3\mathbf{1}8\\
831\ 750\ 42\mathbf{6},
\end{array}
\quad
{}^4L^c_8 = \begin{array}{l}
0\boxed{\mathbf{8}}4\ 273\ 516\\
1\mathbf{6}5\ 084\ 327\\
27\mathbf{3}\ 165\ 408\\
327\ \mathbf{8}40\ 165\\
408\ 6\mathbf{5}1\ 273\\
516\ 73\mathbf{2}\ 084\\
651\ 408\ \mathbf{7}32\\
732\ 516\ 8\mathbf{4}0\\
840\ 327\ 65\mathbf{1}
\end{array}
$$

Fig. 20

We see that these systems of Latin squares have the following properties of the Hughes plane: the associated ternary ring is not linear but couples of Latin squares with opposite slope have up to order the same columns. The columns of every Latin square are always formed by three triples of even permutations of cosets of the elementary 3-group of order nine with respect to the cyclic subgroup $(S_0, +, 0)$. The ternary ring is "piecewise linear". From the first rows of Latin squares of strongly canonical systems it is possible to rewrite multiplication tables of induced operations:

$\triangle_2$	
	12 345 678
	21 687 354
	36 471 582
	48 156 723
	57 813 246
	63 528 417
	75 264 831
	84 732 165

System $(S/T)^T$

$\triangle_3$	
	12 345 678
	21 687 354
	36 714 825
	48 561 237
	57 138 462
	63 852 741
	75 426 183
	84 273 516

System $(T/S)^T$

Fig. 21

Both operations $\triangle_2$ and $\triangle_3$ are loop operations and it is easily seen that the first loop passes to the second under the isomorphism $\varrho = (12)(36)(48)(57)$ so that the corresponding projective planes must be isomorphic. Further, it can be shown that the isomorphism $\delta = (036)(147)(258)$ maps the complete system of mutually orthogonal Latin squares ${}^4\mathbf{L}^a = \{L_1, L_2, {}^4L^a_3, \dots, {}^4L^a_8\}$ onto the strongly canonical system ${}^4\mathbf{L}^c = \{L_1, L_2, {}^4L^c_3, \dots, {}^4L^c_8\}$ and $\delta({}^4\mathbf{L}^c) = {}^4\mathbf{L}^b$, so that these three Latin square representations correspond to the same plane. Remember that the starting addition $+$ is always the same and is determined by L_1 of Fig. 4. If we denote the multiplication of the quaternion group (System R^T) by $\triangle_1$, then we get three equivalent descriptions of the Hughes plane. Then the ternary operations ${}^aT$, ${}^bT$, ${}^cT$ on S defined by

$$v = {}^aT(u,x,y) = \begin{matrix} u\triangle_1 x + y & \text{for} & y \in \{0,1,2\} = S_0 \\ u\triangle_2 x + y & \text{for} & y \in \{3,4,5\} = S_1 \\ u\triangle_3 x + y & \text{for} & y \in \{6,7,8\} = S_2 \end{matrix} \tag{4.6}$$

$$v = {}^bT(u,x,y) = \begin{matrix} u\triangle_2 x + y & \text{for} & y \in S_0 \\ u\triangle_3 x + y & \text{for} & y \in S_1 \\ u\triangle_1 x + y & \text{for} & y \in S_2 \end{matrix} \tag{4.7}$$

$$v = {}^cT(u,x,y) = \begin{matrix} u\triangle_3 x + y & \text{for} & y \in S_0 \\ u\triangle_1 x + y & \text{for} & y \in S_1 \\ u\triangle_2 x + y & \text{for} & y \in S_2 \end{matrix} \tag{4.8}$$

determine planar ternary rings of the same plane, namely of the Hughes plane. Due to three expressions in the formulae for ternary operations ${}^aT$, ${}^bT$, ${}^cT$ they are said to be *piecewise linear.*

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