A PRINCIPLE OF LINEARIZATION IN THEORY OF STABILITY OF SOLUTIONS OF VARIATIONAL INEQUALITIES

Jiří Neustupa,* Praha

(Received April 14, 1992)

Abstract. It is shown that the uniform exponential stability and the uniform stability at permanently acting disturbances of a sufficiently smooth but not necessarily steady-state solution of a general variational inequality is a consequence of the uniform exponential stability of a zero solution of another (so called linearized) variational inequality.

Keywords: stability, variational inequalities

MSC 1991: 34D05, 34G99, 58E35

The principle of linearization and its correctness represents an interesting problem in the theory of stability of solutions of differential equations. In [8], P. Quittner deals with stability of steady solutions of variational inequalities finding sufficient conditions for stability which do not depend on existing nonlinearities. We deal with a similar question, also in the case of variational inequalities. We study the uniform exponential stability and the uniform stability at permanently acting disturbances of a sufficiently smooth solution $\widetilde{U}$ of a certain variational inequality in a Banach space and prove that both these properties follow from the uniform exponential stability of the zero solution of a so called linearized inequality. The main differences between this paper and [8] are: a) Our linearized inequality can be nonautonomous, b) instead of assumptions which guarantee the zero solution of a linearized inequality to be uniformly exponentially stable we assume straight the uniform exponential stability of the zero solution of a linearized inequality, c) we deal also with the uniform stability with respect to permanently acting disturbances.

* The research was supported by the Grant Agency of the Czech Republic (grant No. 201/93/2177).

Other ideas and methods which concern stability or instability of solutions of variational inequalities can be found for example in [3], [4] and [7].

Let H be a Hilbert space with a scalar product $(\,.\,,\,.\,)$ and an associated norm $\|\,.\,\|$, $B_\varepsilon(x)$ will denote the ε–neighbourhood of the element x in H and $\overline{B}_\varepsilon(x)$ will be its closure in H.

Let B be a reflexive Banach space which is continuously imbedded into the Hilbert space H and which is dense in H. If we identify H with its dual and if B' is a dual to B then we have $B \subset H \subset B'$. We shall denote by $\langle\,.\,,\,.\rangle$ such a duality between elements of B' and B that $\langle x', x\rangle = (x', x)$ if $x' \in H$. Let K be a closed convex set in B.

We shall deal with various time-dependent solutions of certain variational inequalities in B. If U is such a solution on a time interval I then we shall use the notation $I = I(U)$. Each solution U will be supposed to satisfy the variational inequality a.e. in $I(U)$ and to be maximal in that sense that it cannot be extended as a solution beyond the right end point of $I(U)$. Under solutions, we understand only functions with the following smoothness: $U(t) \in L^2(J; B)$ and $\mathrm{d}U/\mathrm{d}t \in L^2(J; B')$ for each bounded interval J in $I(U)$. Then U belongs to $C(J; H)$ (after a possible change on a set of measure zero in J) and

$$(1)\qquad \Big\langle \frac{\mathrm{d}U}{\mathrm{d}t}(t), U(t)\Big\rangle = \tfrac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\big(\|U(t)\|^2\big) \qquad \text{for a.a. } t \in I(U)$$

(see e.g. [5]).

Let $F\colon [0, +\infty) \times B \to B'$. Assume that $\widetilde{U}$ is a solution (not necessarily steady-state) of the problem given by

$$(2)\qquad \Big\langle \frac{\mathrm{d}U}{\mathrm{d}t} + F(t, U), W - U\Big\rangle \geqslant 0 \qquad \text{for all } W \in K,$$

$$(3)\qquad U(t) \in K \qquad \text{for all } t \in I(U)$$

on the time interval $[0, +\infty)$. Assume that F has a Fréchet differential $\mathrm{D}_2F(t, \widetilde{U}(t))$ (D_2 denotes the differential with respect to the second variable) for all $t \in [0, +\infty)$. We shall denote this differential by $A(t)$. $A(t)$ is a linear operator from B into B'. Assume further that there exists $c_1 > 0$ such that

$$(4)\qquad \langle A(t)x, x\rangle \geqslant -c_1\|x\|^2 \qquad \text{for all } x \in B \text{ and all } t \in [0, +\infty).$$

We do not investigate the question of existence of solutions of the problem (2), (3) or other analogous problems in this paper. We are going to derive some estimates of those solutions U of the problem (2), (3) which are "sufficiently near" to the solution $\widetilde{U}$ at a time instant τ. These estimates will have such a character that they will be valid as long as the solution U exists (it means for all $t \in I(U) \cap [\tau, +\infty)$). Since the estimates will guarantee that the solution U cannot end by a "blow up" at the right end point of the interval $I(U)$, it is natural to expect that $I(U) = [\tau, +\infty)$. But in fact, to prove it, it would be necessary to use some additional assumptions about the operator F and the set K (see e.g. [1] and [2]). In order not to complicate this paper, we decided not to do it here.

Definition 1. *We say that a solution $\widetilde{U}$ of* (2), (3) *is uniformly exponentially stable (with respect to the norm $\|\,.\,\|$) if there exist $\Delta > 0$, $C > 0$ and $\delta > 0$ such that if U is any solution of the problem* (2), (3), *$\tau \in I(U)$ and $\|U(\tau) - \widetilde{U}(\tau)\| \leqslant \Delta$ then*

$$\|U(t) - \widetilde{U}(t)\| \leqslant C\|U(\tau) - \widetilde{U}(\tau)\|\mathrm{e}^{-\delta(t-\tau)}$$

for all $t \in [\tau, +\infty) \cap I(U)$.

Definition 2. *We say that a solution $\widetilde{U}$ of* (2), (3) *is uniformly stable at permanently acting disturbances if for any given $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ so that if*

a) *$G(t,\,.\,)$ is for each $t \in [0, +\infty)$ an operator from B to B' satisfying the inequality $|\langle G(t,V), W\rangle| \leqslant \delta_1\|W\|$ for all $V \in K \cap \overline{B}_\varepsilon(\widetilde{U}(t))$ and for all $W \in B$,*

b) *U is a solution of the problem given by the condition* (3) *and the variational inequality*

$$\Big\langle \frac{\mathrm{d}U}{\mathrm{d}t} + F(t,U) + G(t,U), W - U \Big\rangle \geqslant 0 \qquad \text{for all } W \in K,$$

c) $\tau \in I(U)$, $\|U(\tau) - \widetilde{U}(\tau)\| \leqslant \delta_2$

then $\|U(t) - \widetilde{U}(t)\| \leqslant \varepsilon$ for all $t \in [0, +\infty) \cap I(U)$.

The uniform exponential stability of the solution $\widetilde{U}$ is the property of those solutions U of (2), (3) which lie in a neighbourhood of $\widetilde{U}$—let us write any of these solutions in the form $\widetilde{U} + u$. If we substitute this form of U into (2), (3), we get

$$\Big\langle \frac{\mathrm{d}\widetilde{U}}{\mathrm{d}t} + \frac{\mathrm{d}u}{\mathrm{d}t} + F(t, \widetilde{U} + u), W - \widetilde{U} - u \Big\rangle \geqslant 0 \qquad \text{for all } W \in K, \tag{5}$$

$$\widetilde{U}(t) + u(t) \in K \qquad \text{for all } t \in I(\widetilde{U} + u). \tag{6}$$

Let $K_1(t) = K - \widetilde{U}(t)$ (i.e. $K_1(t) = \{w \in B;\ \exists\, W \in K\colon w = W - \widetilde{U}(t)\}$). Statements (5) and (6) can be rewritten:

$$\Big\langle \frac{\mathrm{d}\widetilde{U}}{\mathrm{d}t} + \frac{\mathrm{d}u}{\mathrm{d}t} + F(t, \widetilde{U} + u), w - u \Big\rangle \geqslant 0 \qquad \text{for all } w \in K_1(t), \tag{7}$$

$$u(t) \in K_1(t) \qquad \text{for all } t \in I(u) \tag{8}$$

where $I(u) = I(\widetilde{U} + u)$ is the domain of definition of the solution u of (7), (8). It is easy to see that the uniform exponential stability of the solution $\widetilde{U}$ of (2), (3) is equivalent to the uniform exponential stability of the zero solution $u \equiv 0$ of (7), (8). Similarly, it can be easily shown that the uniform stability at permanently acting disturbances of the solution $\widetilde{U}$ of (2), (3) is equivalent to the same property of the zero solution of the problem (7), (8).

The term $F(t, \widetilde{U}(t) + u)$ can be expressed in the form $F(t, \widetilde{U}(t)) + A(t)u + N(t, u)$ where $N(t, .)$ is a nonlinear operator from B into B'. Assume that

$$|\langle N(t,x), y\rangle| \leqslant g(\|x\|) \cdot \|y\| \quad \text{for all } x, y \in B \tag{9}$$

where g is a nondecreasing function on $[0, +\infty)$ such that

$$g(h) = o(h) \ \text{ if } \ h \to 0+ . \tag{10}$$

This is an important assumption which characterizes the nonlinear operator $N(t, .)$.

Let us denote $f(t) = \mathrm{d}\widetilde{U}/\mathrm{d}t(t) + F(t, \widetilde{U}(t))$ in the following. Then the inequality (7) is identical with

$$\Big\langle \frac{\mathrm{d}u}{\mathrm{d}t} + A(t)u + N(t,u) + f(t), w - u \Big\rangle \geqslant 0 \qquad \text{for all } w \in K_1(t). \tag{11}$$

A so called linearized inequality arises from (11) by omitting the nonlinear term $N(t, u)$:

$$\Big\langle \frac{\mathrm{d}v}{\mathrm{d}t} + A(t)v + f(t), w - v \Big\rangle \geqslant 0 \qquad \text{for all } w \in K_1(t). \tag{12}$$

It will be considered with the condition

$$v(t) \in K_1(t) \qquad \text{for all } t \in I(v). \tag{13}$$

In fact, (12), (13) does not represent a linear problem, but it has a zero solution. It is not difficult to show that v is a solution of (12), (13) on the time interval $I(v)$ if and only if the function $V = \widetilde{U} + v$ is a solution of the problem

$$\Big\langle \frac{\mathrm{d}V}{\mathrm{d}t} + A(t)V + f_1(t), W - V \Big\rangle \geqslant 0 \qquad \text{for all } W \in K, \tag{14}$$

$$V(t) \in K \qquad \text{for all } t \in I(V), \tag{15}$$

where $f_1(t) = F(t, \widetilde{U}(t)) - A(t)\widetilde{U}(t)$. Moreover, the uniform exponential stability of the zero solution of (12), (13) is equivalent to the uniform exponential stability of the solution $\widetilde{U}$ of (14), (15).

Let us assume that

(16) if $x_0 \in H$ and $\tau \geqslant 0$ then there exists a solution v of (12), (13) on $[\tau, +\infty)$ such that $v(\tau) = x_0$.

The validity of this condition can be usually verified by various methods in special cases. See e.g. [1] and [2] for more details.

Theorem 1. *Let the zero solution of the problem* (12), (13) *be uniformly exponentially stable and let the conditions* (4), (9), (10) *and* (16) *be satisfied. Then the zero solution of the problem* (11), (8) *is uniformly exponentially stable, too, and consequently, also the solution $\widetilde{U}$ of the problem* (2), (3) *is uniformly exponentially stable.*

Proof. Let u be a solution of (11), (8), $\tau, t \in I(u)$, $\tau < t$ and let v be a solution of (12), (13) on $[\tau, +\infty)$ such that $v(\tau) = u(\tau)$. We shall use the notation $|\!|\!| u |\!|\!|_{[\tau,t]} = \max_{t' \in [\tau,t]} \|u(t')\|$. First, we shall derive an estimate of $\|u(t) - v(t)\|$. Choosing $w = v(t)$ in (11), $w = u(t)$ in (12) and adding (11) and (12), we obtain

$$\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}(u-v) + A(t)(u-v) + N(t,u), u-v \Big\rangle \leqslant 0,$$

$$\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}(u-v), u-v \Big\rangle \leqslant \langle -A(t)(u-v), u-v \rangle - \langle N(t,u), u-v \rangle.$$

Using (1), (4) and (9), we obtain

$$\begin{aligned}
\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)-v(t)\|^2 &\leqslant c_1\,\|u(t)-v(t)\|^2 + g(\|u(t)\|)\|u(t)-v(t)\| \\
&\leqslant (c_1+1)\|u(t)-v(t)\|^2 + \tfrac{1}{4}\,g^2(\|u(t)\|), \\
\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-2(c_1+1)t}\|u(t)-v(t)\|^2\right] &\leqslant \mathrm{e}^{-2(c_1+1)t}\,\tfrac{1}{4}\,g^2(\|u(t)\|), \\
\|u(t)-v(t)\|^2 &\leqslant \tfrac{1}{2}\int_\tau^t \mathrm{e}^{2(c_1+1)(t-\xi)}\,g^2(\|u(\xi)\|)\,\mathrm{d}\xi \\
&\leqslant \frac{1}{4(c_1+1)}\left[\mathrm{e}^{2(c_1+1)(t-\tau)} - 1\right] g^2(|\!|\!| u |\!|\!|_{[\tau,t]}).
\end{aligned}$$

If we denote

$$g_1(\alpha,\beta) = \left[\frac{1}{4(c_1+1)}\left(\mathrm{e}^{2(c_1+1)\alpha} - 1\right)\right]^{1/2} g(\beta)$$

then we can write

$$\|u(t)-v(t)\| \leqslant g_1\big(t-\tau, |\!|\!| u |\!|\!|_{[\tau,t]}\big). \tag{17}$$

If u is a solution of (11), (8) and there exists $\tau \in I(u)$ such that $\|u(\tau)\| = 0$ then $[\tau, +\infty) \subset I(u)$ and $\|u(t)\| = 0$ for all $t \in [\tau, +\infty)$. This follows from (17) (where we use $v \equiv 0$) and (10).

Let Δ, δ and C be numbers which are connected with the uniform exponential stability of the zero solution of (12), (13) (see Definition 1). We shall prove that

(18) there exists $\Delta' > 0$ such that if u is a solution of (11), (8), $\tau \in I(u)$, $0 < \|u(\tau)\| \leqslant \Delta'$ then

$$\|u(t)\| < 2C\,\|u(\tau)\|\mathrm{e}^{-\delta(t-\tau)/2}$$

for $t \in [\tau, +\infty) \cap I(u)$.

Suppose that (18) is false. Then for each $\Delta' > 0$ there exists a solution u of (11), (8), $\tau \in I(u)$ and $t_0 \in [\tau, +\infty) \cap I(u)$ such that $0 < \|u(\tau)\| \leqslant \Delta'$,

$$\|u(t_0)\| = 2C\,\|u(\tau)\|\mathrm{e}^{-\delta(t_0-\tau)/2}, \tag{19}$$

$$\|u(t)\| < 2C\,\|u(\tau)\|\mathrm{e}^{-\delta(t-\tau)/2} \qquad \text{for } t \in [\tau, t_0). \tag{20}$$

Let γ be such a positive number that $C\mathrm{e}^{-\delta\gamma/2} \leqslant \frac{1}{2}$. Suppose that Δ' is so small that $2C\Delta' < \Delta$.

There exists $n \in \mathbb{N} \cup \{0\}$ and $h \in [0, \gamma)$ such that $t_0 = \tau + \gamma n + h$. It follows from (16) that there exists a solution v_n of (12), (13) on the interval $[\tau + \gamma n, +\infty)$ such that $u(\tau + \gamma n) = v_n(\tau + \gamma n)$. Using the inequality (17), we obtain

$$\begin{aligned}
\|u(t_0) - v_n(t_0)\| &\leqslant g_1\bigl(h, |||u|||_{[\tau+\gamma n, t_0]}\bigr),\\
\|u(t_0)\| &\leqslant \|v_n(t_0)\| + g_1\bigl(h, |||u|||_{[\tau+\gamma n, t_0]}\bigr),\\
\|u(t_0)\| &\leqslant C\|v_n(\tau+\gamma n)\|\mathrm{e}^{-\delta h} + g_1\bigl(h, |||u|||_{[\tau+\gamma n, t_0]}\bigr)\\
&\leqslant C\|u(\tau+\gamma n)\|\mathrm{e}^{-\delta h} + g_1\bigl(h, 2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma n/2}\bigr).
\end{aligned}$$

Similarly, we can derive the estimate

$$\|u(\tau+\gamma i)\| \leqslant C\,\|u(\tau+\gamma(i-1))\|\mathrm{e}^{-\delta\gamma} + g_1\bigl(\gamma, 2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma(i-1)/2}\bigr)$$

for $i = 1, 2, \ldots, n$. Thus, using the inequality $C\mathrm{e}^{-\delta\gamma/2} \leqslant \frac{1}{2}$, we can write

$$\begin{aligned}
2C\,\|u(\tau)\|\mathrm{e}^{-\delta(t_0-\tau)/2} &= \|u(t_0)\|\\
&\leqslant C\,\|u(\tau+\gamma n)\|\mathrm{e}^{-\delta h} + g_1\bigl(h, 2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma n/2}\bigr)\\
&\leqslant C\,\|u(\tau+\gamma n)\| + g_1\bigl(\gamma, 2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma n/2}\bigr)\\
&\leqslant C\{C\|u(\tau+\gamma(n-1))\|\mathrm{e}^{-\delta\gamma}\\
&\qquad + g_1\bigl(\gamma, 2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma(n-1)/2}\bigr) + g_1\bigl(\gamma, 2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma n/2}\bigr)\\
&\leqslant \tfrac{1}{2}\,C\mathrm{e}^{-\delta\gamma/2}\|u(\tau+\gamma(n-1))\|\\
&\qquad + C\,g_1\bigl(\gamma, 2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma(n-1)/2}\bigr) + g_1\bigl(\gamma, 2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma n/2}\bigr) \;\leqslant \ldots
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left(\tfrac{1}{2}\right)^n C\mathrm{e}^{-\delta\gamma n/2}\|u(\tau)\| + C\mathrm{e}^{\delta\gamma}\sum_{j=0}^{n}\left(\tfrac{1}{2}\right)^j \mathrm{e}^{-\delta\gamma j/2} g_1\big(\gamma, 2C\|u(\tau)\|\mathrm{e}^{-\delta\gamma(n-j)/2}\big)\\
&= \left(\tfrac{1}{2}\right)^n C\mathrm{e}^{-\delta\gamma n/2}\|u(\tau)\|\\
&\qquad + C\mathrm{e}^{\delta\gamma}\sum_{j=0}^{n}\left(\tfrac{1}{2}\right)^j \mathrm{e}^{-\delta\gamma j/2}\frac{g_1\big(\gamma, 2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma(n-j)/2}\big)}{2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma(n-j)/2}}\,2C\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma(n-j)/2}\\
&\leqslant \left(\tfrac{1}{2}\right)^n C\mathrm{e}^{-\delta\gamma n/2}\|u(\tau)\| + C\mathrm{e}^{\delta\gamma}\sum_{j=0}^{n}\left(\tfrac{1}{2}\right)^j g_2(\gamma, 2C\Delta')2C\|u(\tau)\|\mathrm{e}^{-\delta\gamma n/2}
\end{aligned}
$$

where $g_2(\eta) = \sup\limits_{\sigma\in(0,\eta)} [g_1(\gamma,\sigma)/\sigma]$. The function g_2 is nondecreasing and it follows from (10) that

$$\lim_{\eta\to 0+} g_2(\eta) = 0.$$

Hence

$$
\begin{aligned}
2C\,\|u(\tau)\|\mathrm{e}^{-\delta(t_0-\tau)/2} &\leqslant \left(\tfrac{1}{2}\right)^n C\mathrm{e}^{-\delta\gamma n/2}\|u(\tau)\|\\
&\qquad + 4C^2\mathrm{e}^{\delta\gamma}\,\|u(\tau)\|\mathrm{e}^{-\delta\gamma n/2} g_2(\gamma, 2C\Delta'),\\
2C\mathrm{e}^{-\delta(t_0-\tau)/2} - \left(\tfrac{1}{2}\right)^n C\mathrm{e}^{-\delta\gamma n/2} &\leqslant 4C^2\mathrm{e}^{\delta\gamma}\mathrm{e}^{-\delta\gamma n/2} g_2(2C\Delta'),\\
2\mathrm{e}^{-\delta h/2} - \left(\tfrac{1}{2}\right)^n &\leqslant 4C\mathrm{e}^{\delta\gamma} g_2(2C\Delta').
\end{aligned}
$$

Choosing Δ' sufficiently small, we can make the right hand side of this inequality so small that the inequality does not hold. This is a desired contradiction and hence (18) is true, which implies the uniform exponential stability of the zero solution of (11), (8). $\square$

A theorem about differential equations in Banach spaces which is analogous to Theorem 1 is proved for example in [6].

Lemma 1. *Let the zero solution of the problem* (12), (13) *be uniformly exponentially stable and let the conditions* (4) *and* (16) *be fulfilled. Then the zero solution of* (12), (13) *is also uniformly stable at permanently acting disturbances.*

P r o o f. Suppose that $G(t)\colon B \to B'$ for all $t \geqslant 0$. Let u be a solution of the problem given by

$$\left\langle \frac{\mathrm{d}u}{\mathrm{d}t} + A(t)u + f(t) + G(t)u, w - u\right\rangle \geqslant 0 \qquad \text{for all } w \in K_1(t) \tag{21}$$

and the condition (8) and let v be a solution of (12), (13). Finally, let τ, $t \in I(u)\cap I(v)$, $\tau < t$. In a way similar to that which was used to derive (17), we can get the inequality

$$\tfrac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)-v(t)\|^2 \leqslant c_1\|u(t)-v(t)\|^2 + \langle G(t)u(t), u(t)-v(t)\rangle. \tag{22}$$

Let Δ, δ and C be numbers which are connected with the uniform exponential stability of the zero solution of (12), (13) (see Definition 1). Let $\varepsilon > 0$ be given. Put $\delta_2 = \min\{\Delta;\ \varepsilon/(2C)\}$. There exist $T > 0$ and $\delta_1 > 0$ such that $C\mathrm{e}^{-\delta T} \leqslant \frac{1}{4}$ and $\delta_1\mathrm{e}^{(c_1+1)T} \leqslant \sqrt{c_1+1}\,\delta_2$. Assume that $|\langle G(t)x, y\rangle| \leqslant \delta_1\|y\|$ for all $t \geqslant 0$ and $x \in K_1(t) \cap \overline{B}_\varepsilon(0)$.

Suppose that u is a solution of (21), (8), $\tau \in I(u)$ and $\|u(\tau)\| < \delta_2$. Let v be a solution of (12), (13) on $[\tau, +\infty)$ such that $v(\tau) = u(\tau)$. It follows from (22) that if $t \in [\tau, \tau+T) \cap I(u)$ and $|\!|\!|u|\!|\!|_{[\tau,t]} \leqslant \varepsilon$ then

$$\begin{aligned}\tfrac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)-v(t)\|^2 &\leqslant c_1\|u(t)-v(t)\|^2 + \delta_1\|u(t)-v(t)\| \\ &\leqslant (c_1+1)\|u(t)-v(t)\|^2 + \tfrac{1}{4}\delta_1^2,\end{aligned}$$

$$\tfrac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-2(c_1+1)t}\|u(t)-v(t)\|^2\right] \leqslant \mathrm{e}^{-2(c_1+1)t}\,\tfrac{1}{4}\,\delta_1^2,$$

$$\begin{aligned}\|u(t)-v(t)\|^2 &\leqslant \tfrac{1}{2}\int_\tau^t \mathrm{e}^{2(c_1+1)(t-\xi)}\delta_1^2\,\mathrm{d}\xi = \frac{1}{4(c_1+1)}\left[\mathrm{e}^{2(c_1+1)(t-\tau)} - 1\right]\delta_1^2, \\ \|u(t)-v(t)\| &\leqslant \frac{1}{2\sqrt{c_1+1}}\,\mathrm{e}^{(c_1+1)(t-\tau)}\delta_1, \\ \|u(t)\| &\leqslant \|v(t)\| + \frac{1}{2\sqrt{c_1+1}}\,\mathrm{e}^{(c_1+1)(t-\tau)}\delta_1.\end{aligned}$$

Using the uniform exponential stability of the zero solution of (12), (13), one has

$$\begin{aligned}\|u(t)\| &\leqslant C\|v(\tau)\|\mathrm{e}^{-\delta(t-\tau)} + \frac{\delta_1}{2\sqrt{c_1+1}}\,\mathrm{e}^{(c_1+1)(t-\tau)} \\ &\leqslant C\delta_2 + \tfrac{1}{2}\,\delta_2 \leqslant (C+\tfrac{1}{2})\,\frac{\varepsilon}{2C} \leqslant \tfrac{3}{4}\varepsilon.\end{aligned}$$

This inequality was derived only for $t \in [\tau, \tau+T] \cap I(u)$ such that $|\!|\!|u|\!|\!|_{[\tau,\tau+T]} \leqslant \varepsilon$. Since $\|u(t)\|$ depends continuously on t, the above inequality must hold for all $t \in [\tau, \tau+T]$. Moreover, we have

$$\|u(\tau+T)\| \leqslant C\delta_2\mathrm{e}^{-\delta T} + \frac{\delta_1}{2\sqrt{c_1+1}}\,\mathrm{e}^{(c_1+1)T} \leqslant \tfrac{1}{4}\delta_2 + \tfrac{1}{2}\delta_2 = \tfrac{3}{4}\delta_2.$$

Applying the same considerations on the time intervals $[\tau+iT, \tau+(i+1)T] \cap I(u)$ for $i = 1, 2, \ldots$, one obtains the estimate $\|u(t)\| \leqslant \varepsilon$ for t from all these intervals. Thus, $\|u(t)\| \leqslant \varepsilon$ for $t \in [\tau, +\infty) \cap I(u)$, which we wanted to prove. $\square$

Theorem 2. *Let the zero solution of the problem* (12), (13) *be uniformly exponentially stable and let the conditions* (4), (9), (10) *and* (16) *be satisfied. Then the zero solution of the problem* (11), (8) *is uniformly stable at permanently acting disturbances and consequently, also the solution* $\widetilde{U}$ *of the problem* (2), (3) *is uniformly stable at permanently acting disturbances.*

Proof. Due to Lemma 1, the zero solution of (12), (13) is uniformly stable at permanently acting disturbances. Let $\varepsilon > 0$ be given. Let the numbers T, δ_1 and δ_2 be defined (in dependence on ε) in the same way as in the proof of Lemma 1. Moreover, let ε be so small that $g(\varepsilon) < \delta_1$. Put $\delta_1' = \delta_1 - g(\varepsilon)$.

Let u be a solution of the problem given by

$$\left\langle \frac{\mathrm{d}u}{\mathrm{d}t} + A(t)u + N(t)u + f(t) + G(t)u, w - u \right\rangle \geqslant 0 \qquad \text{for all } w \in K_1(t) \tag{23}$$

and the condition (8), $\tau \in I(u)$ and $\|u(\tau)\| < \delta_2$. Assume that $|\langle G(t)x, y\rangle| \leqslant \delta_1' \|y\|$ for all $t \geqslant 0$, $y \in B$ and $x \in K_1(t) \cap \overline{B}_\varepsilon(0)$. Then

$$|\langle N(t)x + G(t)x, y\rangle| \leqslant [g(\|x\|) + \delta_1'] \cdot \|y\| \leqslant [g(\varepsilon) + \delta_1'] \cdot \|y\| = \delta_1 \|y\|.$$

If we view the term $N(t)u + G(t)u$ in the inequality (23) as a disturbance and use the uniform stability at permanently acting disturbances of the zero solution of (12), (13) and Lemma 1, we obtain $\|u(t\| < \varepsilon$ for all $t \in [\tau, +\infty) \cap I(u)$. This completes the proof. □

References

[1] *J. P. Aubin, A. Cellina*: Differential Inclusions. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.

[2] *H. Brézis*: Problèmes unilatéraux,. J. Math. Pures Appl. *51* (1972), 1–168.

[3] *P. Drábek, M. Kučera*: Eigenvalues of inequalities of reaction-diffusion type and destabilizing effect of unilateral conditions. Czech. Math. J. *36 (111)* (1986), 116–130.

[4] *P. Drábek, M. Kučera*: Reaction-diffusion systems: Destabilizing effect of unilateral conditions. Nonlinear Analysis, Theory, Methods & Applications *12* (1988), no. 11, 1173–1192.

[5] *H. Gajewski, K. Gröger, K. Zacharias*: Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Akademie-Verlag, Berlin, 1974.

[6] *J. Neustupa*: The linearized uniform asymptotic stability of evolution differential equations. Czech. Math. J. *34 (109)* (1984), 257–284.

[7] *M. Kučera, J. Neustupa*: Destabilizing effect of unilateral conditions in reaction-diffusion systems. Commentationes Math. Univ. Carolinae *27* (1986), no. 1, 171–187.

[8] *P. Quittner*: On the principle of linearized stability for variational inequalities. Math. Ann. vol 283 (1989), 257–270.

Author's address: *Jiří Neustupa*, Czech Technical University, Faculty of Mechanical Engineering, Department of Technical Mathematics, Karlovo nám. 13, 121 35 Praha 2, e-mail: `neustupa@fsid.cvut.cz`.