PACKINGS OF PAIRS WITH A MINIMUM KNOWN NUMBER OF QUADRUPLES

Jiří Novák, Liberec

(Received January 19, 1993)

Abstract. Let E be an n-set. The problem of packing of pairs on E with a minimum number of quadruples on E is settled for $n < 15$ and also for $n = 36t + i$, $i = 3, 6, 9, 12$, where t is any positive integer. In the other cases of n methods have been presented for constructing the packings having a minimum known number of quadruples.

Keywords: packing of pairs with quadruples, system of quadruples, configuration, packing of K_4's into K_n.

MSC 1991: 05B40, 05B05

1. Introduction

A packing of pairs with quadruples is a system of quadruples on an n-set in which any two quadruples have at most one element in common and no further quadruple can be added to the system which is therefore maximal. We denote the system by $P(n, 4, 2)$.

If the number of quadruples, called the size of the packing, is maximum (minimum) possible, then such a system will be called a maximum (minimum) packing, respectively. Maximum packings $P_{\mathrm{MAX}}(n, 4, 2)$ are known thanks to the results of Brouwer in [1], but the construction of minimum packings $P_{\mathrm{MIN}}(n, 4, 2)$ remains an open problem. Therefore this paper deals with constructions of packings that have minimum known sizes. The author believes that this start will initiate an investigation of minimum packings. The paper also contains some such constructions for $n = 36t + i$, $i = 3, 6, 9, 12$, where t is any positive integer. Note that the analogous problem of packings of pairs with triples is completely solved in [3, 4].

The constructions used are based on the following famous theorem of Turán on minimum representations of k-tuples with pairs.

Theorem 1.1 [5]. *Let $n > k > 2$ be integers. A pair D from an n-set E represents a k-tuple $K \subset E$, if $D \subset K$. The minimum representation of all k-tuples $K \subset E$ is formed by all pairs of mutually disjoint classes T_i, $i = 1, 2, \dots, k-1$, where $\bigcup_{i=1}^{k-1} T_i = E$ and $\big||T_i| - |T_j|\big| \leqslant 1$ for any two classes T_i, T_j.*

2. Theorems

Theorem 2.1. *Let $P(n, 4, 2)$ be a packing on an n-set E, let $q(n)$ be its size. Let T_1, T_2, T_3 be the classes of the Turán partition of the set E for $k = 4$. Denote $n_i = |T_i|$, $i = 1, 2, 3$. Then the following estimate holds:*

$$q(n) \geqslant \tfrac{1}{6} \sum_{i=1}^{3} \binom{n_i}{2} = L(n).$$

Proof. The pairs contained in the quadruples of $P(n, 4, 2)$ represents all quadruples on the set E. This follows from the fact that any further quadruple has at least two elements in common with a quadruple of the packing. The number of pairs in any representation of quadruples with pairs is at least equal to $\sum_{i=1}^{3} \binom{n_i}{2}$. On the other hand, any quadruple contains six pairs and thus we obtain the above estimate. □

Theorem 2.2 [2]. *Let $n \equiv 1$ or $4 \pmod{12}$, $n > 3$. Then there exists a $P_{\mathrm{MAX}}(n, 4, 2)$ containing all pairs on the n-set E. Therefore its size is equal to $\frac{1}{6}\binom{n}{2}$. This packing will be called exact.*

Theorem 2.3. *Let i, t, n be positive integers, $n = 36 + i$. Then*

a) $|P_{\mathrm{MIN}}(36t + 3, 4, 2)| = 36t^2 + 3t$,

b) $|P_{\mathrm{MIN}}(36t + 6, 4, 2)| = 36t^2 + 9t + 1$,

c) $|P_{\mathrm{MIN}}(36t + 9, 4, 2)| = 36t^2 + 15t + 2$,

d) $|P_{\mathrm{MIN}}(36t + 12, 4, 2)| = 36t^2 + 21t + 3$.

Proof. a) Let us calculate the lower bound $L(n)$. According to Theorem 2.1 we have $L(n) = \frac{1}{6} \sum_{i=1}^{3} \binom{n_i}{2} = \frac{1}{6} \sum_{i=1}^{3} \binom{12t+1}{2} = 36t^2 + 3t$. The packing having $36t^2 + 3t$

quadruples will be constructed as the union of three exact packings on the classes T_1, T_2, T_3 of Turán's partition of E, where $|T_i| = 12t+1$, $i = 1, 2, 3$.

b) The lower bound is $L(n) = \frac{1}{6}\sum_{i=1}^{3}\binom{12t+2}{2} = 36t^2+9t+\frac{1}{2} < 36t^2+9t+1 = L'$.

The packing having L' quadruples will be constructed as the union of exact packings on the classes E_i of a partition on E, where $|E_1| = |E_2| = 12t+1$, $|E_3| = 12t+4$. It is clear that no further quadruple can be added to the union of three exact packings and that its size equals L'.

c) The lower bound is $L(n) = \frac{1}{6}\sum_{i=1}^{3}\binom{12t+3}{2} = 36t^2+15t+\frac{3}{2} < 36t^2+15t+2 = L'$.

The packing having L' quadruples will be constructed as the union of exact packings on the classes E_i of a partition on E, where $|E_1| = 12t+1$, $|E_2| = |E_3| = 12t+4$.

d) The lower bound is $L(n) = \frac{1}{6}\sum_{i=1}^{3}\binom{12t+4}{2} = 36t^2+21t+3$.

The packing having $L(n)$ quadruples will be constructed as the union of exact packings on the classes T_i of Turán's partition of E, where $|T_i| = 12t+4$, $i = 1$, 2, 3. $\square$

Theorem 2.4. *Let $3 < n < 13$. Denote $|P_{\mathrm{MIN}}(n,4,2)| = m(n)$. Then $m(4) = m(5) = m(6) = 1$, $m(7) = m(8) = m(9) = 2$, $m(10) = m(11) = m(12) = 3$.*

Proof. Let $E = \{1,2,\ldots,n\}$. Denote a quadruple $\{a,b,c,d\} \subset E$ briefly by $a\,b\,c\,d$, a packing $P(n,4,2)$ having $q(n)$ quadruples by $P_{q(n)}(n,4,2)$. Let us introduce packings $P_{m(n)}(n,4,2)$ on E that are evidently minimum possible:

$$
\begin{aligned}
P_1(4,4,2) &= P_1(5,4,2) = P_1(6,4,2) = \{1\,2\,3\,4\},\\
P_2(7,4,2) &= \{1\,2\,3\,4, 4\,5\,6\,7\},\\
P_2(8,4,2) &= P_2(9,4,2) = \{1\,2\,3\,4, 5\,6\,7\,8\},\\
P_3(10,4,2) &= \{1\,2\,3\,4, 5\,6\,7\,8, 1\,5\,9\,10\},\\
P_3(11,4,2) &= \{1\,2\,3\,4, 5\,6\,7\,8, 1\,9\,10\,11\},\\
P_3(12,4,2) &= \{1\,2\,3\,4, 5\,6\,7\,8, 9\,10\,11\,12\}.
\end{aligned}
$$

$\square$

Definition 2.1. A system of quadruples on the n-set E having mutually at most one element in common will be called a configuration of quadruples. Assign to each quadruple one line with 4 points that correspond to the elements of the quadruple. Two lines can meet in at most one point and the number of all points must be $\leqslant n$. So we obtain a graphical representation of the configuration. Two configurations are said to be isomorphic if they have the same graphical representation.

Theorem 2.5. *Let $n = 13$, 14. Then $|P_{\mathrm{MIN}}(13,4,2)| = 6$, $|P_{\mathrm{MIN}}(14,4,2)| = 6$.*

Proof. First we introduce two configurations A and B for $n = 13$ and $n = 14$, respectively, each containing six quadruples, and we will prove that they are really packings,

$$A = \{1\,2\,3\,4, 1\,5\,6\,7, 2\,5\,8\,10, 3\,7\,9\,10, 4\,6\,8\,9, 10\,11\,12\,13\},$$
$$B = \{1\,2\,3\,4, 1\,5\,6\,7, 2\,5\,8\,10, 3\,7\,9\,10, 4\,6\,8\,9, 11\,12\,13\,14\}.$$

We shall prove that no further quadruple can be added to the configurations A, B.

Decompose the sets $E = \{1, 2, 3, \ldots, 13\}$ and $E' = \{1, 2, \ldots, 14\}$ into four classes:

$$E = \bigcup_{i=1}^{4} E_i,\ E_1 = \{1,2,3,4\},\ E_2 = \{5,6,7\},\ E_3 = \{8,9,10\},\ E_4 = \{11,12,13\};$$
$$E' = \bigcup_{i=1}^{4} E_i',\ E_1' = \{1,2,3,4\},\ E_2' = \{5,6,7\},\ E_3' = \{8,9,10\},\ E_4' = \{11,12,13,14\}.$$

Suppose there is a quadruple $a_1a_2a_3a_4$ which can be added to the configuration A. Then, necessarily, $a_i \in E_i$ for $i = 1, \ldots, 4$. Let $a_1 = 1$. Because the element 1 is joined with all elements of the class E_2 in the $1\,5\,6\,7$, no quadruple with 1 can be added to A. Let $a_1 = 2$. The element 2 could be joined with 6 or 7 from the class E_2. In the case of joining 2 with 6 it is necessary to add one element of the class $E_3 = \{8, 9, 10\}$. But the triples $2\,6\,8$, $2\,6\,9$, $2\,6\,10$ are not possible. The same consideration holds for the pair $2\,7$, thus no quadruple with the element 2 can be added to A.

In the same way, no quadruple with 3 and 4 can be added to A. Thus the configuration A is really a packing having 6 quadruples.

The same consideration holds for the configuration B. Thus it is really a packing having 6 quadruples.

b) $L(13) = \frac{1}{6}\left(\binom{4}{2} + \binom{4}{2} + \binom{5}{2}\right) = \frac{22}{6} < 4$.
$L(14) = \frac{1}{6}\left(\binom{4}{2} + \binom{5}{2} + \binom{5}{2}\right) = \frac{26}{6} < 5$.

Therefore we first have to prove that no packing $P_4(13,4,2)$ exists and then that no packings $P_5(13,4,2)$ and $P_5(14,4,2)$ exist. We have to investigate all nonisomorphic configurations formed by four quadruples for $n \leqslant 13$.

Using the graphical representations of quadruples and their elements, we easily obtain the result that four quadruples form 11 nonisomorphic configurations:

1) $\{1\,2\,3\,4, 5\,6\,7\,8, 9\,10\,11\,12, 1\,5\,9\,13\}$,
2) $\{1\,2\,3\,4, 4\,5\,6\,7, 7\,8\,9\,10, 10\,11\,12\,13\}$,
3) $\{1\,2\,3\,4, 4\,5\,6\,7, 7\,8\,9\,10, 7\,11\,12\,13\}$,

4) $\{1\,2\,3\,4, 5\,6\,7\,8, 8\,9\,10\,11, 5\,11\,12\,13\}$,
5) $\{1\,2\,3\,4, 5\,6\,7\,8, 8\,9\,10\,11, 1\,5\,9\,12\}$,
6) $\{1\,2\,3\,4, 4\,5\,6\,7, 7\,8\,9\,10, 1\,10\,11\,12\}$,
7) $\{1\,2\,3\,4, 4\,5\,6\,7, 1\,7\,8\,9, 1\,10\,11\,12\}$,
8) $\{1\,2\,3\,4, 4\,5\,6\,7, 1\,7\,8\,9, 2\,5\,8\,10\}$,
9) $\{1\,2\,3\,4, 4\,5\,6\,7, 1\,7\,8\,9, 2\,7\,10\,11\}$,
10) $\{1\,2\,3\,4, 4\,5\,6\,7, 1\,7\,8\,9, 5\,8\,10\,11\}$,
11) $\{1\,2\,3\,4, 1\,5\,6\,7, 1\,8\,9\,10, 1\,11\,12\,13\}$.

However, no configuration is simultaneously a packing $P(13,4,2)$ because we can always add a further quadruple to any configuration, as shown below. Added quadruples are:

ad 1) 1 6 10 13, ad 2) 2 6 9 11, ad 3) 2 6 9 11,
ad 4) 1 6 9 12, ad 5) 2 6 10 12, ad 6) 2 5 8 12,
ad 7) 2 5 8 10, ad 8) 3 6 8 11, ad 9) 3 5 8 10,
ad 10) 3 5 9 12, ad 11) 3 5 8 11.

c) Analogously we can find all nonisomorphic configurations formed by five quadruples. Let us consider only configurations containing at most 14 points. We obtain 50 nonisomorphic configurations and then we can state the following result: it is possible to add a further quadruple to each configuration.

Therefore no $P_5(13,4,2)$ and no $P_5(14,4,2)$ exist. The 50 above mentioned configurations have been omitted in order to reduce the extent of the paper. Thus Theorem 2.5 holds. □

3. Construction of the packings $P(n,4,2)$ with minimum known sizes

Consider the sequence $\{m(n)\}$, $n = 4, 5, \ldots$, of minimum sizes of packings $P(n,4,2)$. The values of minimum sizes determinated until now indicate that the following conjecture is highly probable:

Conjecture 3.1. *The sequence $\{m(n)\}$ is never decreasing, i.e. $m(n) \leqslant m(n+1)$ for any $n > 3$.*

Denote by $\{m^*(n)\}$ a sequence of not necessarily minimum sizes of packings. If the value $m(n)$ was already determinated for some n, then put $m^*(n) = m(n)$. The purpose of this paper is to determine such values of members $m^*(n)$ that the relation $m^*(n) \leqslant m^*(n+1)$ holds for any $n > 3$. Then the members $m^*(n)$ will be called minimum known sizes of packings $P(n,4,2)$. This term is justified by the fact that our investigation is, as far as I now, the first attempt at finding such packings. It is probable that the values found can be made smaller in future.

First we shall describe methods for construction of packings $P(n,4,2)$ with minimum known sizes.

Method 3.1. Suppose that the basic set E of order $n > 13$ can be decomposed into two disjoint classes E_1, E_2 such that $|E_1| = n_1 \equiv 1$ or $4 \pmod{12}$, $|E_2| = n_2 \equiv 0$ or 1 or 3 or $4 \pmod{12}$. On the class E_1 we construct the exact packing $P_{\mathrm{MAX}}(n_1,4,2)$. If $n_2 \equiv 1$ or $4 \pmod{12}$ then we construct the exact packing $P_{\mathrm{MAX}}(n_2,4,2)$ on the class E_2 and the desired packing $P(n,4,2) = P_{\mathrm{MAX}}(n_1,4,2) \cup P_{\mathrm{MAX}}(n_2,4,2)$. If $n_2 \equiv 0$ or $3 \pmod{12}$, then we add one element $x_1 \in E_1$ to the class E_2 and construct on the set $E_2' = \{x_1\} \cup E_2$ the exact packing $P_{\mathrm{MAX}}(n_2+1,4,2)$. Then the desired packing is $P(n,4,2) = P_{\mathrm{MAX}}(n_1,4,2) \cup P_{\mathrm{MAX}}(n_2+1,4,2)$.

Method 3.2. Suppose that the basic set E can be decomposed into three mutually disjoint classes E_i such that $|E_i| = n_i \equiv 0$ or 1 or 3 or $4 \pmod{12}$, $i = 1, 2, 3$. If the cardinality is $n_i \equiv 1$ or $4 \pmod{12}$ for some i, then there exists an exact packing $P(n_i,4,2)$ on the class E_i. If the cardinality $n_i \equiv 0$ or $3 \pmod{12}$ for some i, then we add to the class E_i an element $x \in E_j$, $i \neq j$ and construct an exact packing $P(n_i+1,4,2)$ on the set $E_i' = \{x\} \cup E_i$. In this manner we can extend all classes E_i, $i = 1, 2, 3$. The desired packing $P(n,4,2)$ is the union of three exact packings on three sets having cardinalities either n_i or $n_i + 1$.

If some class E_i contains only one element then the packing $P(n,4,2)$ will be formed only by packings on the other two classes.

Method 3.3. Suppose that the basic set E can be decomposed into three classes E_1, E_2, E_3 such that $|E_1| = n_1 \equiv 1$ or $4 \pmod{12}$, $|E_2| = n_2 \equiv 1$ or $4 \pmod{12}$, $|E_3| = n_3 \equiv -1$ or $2 \pmod{12}$. On the classes E_1 and E_2 we construct exact packings $P_{\mathrm{MAX}}(n_1,4,2)$ and $P_{\mathrm{MAX}}(n_2,4,2)$, respectively. To the class E_3 we add one element $x_1 \in E_1$ and one element $x_2 \in E_2$ so that $n_3 + 2 \equiv 1$ or $4 \pmod{12}$. We obtain the set $E_3' = E_3 \cup \{x_1\} \cup \{x_2\}$, on which an exact packing $P_{\mathrm{MAX}}(n_3+2,4,2)$ exists. The desired packing is $P(n,4,2) = P_{\mathrm{MAX}}(n_1,4,2) \cup P_{\mathrm{MAX}}(n_2,4,2) \cup P_{\mathrm{MAX}}(n_3+2,4,2)$.

Method 3.4. Suppose that the basic set E can be decomposed into three classes E_i such that $|E_1| = n_1 \equiv 1$ or $4 \pmod{12}$, $|E_2| = n_2 \equiv 1$ or $4 \pmod{12}$, $|E_3| = n_3 = 5$ or 6 or 7 or 9. Let $n_i > 3$, $i = 1, 2, 3$.

a) Let $E_3 = \{a_i, i = 1,\dots,5\}$. Form two exact packings on classes E_1 and E_2. On the class E_3 form one quadruple $a_1a_2a_3a_4$ and four pairs a_1a_5, a_2a_5, a_3a_5, a_4a_5. Let x_i be mutually different elements of E_1, let Y_i be mutually different elements of E_2, $i = 1,\dots,4$. Form a system S of four quadruples, $S = \{x_1y_1a_1a_5, x_2y_2a_2a_5, x_3y_3a_3a_5, x_4y_4a_4a_5\}$. Then the desired packing is

$$P(n,4,2) = P_{\mathrm{MAX}}(|E_1|,4,2) \cup P_{\mathrm{MAX}}(|E_2|,4,2) \cup a_1a_2a_3a_4 \cup S.$$

b) Let $E_3 = \{a_i, i = 1, \ldots, 6\}$ or $E_3 = \{a_i, i = 1, \ldots, 7\}$. If $|E_3| = 6$, then form $E_3' = E_3 \cup \{x_1\}$, where $x_1 \in E_1$. Thus $|E_3'| = 7$ and we can form a Steiner triple system on E_3' having 7 triples. Add to each triple one element $y_i \in E_1 \cup E_2$, $i = 1, \ldots, 7$. The elements added must be mutually different and different from x_1. Only the elements $y_i \in E_2$ can be added to the triples containing x_1. We obtain a quadruple system $S = \{a_1a_2a_3y_1, a_1a_6x_1y_2, a_1a_4a_5y_3, a_2a_4a_6y_4, a_2a_5x_1y_5, a_3a_4x_1y_6, a_3a_5a_6y_7\}$. The desired packing then is

$$P(n,4,2) = P_{\mathrm{MAX}}(|E_1|,4,2) \cup P_{\mathrm{MAX}}(|E_2|,4,2) \cup S. \tag{1}$$

If $|E_3| = 7$, then $|E_3'| = E_3$, the quadruple system is $S = \{a_1a_2a_3y_1, a_1a_6a_7y_2, a_1a_4a_5y_3, a_2a_4a_6y_4, a_2a_5a_7y_5, a_3a_4a_7y_6, a_3a_5a_6y_7\}$ and the desired packing is (1).

c) Suppose $|E_1 \cup E_2| \geqslant 12$. If $E_3 = \{a_i, i = 1, \ldots, 9\}$, then form the Steiner triple system T on E_3 having 12 triples and add one element $x_i \in E_1 \cup E_2$ to each triple of T. The elements added must be mutually different. We obtain a quadruple system $S = \{a_1a_2a_3x_1,\ a_4a_5a_6x_2,\ a_7a_8a_9x_3,\ a_1a_4a_7x_4,\ a_1a_5a_8x_5,\ a_1a_6a_9x_6,\ a_2a_4a_8x_7,\ a_2a_5a_9x_8, a_2a_6a_7x_9, a_3a_4a_9x_{10}, a_3a_5a_7x_{11}, a_3a_6a_8x_{12}\}$. The desired packing is again (1).

Theorem 3.1. *Let $13 < n < 36$. Then the minimum known sizes $m^*(n)$ of packings $P(n,4,2)$ are given by the following values:*

n:	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$m^*(n)$:	9	12	14	14	15	15	15	19	21	21	26	26	26	27
n:	29	30	31	32	33	34	35							
$m^*(n)$:	27	27	31	33	33	38	38.							

Proof. a) $n = 15 = 4 + 4 + 7$. $E_1 = \{1,2,3,4\}$, $E_2 = \{5,6,7,8\}$, $E_3 = \{9,10,11,12,13,14,15\}$. We can apply method 3.4 b). The desired packing is $P_9(15,4,2) = \{1\,2\,3\,4, 5\,6\,7\,8, 1\,9\,10\,11, 2\,9\,14\,15, 3\,9\,12\,13, 4\,10\,12\,14, 5\,10\,13\,15, 6\,11\,12\,15, 7\,11\,13\,14\}$, $m^*(15) = 9$.

b) $n = 16 = 13 + 3$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14,15,16\}$. Applying the method 3.1 we obtain $P_{14}(16,4,2) = P_{13}(13,4,2) \cup 13\,14\,15\,16$. $P_{13}(13,4,2)$ denotes the maximum packing on E_1 having 13 quadruples (according to Theorem 2.2).

However, the size 14 can be improved if we investigate the configuration of 12 quadruples on 16 points. Let us introduce this configuration: $C = \{1\,2\,3\,4, 1\,5\,9\,13, 1\,7\,12\,14, 2\,6\,10\,14,\ 2\,8\,11\,15,\ 3\,7\,15\,16,\ 3\,8\,12\,13,\ 4\,6\,9\,15,\ 4\,10\,13\,16,\ 5\,6\,7\,8,\ 5\,11\,14\,16, 9\,10\,11\,12\}$. We will show that no quadruple with element 1 can be added. This element is in C separated from all elements of $S = \{6,8,10,11,15,16\}$. Form 20 quadruples each containing element 1 and one of 20 triples on the set S and determine whether they can be added to C. The answer is negative. In this way we prove that no quadruple with any element of $\{2, 3, \ldots, 16\}$ can be added to C. Therefore C is a packing and we have $m^*(16) = 12$.

$n = 17 = 13 + 4$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, 15, 16, 17\}$. Applying the method 3.1 we obtain $P_{14}(17, 4, 2) = P_{13}(13, 4, 2) \cup 14\,15\,16\,17$. Thus $m^*(17) = 14$.

$n = 18 = 13+4+1$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, \ldots, 17\}$, $E_3 = \{18\}$. Applying the method 3.2 (the last proposition) we obtain $P_{14}(18, 4, 2) = P_{13}(13, 4, 2) \cup 14\,15\,16\,17$. Thus $m^*(18) = 14$.

c) $n = 19 = 13+4+2$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, 15, 16, 17\}$, $E_3 = \{18, 19\}$. Applying the method 3.3 we obtain $P_{15}(19, 4, 2) = P_{13}(13, 4, 2) \cup 14\,15\,16\,17 \cup 13\,14\,18\,19$. Thus $m^*(19) = 15$.

$n = 20 = 13+4+3$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, 15, 16, 17\}$, $E_3 = \{18, 19, 20\}$. Applying the method 3.2 we obtain $P_{15}(20, 4, 2) = P_{13}(13, 4, 2) \cup 14\,15\,16\,17 \cup 17\,18\,19\,20$. Thus $m^*(20) = 15$.

$n = 21 = 13 + 4 + 4$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, 15, 16, 17\}$, $E_3 = \{18, 19, 20, 21\}$. Applying the method 3.2 we obtain $P_{15}(21, 4, 2) = P_{13}(13, 4, 2) \cup 14\,15\,16\,17 \cup 18\,19\,20\,21$. Thus $m^*(21) = 15$.

d) $n = 22 = 13 + 4 + 5$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, 15, 16, 17\}$, $E_3 = \{18, 19, 20, 21, 22\}$. Applying the method 3.4 a) we obtain $P_{19}(22, 4, 2) = P_{13}(13, 4, 2) \cup 14\,15\,16\,17 \cup \{18\,19\,20\,21, 1\,14\,18\,22, 2\,5\,19\,22, 4\,17\,21\,22\}$. Thus $m^*(22) = 19$.

e) $n = 23 = 13 + 4 + 6$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, 15, 16, 17\}$, $E_3 = \{18, 19, 20, 21, 22, 23\}$. Form $E'_3 = E_3 \cup \{17\}$, then $|E'_3| = 7$ and we can apply the method 3.4 b). The desired packing is $P_{21}(23, 4, 2) = P_{13}(13, 4, 2) \cup 14\,15\,16\,17 \cup \{1\,18\,19\,20, 2\,18\,21\,22, 3\,19\,21\,23, 4\,20\,22\,23, 5\,17\,19\,20, 6\,17\,20\,21, 7\,17\,18\,21\}$.

Thus $m^*(23) = 21$.

$n = 24 = 13 + 4 + 7$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, 15, 16, 17\}$, $E_3 = \{18, 19, 20, 21, 22, 23, 24\}$. Applying the method 3.4 b) we obtain $P_{21}(24, 4, 2) = P_{13}(13, 4, 2) \cup 14\,15\,16\,17 \cup \{1\,18\,19\,20,\ 2\,18\,23\,24,\ 3\,18\,21\,22,\ 4\,19\,21\,23,\ 5\,19\,22\,24,\ 6\,20\,21\,24,\ 7\,20\,22\,23\}$.

Thus $m^*(24) = 21$.

f) $n = 25 = 13 + 12$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, \ldots, 25\}$, $E'_2 = E_2 \cup \{13\}$. Applying the method 3.1 we obtain $P_{26}(25, 4, 2) = P_{13}(|E_1|, 4, 2) \cup P_{13}(|E'_2|, 4, 2)$. Thus $m^*(25) = 26$.

$n = 26 = 13 + 13$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, \ldots, 26\}$, $E'_2 = E_2$. Applying the method 3.1 we obtain $P_{26}(26, 4, 2) = P_{13}(|E_1|, 4, 2) \cup P_{13}(|E_2|, 4, 2)$. Thus $m^*(26) = 26$.

$n = 27 = 13 + 13 + 1$. $E_1 = \{1, \ldots, 13\}$, $E_2 = \{14, \ldots, 26\}$, $E_3 = \{27\}$. Applying the method 3.2 (the last proposition) we obtain $P_{26}(27, 4, 2) = P_{13}(|E_1|, 4, 2) \cup P_{13}(|E_2|, 4, 2)$. Thus $m^*(27) = 26$.

g) $n = 28 = 13+13+2$. $E_1 = \{1,\dots,13\}$, $E_2 = \{14,\dots,26\}$, $E_3 = \{27,28\}$, $E_3' = \{13,26,27,28\}$. Applying the method 3.3 we obtain $P_{27}(28,4,2) = P_{13}(|E_1|,4,2) \cup P_{13}(|E_2|,4,2) \cup 13\,26\,27\,28$. Thus $m^*(28) = 27$.

$n = 29 = 13+13+3$. $E_1 = \{1,\dots,13\}$, $E_2 = \{14,\dots,26\}$, $E_3 = \{27,28,29\}$. Applying the method 3.2 we obtain $E_3' = \{26,27,28,29\}$. $P_{27}(29,4,2) = P_{13}(|E_1|,4,2) \cup P_{13}(|E_2|,4,2) \cup 26\,27\,28\,29$. Thus $m^*(29) = 27$.

$n = 30 = 13+13+4$. $E_1 = \{1,\dots,13\}$, $E_2 = \{14,\dots,26\}$, $E_3 = \{27,28,29,30\}$. Applying the method 3.2 we obtain $P_{27}(30,4,2) = P_{13}(|E_1|,4,2) \cup P_{13}(|E_2|,4,2) \cup 27\,28\,29\,30$. Thus $m^*(30) = 27$.

h) $n = 31 = 13+13+5$. $E_1 = \{1,\dots,13\}$, $E_2 = \{14,\dots,26\}$, $E_3 = \{27,28,29,30,31\}$. Using the method 3.4 a) we obtain $P_{31}(31,4,2) = P_{13}(|E_1|,4,2) \cup P_{13}(|E_2|,4,2) \cup 27\,28\,29\,30 \cup \{1\,14\,27\,31, 2\,15\,28\,31, 3\,16\,29\,31, 4\,17\,30\,31\}$.

Thus $m^*(31) = 31$.

i) $n = 32 = 13+13+6$. $E_1 = \{1,\dots,13\}$, $E_2 = \{14,\dots,26\}$, $E_3 = \{27,28,29,30,31,32\}$. $E_3' = E_3 \cup \{26\}$. Applying the method 3.4 b) we obtain $P_{33}(32,\ 4,\ 2) = P_{13}(|E_1|,\ 4,\ 2) \cup P_{13}(|E_2|,\ 4,\ 2) \cup \{1\,26\,27\,28,\ 2\,26\,31\,32, 4\,27\,29\,31, 4\,27\,29\,31, 5\,27\,30\,32, 6\,28\,29\,32, 7\,28\,30\,31\}$. Thus $m^*(32) = 33$.

$n = 33 = 13+13+7$. $E_1 = \{1,\dots,13\}, E_2 = \{14,\dots,26\}$, $E_3 = \{27,28,29,30,31,32,33\}$. Applying the method 3.4 b) we obtain $P_{33}(33,4,2) = P_{13}(|E_1|,4,2) \cup P_{13}(|E_2|,4,2) \cup \{1\,27\,28\,29,\ 2\,27\,32\,33,\ 3\,27\,30\,31,\ 4\,28\,30\,32,\ 5\,28\,31\,33,\ 6\,29\,30\,33, 7\,29\,31\,32\}$. Thus $m^*(33) = 33$.

j) $n = 34 = 16+13+5$. $E_1 = \{1,\dots,16\}$, $E_2 = \{17,\dots,29\}$, $E_3 = \{30,31,32,33,34\}$. Applying the method 3.4 a) we obtain $P_{38}(34,4,2) = P_{20}(|E_1|,4,2) \cup P_{13}(|E_2|,4,2) \cup \{30\,31\,32\,33, 1\,17\,30\,34, 2\,18\,31\,34, 3\,19\,32\,34, 4\,20\,33\,34\}$.

Thus $m^*(34) = 38$.

$n = 35 = 13+13+9$. $E_1 = \{1,\dots,13\}$, $E_2 = \{14,\dots,26\}$, $E_3 = \{27,28,29,30,31,32,33,34,35\}$. Applying the method 3.4 c) we obtain $P_{38}(35,4,2) = P_{13}(|E_1|,4,2) \cup P_{13}(|E_2|,4,2) \cup \{1\,27\,28\,29,\ 2\,30\,31\,32,\ 3\,33\,34\,35,\ 4\,27\,30\,33,\ 5\,27\,31\,34, 6\ \ 27\,32\,35, 7\,28\,30\,34, 8\,28\,31\,35, 9\,28\,32\,33, 10\,29\,30\,35, 11\,29\,31\,33, 12\,29\,32\,34\}$. Thus $m^*(35) = 38$. $\square$

Theorem 3.2. *Let $j \geqslant 0$, $t > 0$, $n > 0$ be nonnegative integers such that $n = 36t + j$, $j \in \{0,1,2,4,5,7,8,10,11\} \cup \{13,14,\dots,35\}$. Then the minimum known sizes $m^*(n)$ in the packings $P(n,4,2)$ are given by the following values:*

$n:$	$36t$	$36t+1$	$36t+2$	$36t+4$	$36t+5$
$m^*(n):$	$36t^2+3t$	$36t^2+3t$	$36t^2+3t$	$36t^2+9t+1$	$36t^2+9t+1$
$n:$	$36t+7$	$36t+8$	$36t+10$	$36t+11$	$36t+13$
$m^*(n):$	$36t^2+15t+2$	$36t^2+15t+2$	$36t^2+21t+3$	$36t^2+21t+3$	$36t^2+27t+13$
$n:$	$36t+14$	$36t+15$	$36t+16$	$36t+17$	$36t+18$

$m^*(n)$:	$36t^2+27t+13$	$36t^2+27t+13$	$36t^2+33t+14$	$36t^2+33t+14$	$36t^2+36t+14$
n :	$36t+19$	$36t+20$	$36t+21$	$36t+22$	$36t+23$
$m^*(n)$:	$36t^2+39t+15$	$36t^2+39t+15$	$36t^2+39t+15$	$36t^2+45t+22$	$36t^2+45t+22$
n :	$36t+24$	$36t+25$	$36t+26$	$36t+27$	$36t+28$
$m^*(n)$:	$36t^2+45t+22$	$36t^2+51t+26$	$36t^2+51t+26$	$36t^2+51t+26$	$36t^2+57t+27$
n :	$36t+29$	$36t+30$	$36t+31$	$36t+32$	$36t+33$
$m^*(n)$:	$36t^2+57t+27$	$36t^2+57t+27$	$36t^2+63t+40$	$36t^2+63t+40$	$36t^2+63t+40$
n :	$36t+34$	$36t+35$			
$m^*(n)$:	$36t^2+69t+41$	$36t^2+69t+41$.			

Proof. For each order n of a packing we state the decomposition of the basic set E into three disjoint classes E_i, giving their cardinalities $n_i = |E_i|$, $i = 1, 2, 3$. Then we state the method of construction of three exact packings the union of which is the desired packing of order n and of size $m^*(n)$. The calculation of the size $m^*(n)$ will be done only as an example. The results for the orders n are as follows.

$$n = 36t = 12t + 12t + 12t, \quad n_i = 12t, \quad i = 1,2,3, \text{ method 3.2.}$$

$$\text{Size } m^*(n) = 3\binom{12t+1}{2}\cdot\frac{1}{6} = 36t^2 + 3t.$$

$$n = 36t + 1 = (12t+1) + 12t + 12t,$$

$$n_1 = 12t+1,\ n_2 = 12t,\ n_3 = 12t, \text{ method 3.2.}$$

$$n = 36t + 2 = (12t+1) + (12t+1) + 12t, \text{ method 3.2.}$$

$$n = 36t + 4 = (12t+1) + (12t+1) + (12t+2), \text{ method 3.3.}$$

$$m^*(n) = 2\binom{12t+1}{2}\cdot\frac{1}{6} + \binom{12t+4}{2}\cdot\frac{1}{6} = 36t^2 + 9t + 1.$$

$$n = 36t + 5 = (12t+4) + (12t+1) + 12t, \text{ method 3.2.}$$

$$n = 36t + 7 = (12t+4) + (12t+3) + 12t, \text{ method 3.2.}$$

$$n = 36t + 8 = (12t+4) + (12t+4) + 12t, \text{ method 3.2.}$$

$$n = 36t + 10 = (12t+4) + (12t+3) + (12t+3), \text{ method 3.2.}$$

$$n = 36t + 11 = (12t+4) + (12t+4) + (12t+3), \text{ method 3.2.}$$

$$n = 36t + 13 = (12t+13) + 12t + 12t, \text{ method 3.2.}$$

$$n = 36t + 14 = (12t+13) + (12t+1) + 12t, \text{ method 3.2.}$$

$$n = 36t + 15 = (12t+13) + (12t+1) + (12t+1), \text{ method 3.2.}$$

$$n = 36t + 16 = (12t+13) + (12t+3) + 12t, \text{ method 3.2.}$$

$$n = 36t + 17 = (12t+13) + (12t+4) + 12t, \text{ method 3.2.}$$

$$n = 36t + 18 = (12t+13) + (12t+4) + (12t+1), \text{ method 3.2.}$$

$$n = 36t + 19 = (12t+13) + (12t+3) + (12t+3), \text{ method 3.2.}$$

$$n = 36t + 20 = (12t+13) + (12t+4) + (12t+3), \text{ method 3.2.}$$

$$n = 36t + 21 = (12t + 13) + (12t + 4) + (12t + 4), \text{ method 3.2.}$$

$$n = 36t + 22 = (12t + 4) + (12t + 4) + (12t + 14), \text{ method 3.3.}$$

$$\text{Size } m^*(n) = 2\binom{12t+4}{2} \cdot \frac{1}{6} + \binom{12t+16}{2} \cdot \frac{1}{6} = 36t^2 + 45t + 22.$$

$$n = 36t + 23 = (12t + 4) + (12t + 4) + (12t + 15), \text{ method 3.2.}$$

$$n = 36t + 24 = (12t + 4) + (12t + 4) + (12t + 16), \text{ method 3.2.}$$

$$n = 36t + 25 = (12t + 13) + (12t + 12) + 12t, \text{ method 3.2.}$$

$$n = 36t + 26 = (12t + 13) + (12t + 13) + 12t, \text{ method 3.2.}$$

$$n = 36t + 27 = (12t + 13) + (12t + 13) + (12t + 1), \text{ method 3.2.}$$

$$n = 36t + 28 = (12t + 13) + (12t + 13) + (12t + 2), \text{ method 3.3.}$$

$$m^*(n) = 36t^2 + 57t + 27.$$

$$n = 36t + 29 = (12t + 13) + (12t + 13) + (12t + 3), \text{ method 3.2.}$$

$$n = 36t + 30 = (12t + 13) + (12t + 13) + (12t + 4), \text{ method 3.2.}$$

$$n = 36t + 31 = (12t + 16) + (12t + 15) + 12t, \text{ method 3.2.}$$

$$n = 36t + 32 = (12t + 16) + (12t + 16) + 12t, \text{ method 3.2.}$$

$$n = 36t + 34 = (12t + 16) + (12t + 16) + (12t + 2), \text{ method 3.3.}$$

$$n = 36t + 35 = (12t + 16) + (12t + 16) + (12t + 3), \text{ method 3.2.}$$

$$m^*(n) = 36t^2 + 69t + 41.$$

□

Remark. Note that $m^*(36t+39) = m(36t+39) = \binom{12t+13}{2} \cdot \frac{3}{6} = 36t^2 + 75t + 39$.

References

[1] *A.E. Brouwer*: Optimal packings of K_4's into a K_n. J. Combinatorial Theory *26* (1979), 278–297.

[2] *H. Hanani*: The existence and construction of balanced incomplete block design. Ann. Math. Statist. *32* (1961), 361–386.

[3] *J. Novák*: Edge-bases of complete uniform hypergraphs. Mat. čas. *24* (1974), 43–57.

[4] *C. Colbourn, A. Rosa, Š. Znám*: The spectrum of maximal partial Steiner triple systems. Math. Reports Mc. Master University. 1991.

[5] *P. Turán*: On the theory of graphs. Colloq. Math. *3* (1955), 19–30.

Author's address: *Jiří Novák*, Hálkova 1264/1, 460 01 Liberec I, Czech Republic.