SUBTRACTION SEMIGROUPS

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Abstract. A subtraction semigroup is a semigroup $(A, ., -)$ with a further operation "$-$" added, called subtraction and satisfying certain axioms. The paper concerns a problem by B. M. Schein concerning the structure of multiplication in a subtraction semigroup.

Keywords: subtraction semigroup, subtraction algebra, Boolean algebra

MSC 1991: 20M20, 06E05

This paper concerns a problem by B. M. Schein [2] concerning subtraction semigroups.

A subtraction algebra is a grupoid $(A, -)$ where $-$ is a binary operation called subtraction; this subtraction satisfies the following axioms (for any elements x, y, z):

$$x - (y - x) = x;$$
$$x - (x - y) = y - (y - x);$$
$$(x - y) - z = (x - z) - y.$$

If to a subtraction algebra a semigroup multiplication is added satisfying the distributivity laws

$$x(y - z) = xy - xz,$$
$$(y - z)x = yx - zx$$

then the resulting algebra $(A, ., -)$ is called a subtraction semigroup.

In [2] B. M. Schein proposed the problem to characterize semigroups which can become subtraction semigroups by adding a suitable subtraction. Here we will solve this problem for subtraction algebras of a special type, the so-called atomic subtraction algebras.

In [2] it is proved that in every subtraction algebra $(A, -)$ there exists an element o such that $x - x = o$ for each $x \in A$. In a subtraction semigroup this element o is the zero element for multiplication. If we denote $x \wedge y = x - (x - y)$, then the operation $\wedge$ is idempotent and commutative; therefore $(A, \wedge)$ is a semilattice. The ordering $\leqslant$ in this semilattice is determined so that $x \leqslant y$ if and only if $x - y = o$. This semilattice is a semi-Boolean algebra in the sense of [1], i.e. a semilattice with zero o in which every interval $[o, a]$ is a Boolean algebra.

A subtraction algebra $(A, -)$ or a subtraction semigroup $(A, ., -)$ is called atomic, if for any x, y we have $x - x = o$ and $x - y = x$ for $x \neq y$. The ordering $\leqslant$ in an atomic subtraction algebra is such that $o \leqslant x$ for each x and any two elements x, y such that $x \neq y$, $x \neq o$, $y \neq o$ are incomparable. Therefore the corresponding semilattice consists of the least element o and of atoms.

Theorem. *Let $(S, .)$ be a semigroup with zero o. The semigroup $(S, .)$ can become an atomic subtraction semigroup by adding a suitable subtraction if and only if the following two conditions are satisfied:*

(i) *$ax = ay$ implies $x = y$ or $ax = ay = o$;*

(ii) *$xa = ya$ implies $x = y$ or $xa = ya = o$.*

Proof. Let (i) and (ii) be satisfied. We define the operation "$-$" so that $x - x = o$ for each $x \in S$ and $x - y = x$ for any $x \in S$, $y \in S$, $x \neq y$. Now let x, y, z be elements of S. If $y = z$, then $y - z = o$, $x(y - z) = xo = o$, $xy - xz = xy - xy = o$, $(y - z)x = ox = o$, $yx - zx = yx - yx = o$ and the distributivity laws hold. If $y \neq z$, then $y - z = y$, $x(y - z) = xy$, $(y - z)x = yx$. As (i) holds, we have either $xy \neq xz$ and $xy - xz = xy$, or $xy = xz = o$ and $xy - xz = o = xy$. As (ii) holds, we have either $yx \neq zx$ and $yx - zx = yx$, or $yx = zx = o$ and $yx - zx = o = yx$. Therefore again the distributive laws hold and $(S, ., -)$ is a subtractive semigroup.

Now suppose that (i) does not hold. Then there exist elements a, x, y of S such that $x \neq y$ and $ax = ay \neq o$. At least one of the elements x, y is different from o; without loss of generality suppose that $x \neq o$. We have $x - y = x \neq o$ and $ax - ay = ax - ax = o \neq ax$. Therefore $a(x - y) \neq ax - ay$ and we have not a subtractive semigroup. In the case when (ii) does not hold we proceed analogously. $\square$

Let $\mathcal{S}$ be a family of pairwise disjoint semigroups. We shall define the semigroup $\Sigma(\mathcal{S})$. The set of elements of $\Sigma(\mathcal{S})$ is $\{o\} \cup \bigcup_{s \in \mathcal{S}} S$, where o is an element contained in no semigroup from $\mathcal{S}$. Now let x, y be two elements of $\Sigma(\mathcal{S})$. If a semigroup $S \in \mathcal{S}$ contains both x and y, then the product xy in $\Sigma(\mathcal{S})$ is equal to the product xy in S; otherwise $xy = o$.

Corollary. *Let $\mathcal{S}$ be a family of pairwise disjoint cancellative semigroups. Then $\Sigma(\mathcal{S})$ can become a subtraction semigroup by adding a suitable subtraction.*

We do not exclude the case when $\mathcal{S}$ contains only one element; therefore the assertion is true for every cancellative semigroup with outer zero added and in particular for every group with outer zero added. (An outer zero is the zero element which is a product of no two non-zero elements.)

References

[1] *Abbott J. C.*: Sets, lattices and Boolean algebras. Allyn and Bacon, Boston, 1969.
[2] *Schein B. M.*: Difference semigroups. Communications in Algebra *20* (1992), 2153–2169.

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