# CONVEXITIES OF LATTICE ORDERED GROUPS

Ján Jakubík, Košice

(Received August 3, 1994)

*Abstract.* In this paper an injective mapping of the class of all infinite cardinals into the collection of all convexities of lattice ordered groups is constructed; this generalizes an earlier result on convexities of $d$-groups.

*Keywords*: lattice ordered group, convex $\ell$-subgroup, direct product, convexity of lattice ordered groups

*MSC 1991*: 06F15

The notion of convexity of lattices has been introduced by E. Fried ([9], p. 225; cf. also [4]). By applying analogous postulates we can define convexities also for other types of ordered algebraic structures.

In the present paper the collection $\mathcal{C}(\mathcal{L})$ of all convexities of lattice ordered groups will be investigated.

An injective mapping of the class of all infinite cardinals into the collection $\mathcal{C}(\mathcal{L})$ will be constructed; hence $\mathcal{C}(\mathcal{L})$ is a proper class. This generalizes a result from [6] concerning convexities of $d$-groups.

The notion of torsion class is due to J. Martinez [8]. For some torsion classes (which have been studied in literature) we shall deal with the question whether they are convexities.

Supported by SAV grant 1230/94

## 1. Preliminaries

We shall apply the standard notation for lattice ordered groups. The group operation in a lattice ordered group will be written additively; the commutativity of this operation will not be assumed.

Let $\mathcal{L}$ be the class of all lattice ordered groups. A nonempty subclass of $\mathcal{L}$ will be said to be a convexity of lattice ordered groups if it is closed under homomorphic images, convex $\ell$-subgroups and direct products.

We denote by $\mathcal{C}(\mathcal{L})$ the collection of all convexities of all lattice ordered groups. This collection is partially ordered by inclusion. The least element of $\mathcal{C}(\mathcal{L})$ is the class $X_0$ consisting of all one-element lattice ordered groups.

Let $\emptyset \neq X \subseteq \mathcal{L}$. We denote by

$HX$—the class of all homomorphic images of elements of $X$;
$CX$—the class of all convex $\ell$-subgroups of elements of $X$;
$PX$—the class of all direct products of elements of $X$.

**1.1. Lemma.** *Let $\emptyset \neq X \subseteq \mathcal{L}$. Then*

(i) $HCPX \in \mathcal{C}(\mathcal{L})$;
(ii) *for each $Y \in \mathcal{C}(\mathcal{L})$ with $X \subseteq Y$ the relation $HCPX \subseteq Y$ is valid.*

The proof will be omitted. For analogous results concerning convexities of lattices and convexities of $d$-groups cf. [9], p. 256 and [6].

In view of 1.1. the convexity $HCPX$ will be said to be generated by $X$.

The direct product of lattice ordered groups $A$ and $B$ will be denoted by $A \times B$. If $I$ is any nonempty system of indices and $G_i \in \mathcal{L}$ for each $i \in I$, then $\prod\limits_{i\in I} G_i$ denotes the direct product of the system $\{G\}_{i\in I}$. If $I = \emptyset$, then we put $\prod\limits_{i\in I} G_i = \{0\}$.

When no confusion can occur, then for $j \in I$ the lattice ordered group $G_j$ will be identified with the $\ell$-subgroup of $\prod\limits_{i\in I} G_i$ consisting of all elements $g$ of the direct product under consideration such that $g(i) = 0$ for each $i \in I \setminus \{j\}$.

If $G \in \mathcal{L}$, $g \in G$ and if $D$ is an $\ell$-ideal of $G$, then we put $\overline{x} = x + D$; for $X \subseteq G$ we set $\overline{X} = \{\overline{x}\colon x \in X\}$.

We will apply below the following well-known results:

**1.2. Lemma.** *Let $G \in \mathcal{L}$, $G = A \times B$ and let $D$ be a convex $\ell$-subgroup of $G$. Then $D = (A \cap D) \times (B \cap D)$.*

**1.3. Lemma.** *Let $G$, $A$ and $B$ be as in* 1.2. *Let $D$ be an $\ell$-ideal of $G$. Then $\overline{G} = G/D = \overline{A} \times \overline{B}$.*

## 2. The lattice ordered groups $G_\alpha$

For each infinite cardinal $\alpha$ we denote by $J_\alpha$ the first ordinal having the power $\alpha$.

The additive group of all integers with the natural linear order will be denoted by $Z$. Let $\alpha$ be a fixed infinite cardinal and for each $j \in J_\alpha$ let $P_j = Z$. Now let $Q'(\alpha)$ be the lexicographic product of the system $\{P_j\}(j \in J_\alpha)$. The $\ell$-subgroup of $Q'(\alpha)$ consisting of all elements $q'$ such that the set $\{j \in J(\alpha) \colon q'(j) \neq 0\}$ is finite will be denoted by $Q(\alpha)$.

Let $G_\alpha$ be the set of all triples $(x, y, z)$ such that $x, y \in Q(\alpha)$ and $z \in Z$. For $(x_1, y_1, z_1), (x_2, y_2, z_2) \in G_\alpha$ we put $(x_1, y_1, z_1) \leqslant (x_2, y_2, z_2)$ if either

(i) $z_1 < z_2$,

or

(ii) $z_1 = z_2$ and $x_1 \leqslant x_2$, $y_1 \leqslant y_2$.

Next we define the binary operation $+$ in $G_\alpha$ as follows.

a) If $z_1$ is even, then we put

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

b) If $z_1$ is odd, then we define

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + y_2, y_1 + x_2, z_1 + z_2).$$

Then $G_\alpha$ is a non-abelian lattice ordered group. Clearly $\operatorname{card} G_\alpha = \alpha$.

The class of all infinite cardinals will be denoted by $J$.

**2.1. Lemma.** *Let $\alpha, \beta \in J$, $\beta < \alpha$. Then $G_\beta$ does not belong to the class $HCP\{G_\alpha\}$.*

Proof. By way of contradiction, assume that $G_\beta$ belongs to $HCP\{G_\alpha\}$. Thus there exist $B \in CP\{G_\alpha\}$ and an $\ell$-ideal $D$ of $B$ such that $B/D$ is isomorphic to $G_\beta$. Next, there is an indexed system $\{A_i\}_{i \in I}$ of lattice ordered groups such that $A_i = G_\alpha$ for each $i \in I$ and $B$ is a convex $\ell$-subgroup of the lattice ordered group $A = \prod\limits_{i \in I} A_i$.

For $a \in A$ we denote by $a_i$ the component of $a$ in the direct factor $A_i$. Let $b \in B \setminus D$, $i \in I$, $b_i = (x_i, y_i, z_i)$. If for each such $b$ and each $i \in I$ the relation $z_i = 0$ is valid, then $B/D$ is commutative; since $G_\beta$ fails to be commutative, we obtain a contradiction. Therefore there exists $b \in B \setminus D$ such that $z_i \neq 0$ for some $i \in I$. Denote

$$I_1 = \{i \in I \colon z_i \neq 0\}, \quad I_2 = I \setminus I_1.$$

Hence $I_1 \neq \emptyset$. Put

$$\begin{aligned} A^1 &= \prod A_i \quad (i \in I_1), & A^2 &= \prod A_i \quad (i \in I_2), \\ B^1 &= B \cap A^1 & B^2 &= B \cap A^2. \end{aligned}$$

Then $A = A^1 \times A^2$, hence in view of 1.2 and 1.3,

$$B/D = \overline{B}_1 \times \overline{B}_2. \tag{1}$$

There exist elements $b'$ and $b''$ in $A$ such that

$$b' = (x_i, y_i, 0)_{i \in I}, \quad b'' = (0, 0, z_i)_{i \in I}.$$

For each $i \in I$ we have

$$-|b_i| \leqslant b'_i \leqslant |b_i|, \quad -2|b_i| \leqslant b''_i \leqslant 2|b_i|,$$

hence both $b'$ and $b''$ belong to $B$.

For each $t \in B$ we put $\overline{t} = t + D$. Clearly $b = b' + b''$ and $\overline{b} = \overline{b'} + \overline{b''}$. If $\overline{b''} = \overline{0}$ (i.e., $\overline{b} = \overline{b'}$) for all $b \in B$ with the above mentioned properties, then $B/D$ would be abelian, which is impossible. Hence without loss of generality we can suppose that $b = b''$. Further, we can suppose that $b' > 0$.

We have $b \in B$ and $\overline{b} \neq \overline{0}$, whence $\overline{B^1}$ is a nonzero lattice ordered group. It is obvious that $G_\beta$ is directly indecomposable, thus so is $B/D$. Hence (1) yields that $B/D = \overline{B}_1$. Therefore we can assume without loss of generality that $I = I_1$, whence

$$z_i > 0 \quad \text{for each } i \in I. \tag{2}$$

The relation (2) yields that whenever $b^1 \in A$ such that $b^1_i = (x^1_i, y^1_i, 0)$ for each $i \in I$, then $-b < b^1 < b$, whence $b^1 \in B$.

If for each $b^1$ with the above mentioned properties the relation $b^1 \in D$ holds, then $B/D$ is commutative, which is a contradiction. Hence among the elements $b^1$ under consideration there exists at least one with $b^1 \neq D$. Below we deal with this fixed $b^1$.

Let $b^{11}$ be the element of $A$ with $b^{11}_i = (x^1_i, 0, 0)$ for each $i \in I$; similarly, let $b^{12} \in A$ such that $b^{12}_i = (0, y^1_i, 0)$ for each $i \in I$. Then either $b^{11}$ or $b^{12}$ does not belong to $D$. Without loss of generality we can suppose that $b^{11} \notin D$ and that $b^{11} > 0$.

Let $I_{11} = \{i \in I \colon b^{11}_i \neq 0\}$, $I_{12} = I \setminus I_{11}$. Next, we put

$$\begin{aligned} B^{11} &= \{t \in B \colon t_i = 0 \quad \text{for each } i \in I_{12}\}, \\ B^{12} &= \{t \in B \colon t_i = 0 \quad \text{for each } i \in I_{11}\}. \end{aligned}$$

Then $B = B^{11} \times B^{12}$. Hence in view of 1.3,

$$B/D = \overline{B^{11}} \times \overline{B^{12}}.$$

Clearly $b^{11} \in B^{11} \setminus D$, therefore $\overline{b^{11}} \in \overline{B^{11}}$ and $\overline{b^{11}} \neq \overline{0}$. Thus $\overline{B^{11}} \neq \{\overline{0}\}$. From the fact that $G_\beta$ is directly indecomposable we obtain that $B/D = \overline{B^{11}}$. Now it is obvious that instead of $A$ and $B$ it suffices to take the lattice ordered groups

$$\prod_{i \in I_{11}} A_i, \quad B \cap \prod_{i \in I_{11}} A_i,$$

respectively. This means that without loss of generality we can suppose the validity of the relation $I = I_{11}$. Hence $b_i^{11} > 0$ for each $i \in I$. Hence $x_i^1 > 0$ for each $i \in I$.

Now we apply the fact that $x_i^1$ belongs to $Q_\alpha$. Let $j(i)$ be the least element of $J_\alpha$ with $x_i^1(j(i)) \neq 0$.

In view of the definition of $J_\alpha$ there exists a monotone injection $\psi_i$ of $J_\alpha$ onto $\{j \in J_\alpha \colon j \geqslant j(i)\}$.

Let $J_\alpha^0$ be the set of all elements of $J_\alpha$ which are distinct from the least element of $J_\alpha$. We construct the elements $b^j$ ($j \in J_\alpha^0$) in $A$ as follows. For each $i \in I$ and $j \in J_\alpha$ let $b_i^j = (x_i^j, 0, 0)$ where, for each $j(1) \in J_\alpha$, we have

$x_i^j(j(1)) = 1$ if $j(1) = \psi_i(j)$ and

$x_i^j(j(1)) = 0$ otherwise.

Thus $b^j \in B$ for each $j \in J_\alpha^0$.

If $j(1)$ and $j(2)$ are elements of $J_\alpha^0$ with $j(1) < j(2)$, then

$$|b^1| < b^{j(2)} - b^{j(1)}. \tag{3}$$

Hence if $b^{j(2)} - b^{j(1)} \in D$, we would have $b^1 \in D$, which is a contradiction. Therefore $b^{j(1)} + D$ and $b^{j(2)} + D$ are distinct elements of $B/D$. Thus $\operatorname{card}(B/D) \geqslant \operatorname{card} J_\alpha^0 = \alpha$. On the other hand, $\operatorname{card}(B/D) = \operatorname{card} G_\beta = \beta$ and so we arrived at a contradiction. □

**2.2. Theorem.** *For each infinite cardinal $\alpha$ let $\varphi(\alpha) = HCP\{G_\alpha\}$, where $G_\alpha$ is as above. Then $\varphi$ is an injective mapping of the class $J$ of all infinite cardinals into the collection of all convexities of lattice ordered groups.*

Proof. This is a consequence of 1.1. and 2.1. □

We apply the notion of a $d$-group in the same sense as in the paper of Kopytov and Dimitrov [7]; cf. also [5]. Convexities of $d$-groups were investigated in [6].

Let $\mathcal{D}$ be the class of all $d$-groups and $\mathcal{C}(\mathcal{D})$ the collection of all convexities of $d$-groups. Since the class $\mathcal{L}$ of all lattice ordered groups is a variety in $\mathcal{D}$ (cf. [7])

and since each variety in $\mathcal{D}$ is an element of $\mathcal{C}(\mathcal{D})$, we conclude that each convexity of lattice ordered groups is, at the same time, a convexity of $d$-groups. In fact, $\mathcal{C}(\mathcal{L})$ is an interval of $\mathcal{C}(\mathcal{D})$. Thus 2.2 implies

**2.3. Corollary.** (Cf. [6].) *There exists an injective mapping of the class of all infinite cardinals into the collection $\mathcal{C}(\mathcal{D})$.*

## 3. Concluding remarks; radical classes and torsion classes

**3.1.** Each variety of lattice ordered groups is a convexity. This is an immediate consequence of the definition of convexity.

**3.2.** A convexity of lattice ordered groups need not be closed with respect to $\ell$-subgroups. For example, let $G_\alpha$ and $G_\beta$ be as in Section 2 ($\alpha$ and $\beta$ are infinite cardinals with $\beta < \alpha$). Then $G_\beta$ is isomorphic to an $\ell$-subgroup of $G_\alpha$, but $G_\beta$ does not belong to the convexity generated by $G_\alpha$.

**3.3.** A nonempty class $X$ of lattice ordered groups is said to be closed under joins of convex $\ell$-subgroups if, whenever $G \in \mathcal{L}$ and $\{G_i\}_{i\in I}$ is a system of convex $\ell$-subgroups of $G$ such that $G_i$ belongs to $X$ for each $i \in I$, then the join $\bigvee_{i\in I} G_i$ also belongs to $X$.

A nonempty class $Y$ of lattice ordered groups is called a radical class [3] if it is closed under isomorphisms, convex $\ell$-subgroups and joins of convex $\ell$-subgroups.

A radical class which is closed under direct products is called a product radical class; this notion was studied by Dao Rong Ton [2]. Hence a product radical class which is closed under homomorphic images is a particular case of convexity.

A radical class of lattice ordered groups need not be a convexity. For example, the class of all archimedean lattice ordered groups is a radical class, but it fails to be a convexity (since it is not closed under homomorphic images).

**3.4.** A radical class which is closed under homomorphic images is called a torsion class (Martinez [8]). A torsion class is a convexity iff it is closed under direct products.

The main results of Conrad's paper [1] consist in a detailed investigation of torsion classes **A**, **F**, $\mathbf{F}_v$, **D**, **O**, **R** and **B** (for definitions of these classes cf. below; they have been studied also in other papers). Let us consider the question which of these torsion classes are convexities.

The torsion classes under consideration are defined as follows:

**A**—all hyperarchimedean lattice ordered groups;

**F**—all lattice ordered groups such that each bounded disjoint subset is finite;

$\mathbf{F}_v$—all finite valued lattice ordered groups;

**D**—all lattice ordered groups whose regular subgroups satisfy the descending chain condition;

**O**—all cardinal sums of linearly ordered groups;

**R**—all cardinal sums of archimedean linearly ordered groups;

**B**—all lattice ordered groups such that each prime exceeds a unique minimal prime.

Let $Z$ be as above (cf. Section 2). Then $Z$ belongs to each of the torsion classes under consideration. Let $I$ be an infinite set and for each $i \in I$ let $G_i = Z$, $G = \prod_{i \in I} G_i$.

**3.4.1.** Suppose that $I = \mathbb{N}$ (the set of all positive integers). Let $f$ and $g$ be elements of $G$ such that $f(n) = n$ and $g(n) = 1$ for each $n \in \mathbb{N}$. Then $f \wedge ng < f \wedge (n+1)g$ for each $n \in \mathbb{N}$. Hence in view of Theorem 1.1. in [1] the lattice ordered group $G$ is not hyperarchimedean. Therefore **A** is not a convexity.

**3.4.2.** $G$ does not belong to **F**, hence $\mathbf{F} \notin \mathcal{C}(\mathcal{L})$.

**3.4.3.** Let $g \in G$ be such that $g(i) = 1$ for each $i \in I$. Then $g$ has infinitely many values in $G$, hence $G \notin \mathbf{F}_v \notin \mathcal{C}(\mathcal{L})$.

**3.4.4.** Let $G$ be as in 3.4.1 and for each $j \in I$ let $G^j = \{g \in G \colon g(i) = 0$ for each $i \in I$ with $i < j\}$. Then $G^1 \supset G^2 \supset G^3 \supset \ldots$ and the set $\{G^n\}_{n \in I}$ has no minimal element. Also, all $G^n$ are regular subgroups of $G$. Hence $G \notin \mathbf{D}$ and so $\mathbf{D} \notin \mathcal{C}(\mathcal{L})$.

**3.4.5.** The lattice ordered group $G$ does not belong to **O**, hence $G \notin \mathbf{R}$. Therefore $\mathbf{O} \notin \mathcal{C}(\mathcal{L})$ and $\mathbf{R} \notin \mathcal{C}(\mathcal{L})$.

**3.4.6.** Now we will show that **B** is a convexity. We will apply the following result (cf. [1], p. 492):

$(*)$ For each lattice ordered group $G$ the following are equivalent:

(i) $G \in \mathbf{B}$.

(ii) Each pair of incomparable primes in $G$ generates $G$.

Let $I$ be a nonempty set of indices and for each $i \in I$ let $B_i$ be a lattice ordered group belonging to **B** with $B_i \neq \{0\}$. Put $G = \prod_{i \in i} B_i$. We have to verify that $G$ belongs to **B** as well.

First we consider the question what is the general form of primes in $G$. Let $H^1$ be a prime in $G$. Let $I(H^1)$ be the set of all $i \in I$ having the property that there is $g \in G \setminus H^1$ such that $g(i) \neq 0$. Then $I(H^1) \neq \emptyset$.

Suppose that $i(1)$ and $i(2)$ are distinct elements of $I(H^1)$. Put $\overline{G}_{i(1)} = \{g^1 \in G: g'(i(1)) = 0\}$, and let $\overline{G}_{i(2)}$ be defined analogously. Next, let

$$Q_1 = H^1 + G_{i(1)}, \quad Q_2 = H^1 + G_{i(2)}.$$

Then $Q_1$ and $Q_2$ are convex $\ell$-subgroups of $G$ and $H^1 \subseteq Q_j$ $(j = 1, 2)$. We have neither $Q_1 \subseteq Q_2$ nor $Q_2 \subseteq Q_1$. Nonetheless, since $H^1$ is prime, the system of all convex $\ell$-subgroups $Q$ of $G$ with $H^1 \subseteq Q$ is linearly ordered; hence we arrive at a contradiction. Therefore $I(H^1)$ is a one-element set, $I(H^1) = \{i(1)\}$. Hence $\overline{G}_{i(1)} \subseteq H^1$ and thus

$$H^1 = (H^1 \cap G_{i(1)}) + \overline{G}_{i(1)}. \tag{4}$$

It is easy to verify that $H^1 \cap G_{i(1)}$ is a prime subgroup of $H^1$.

Conversely, if $H^1 \in C\{G\}$, $i(1) \in I$, $H^1 \cap G_{i(1)}$ is a prime in $G_{i(1)}$ and if (4) holds, then $H^1$ is a prime in $G$.

Let $H^2$ be a prime in $G$ such that $H^1$ and $H^2$ are incomporable. There is $i(2) \in I$ such that $I(H^2) = \{i(2)\}$. Analogously as above we have

$$H^2 = (H^2 \cap G_{i(2)}) + \overline{G}_{i(2)}.$$

We distinguish two cases.

(i) First suppose that $i(1) \neq i(2)$. Then $\overline{G}_{i(1)} + \overline{G}_{i(2)} = G$, whence the pair $H^1$ and $H^2$ generates $G$.

(ii) Next suppose that $i(1) = i(2)$. Denote

$$H_0^1 = H^1 \cap G_{i(1)}, \quad H_0^2 = H^2 \cap G_{i(1)}.$$

Then $H_0^1$ and $H_0^2$ are incomparable primes in $G_{i(1)}$. Thus, since $G_{i(1)}$ belongs to **B**, in view of $(*)$ the pair $H_0^1$ and $H_0^2$ generates $G_{i(1)}$. Therefore the pair $H^1$ and $H^2$ generates $G$.

By applying $(*)$ again we infer that $G$ belongs to **B**.

*References*


[1] *P. Conrad*: Torsion radicals of lattice ordered groups. Symposia Math. 31. Academic Press, New York-London, 1977, pp. 479–513.

[2] *Dao-Rong Ton*: Product radical classes of $\ell$-groups. Czechoslovak Math. J. *42* (1992), 129–142.

[3] *J. Jakubík*: Radical classes and radical mappings of lattice ordered groups. Symposia Math. 31. Academic Press, New York-London, 1977, pp. 451–477.

[4] *J. Jakubík*: On convexities of lattices. Czechoslovak Math. J. *42* (1992), 325–330.

[5] *J. Jakubík*: On directed groups with additional operations. Math. Bohem. *118* (1993), 11–17.

[6] *J. Jakubík*: On convexities of $d$-groups. Czechoslovak Math. J. *44* (1994), 305–314.

[7] *V. M. Kopytov and Z. J. Dimitrov*: On directed groups. Sibirsk. Mat. Zh. *30* (1989), 78–86. (In Russian.)

[8] *J. Martinez*: Torsion theory of lattice ordered groups. Czechoslovak Math. J. *25* (1975), 284–299.

[9] *R. Mlitz (ed.)*: General algebra 1980. Proc. Internat. Conf. Krems 1988. North Holland, Amsterdam-New York-Tokyo-Oxford, 1990.



*Author's address*: *Ján Jakubík*, Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia.