ON THEOREMS OF PU & PU AND GRANDE

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Abstract. Given a finite family of cliquish functions, $\mathfrak{A}$, we can find a Lebesgue function α such that $f+\alpha$ is Darboux and quasi-continuous for every $f \in \mathfrak{A}$. This theorem is a generalization both of the theorem by H. W. Pu & H. H. Pu and of the theorem by Z. Grande.

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In 1987 H. W. Pu and H. H. Pu [3] proved the following theorem.

Theorem 1. *Let $\mathfrak{A}$ be a finite family of Baire one functions. Then there is a Baire one function f such that $f+g$ is a Darboux function for each $g \in \mathfrak{A}$.*

This theorem was generalized in 1992 by Z. Grande [1].

Theorem 2. *Let $f_1, \ldots, f_k$ be cliquish functions. There is a Baire one function f such that $f \neq 0$ only on a null set and all sums $f+f_i$, $i \in \{1, \ldots, k\}$, are Darboux functions.*

In this paper I prove that given a finite family of cliquish functions, $\mathfrak{A}$, we can find a Lebesgue function α such that $f+\alpha$ is Darboux and quasi-continuous for every $f \in \mathfrak{A}$. Clearly we cannot require that $\alpha \neq 0$ only on a null set.

First we need some notation. The real line $(-\infty, \infty)$ is denoted by $\mathbb{R}$ and the set of positive integers by $\mathbb{N}$. The word *function* means a mapping from $\mathbb{R}$ into $\mathbb{R}$. The words *measure*, *summable* etc. refer to the Lebesgue measure and integral in $\mathbb{R}$. The Euclidean metric in $\mathbb{R}$ will be denoted by ϱ. For every set $A \subset \mathbb{R}$ let $\operatorname{cl} A$ be its

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closure and $|A|$ its outer Lebesgue measure. A symbol like $\int_A f$ will always mean the Lebesgue integral.

Let f be a function and let $A \subset \mathbb{R}$ be non-empty. We will write $\sup(f, A)$ for $\sup\{f(x) \colon x \in A\}$ and we denote $\inf(f, A) = -\sup(-f, A)$. The *oscillation of f on A* will be denoted by $\omega(f, A)$, i.e., $\omega(f, A) = \sup(f, A) - \inf(f, A)$. Similarly, the *oscillation of f at a point* $x \in \mathbb{R}$ will be denoted by $\omega(f, x)$, i.e., $\omega(f, x) = \lim_{r \to 0+} \omega(f, [x-r, x+r])$. The set of points of continuity of f will be denoted by $C(f)$.

We say that a function f is *quasi-continuous* (*cliquish*) at a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$ and each open set $U \ni x$ we can find a non-empty open set $V \subset U$ such that $\omega(f, \{x\} \cup V) < \varepsilon$ $(\omega(f, V) < \varepsilon$, respectively$)$. We say that f is *quasi-continuous* (*cliquish*) if it is quasi-continuous (cliquish) at each point $x \in \mathbb{R}$. Cliquish functions are also known as *pointwise discontinuous*.

We will use the following well-known (and easy to prove) facts.

- A function f is quasi-continuous iff for each $x \in \mathbb{R}$ there exists a sequence $x_1, x_2, \ldots \in C(f)$ such that $x_n \to x$ and $f(x_n) \to f(x)$.
- A function f is cliquish iff $C(f)$ is residual. In particular, every Baire one function is cliquish.

We say that $x \in \mathbb{R}$ is a *Lebesgue point* of a function α if α is locally summable at x and $\lim_{r \to 0} \int_x^{x+r} |\alpha - \alpha(x)|/r = 0$. We say that α is a *Lebesgue function* if each $x \in \mathbb{R}$ is a Lebesgue point of α.

The proof of the next lemma is straightforward. (Cf. also Lemma 3.3 of [2].)

Lemma 3. *Let I be a compact interval, let functions $g_1, \ldots, g_k$ be cliquish, $K \subset \mathbb{R}$ nowhere dense, $L \geqslant \eta \geqslant \sup\{\omega(g_i, I) \colon i \in \{1, \ldots, k\}\}$, and $\varepsilon > 0$. Then there is a nowhere dense perfect set $F \subset I \cap \bigcap_{i=1}^{k} C(g_i) \setminus \operatorname{cl} K$ and a continuous function α such that $|\alpha| < L + \eta$, $\alpha = 0$ on $\operatorname{cl} K \cup (\mathbb{R} \setminus I)$, $\int_I |\alpha| < \varepsilon$, and $(g_i + \alpha)(F) \supset [\inf(g_i, I) - L, \sup(g_i, I) + L]$ for $i \in \{1, \ldots, k\}$.* $\square$

Theorem 4. *Let $f_1, \ldots, f_k$ be cliquish functions and $\eta > 0$. There is a Lebesgue function α such that $f_i + \alpha$ is Darboux and quasi-continuous for each $i \in \{1, \ldots, k\}$, $C(\alpha) \supset \bigcap_{i=1}^{k} C(f_i)$ and $|\alpha| < \sup\{\omega(f_i, x) \colon i \in \{1, \ldots, k\},\ x \in \mathbb{R}\} + \eta$.*

Proof. Denote $C = \bigcap_{i=1}^{k} C(f_i)$. Set $\eta_0 = \sup\{\omega(f_i, x) \colon i \in \{1, \ldots, k\},\ x \in \mathbb{R}\}$, $\alpha_0 = 0$ and $B_0 = F_0 = \emptyset$. We will proceed by induction. Fix an $n \in \mathbb{N}$.

Put $\eta_n = \eta/2^{n+1}$ and $B_n = \{x \in \mathbb{R} \colon \omega(f_i, x) \geqslant \eta_n \text{ for some } i \in \{1, \ldots, k\}\}$. (Clearly we may assume that $\eta_1 < \eta_0$.) Find a family of non-overlapping compact intervals $\mathcal{I}_n = \{I_{n,m} \colon m \in \mathbb{N}\}$ such that $\bigcup \mathcal{I}_n = \mathbb{R} \setminus B_n$ and each $x \notin B_n$ belongs to the interior of the union of some two elements of $\mathcal{I}_n$. Since each $I_{n,m}$ is compact and $\omega(f_i, x) < \eta_n$ for each $x \in I_{n,m}$ and $i \in \{1, \dots, k\}$, and α_{n-1} is continuous out of B_{n-1}, so we may assume that $\max\bigl\{\omega(f_i + \alpha_{n-1}, I_{n,m}) \colon i \in \{1, \dots, k\}\bigr\} < \eta_n$; we moreover assume that $|I_{n,m}| < \varrho(I_{n,m}, B_n)$.

Fix an $m \in \mathbb{N}$. If $\eta_{n-1} < +\infty$, then set $L_{n,m} = \eta_{n-1}$. Otherwise set $L_{n,m} = 2\max\bigl\{\sup(|f_i + \alpha_{n-1}|, I_{n,m}) \colon i \in \{1, \dots, k\}\bigr\} + m$. (This case is possible only if $n = 1$.) Use Lemma 3 to find a nowhere dense perfect set $F_{n,m} \subset I_{n,m} \cap C \setminus F_{n-1}$ and a continuous function $\alpha_{n,m}$ such that

$$|\alpha_{n,m}| < L_{n,m} + \eta_n \quad \text{on } I_{n,m}, \tag{1}$$

$$\alpha_{n,m} = 0 \quad \text{on } F_{n-1} \cup (\mathbb{R} \setminus I_{n,m}), \tag{2}$$

$$\int_{I_{n,m}} |\alpha_{n,m}| < 2^{-m} |I_{n,m}|, \tag{3}$$

and

$$\begin{aligned} &(f_i + \alpha_{n-1} + \alpha_{n,m})(F_{n,m}) \\ &\qquad \supset \bigl[\inf(f_i + \alpha_{n-1}, I_{n,m}) - L_{n,m}, \sup(f_i + \alpha_{n-1}, I_{n,m}) + L_{n,m}\bigr] \end{aligned} \tag{4}$$

for $i \in \{1, \dots, k\}$.

Define $F_n = F_{n-1} \cup \bigcup\limits_{m=1}^{\infty} F_{n,m}$ and $\alpha_n = \alpha_{n-1} + \sum\limits_{m=1}^{\infty} \alpha_{n,m}$. It is easy to show that each $x \in B_n$ is a Lebesgue point of α_n. Since α_n is continuous on $\mathbb{R} \setminus B_n$, so α_n is a Lebesgue function. Note that $\alpha_n = 0$ on $B_n \cup F_{n-1}$.

By (1), the sequence (α_n) is uniformly convergent, so its sum, which we denote by α, is a Lebesgue function. By the construction, $C(\alpha) \supset \bigcap\limits_{n=1}^{\infty} (\mathbb{R} \setminus B_n) = C$, while $|\alpha| < \eta_0 + 2\sum\limits_{n=1}^{\infty} \eta_n = \eta_0 + \eta$.

Suppose that $f_i + \alpha$ is not Darboux for some $i \in \{1, \dots, k\}$. Let $a, b, y \in \mathbb{R}$ be such that $(f_i + \alpha)(a) < y < (f_i + \alpha)(b)$ but $(f_i + \alpha)(x) = y$ for no x between a and b. Assume that, e.g., $a < b$. (The other case is analogous.) Set $x_0 = \sup\bigl\{x \in [a,b] \colon (f_i + \alpha)(t) < y \text{ for each } t \in [a,x] \cap C\bigr\}$. By the definition, either $(f_i + \alpha)(x_0) < y$ and there is a sequence (t_l) of elements of C such that $t_l \searrow x_0$ and $(f_i + \alpha)(t_l) > y$ for each $l \in \mathbb{N}$, or $(f_i + \alpha)(x_0) > y$ and there is a sequence (t_l) of elements of C such that $t_l \nearrow x_0$ and $(f_i + \alpha)(t_l) < y$ for each $l \in \mathbb{N}$. We will consider the first case only. (The other case is analogous.)

One can easily see that $x_0 \notin C(f_i)$. So $x_0 \in B_n \setminus B_{n-1}$ for some $n \in \mathbb{N}$.

If $\eta_{n-1} = \infty$, then take an $m > |y|$ with $I_{n,m} \subset [a,b]$. Then by (4), there is an $x \in F_{n,m} \subset [a,b]$ with $(f_i + \alpha_n)(x) = y$. However, by (2), $\alpha_l(x) = 0$ for each $l > n$, so $(f_i + \alpha)(x) = y$, contradicting our assumption.

If $\eta_{n-1} < \infty$, then there is a $\delta \in (0, b - x_0)$ with $\omega\big(f_i + \alpha_{n-1}, [x_0, x_0 + \delta]\big) < \eta_{n-1}$. Let $t \in (x_0, x_0 + \delta/4) \cap C$ be such that $(f_i + \alpha)(t) > y$. We will show that

$$(f_i + \alpha_{k-1})(t) > y \quad \text{for each} \quad k \geqslant n. \tag{5}$$

Indeed, suppose that this condition fails. Since by (1), $(f_i + \alpha_k)(t) > y$ for each sufficiently large k, there is a $k \geqslant n$ with

$$(f_i + \alpha_{k-1})(t) \leqslant y < (f_i + \alpha_k)(t). \tag{6}$$

Let $m \in \mathbb{N}$ be such that $t \in I_{k,m}$. Then $y > \inf(f_i + \alpha_{k-1}, I_{k,m}) - L_{k,m}$, so by (5), there is a $z \in F_{k,m}$ with

$$(f_i + \alpha_k)(z) = \sup(f_i + \alpha_{k-1}, I_{k,m}) + L_{k,m} < y. \tag{7}$$

Let $z \in I_{k+1,p}$. Since (1) and (6) yield

$$\begin{aligned}\sup(f_i + \alpha_k, I_{k+1,p}) + \eta_k \geqslant (f_i + \alpha_k)(z) + \eta_k &= \sup(f_i + \alpha_{k-1}, I_{k,m}) + L_{k,m} + \eta_k \\ &\geqslant (f_i + \alpha_{k-1})(t) + L_{k,m} + \eta_k \geqslant (f_i + \alpha_k)(t) > c,\end{aligned}$$

so (7) and (4) imply that there is an $x \in F_{k+1,p} \subset [x_0, b]$ with $(f_i + \alpha_{k+1})(x) = y$. It follows that $(f_i + \alpha)(x) = y$, contradicting our assumption.

The condition (5) implies, in particular, that $(f_i + \alpha_{n-1})(t) > y$. Let $m \in \mathbb{N}$ be such that $t \in I_{n,m}$. Using $t \in (x_0, x_0 + \delta/4) \cap C$, we obtain

$$\begin{aligned}\sup(f_i + \alpha_{n-1}, I_{n,m}) + \eta_{n-1} &> (f_i + \alpha_{n-1})(t) > y > (f_i + \alpha_{n-1})(x_0) \\ &> (f_i + \alpha_{n-1})(t) - \eta_{n-1} \\ &\geqslant \inf(f_i + \alpha_{n-1}, I_{n,m}) - \eta_{n-1}.\end{aligned}$$

Hence by (4), there is an $x \in F_{n,m}$ with $(f_i + \alpha_n)(x) = y$. So $x \in [a, b]$ and $(f_i + \alpha)(x) = y$, contradicting our assumption.

We have shown that $f_i + \alpha$ is Darboux. Now we will prove that for each interval I we have

$$(f_i + \alpha)(I) = (f_i + \alpha)(I \cap C). \tag{8}$$

The inclusion "$\supset$" is evident. To prove the converse inclusion fix an interval I and take an $x_0 \in I \setminus C$. Arguing as above, we can find $n, m \in \mathbb{N}$ and $x \in F_{n,m} \subset I$ such that $(f_i + \alpha_n)(x) = (f_i + \alpha_n)(x_0)$. So (8) holds. This condition yields that $f_i + \alpha$ is quasi-continuous, which completes the proof. $\square$

The next corollary is a generalization of Theorem 4.2 of [2].

Corollary 5. *Given a cliquish function f, we can find a quasi-continuous Lebesgue function α such that $f - \alpha$ is Darboux and quasi-continuous and $C(\alpha) \supset C(f)$. Moreover, we can require α to be bounded provided f is bounded.*

Proof. Use Theorem 4 for the family $\{-f, 0\}$. □

References

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