ON COLOURING PRODUCTS OF GRAPHS

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Abstract. In this paper, we give some results concerning the colouring of the product (cartesian product) of two graphs.

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INTRODUCTION

Graphs, considered here, are finite, undirected, without loops or multiple edges, and [1] is followed for terminology and notation. The *product* (also called *cartesian product* [2]) $G_1 \times G_2$ of two graphs G_1 and G_2 with vertex sets V_1 and V_2, respectively, has the cartesian product $V_1 \times V_2$ as its set of vertices. Two vertices (u_1, u_2) and (v_1, v_2) are adjacent, if $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.

Let $V_1 = \{v_{11}, v_{12}, \ldots, v_{1p_1}\}$, $V_2 = \{v_{21}, v_{22}, \ldots, v_{2p_2}\}$, and let q_i denote the number of edges of G_i, $i = 1, 2$. The graph $G_1 \times G_2$ has $p_1 \cdot p_2$ vertices and $p_1 \cdot q_2 + p_2 \cdot q_1$ edges. This graph, which is isomorphic to $G_2 \times G_1$, contains p_2 disjoint "horizontal" copies G_{11}, G_{12}, …, G_{1p_2} (ordered from top to bottom) of G_1 and p_1 "vertical" copies G_{21}, G_{22}, …, G_{2p_1} (ordered from left to right) of G_2. A horizontal copy G_{1i} and a vertical copy G_{2j} have only one vertex (v_{1j}, v_{2i}) in common.

The *vertex-chromatic number* $\gamma(G)$ of a graph G is the minimum number of colours required to colour the vertices of G in such a way that no two adjacent vertices have the same colour. The *edge-chromatic number* $\gamma'(G)$ is defined similarly. The *total-chromatic number* $\gamma''(G)$ of G is the minimum number of colours required to colour the elements (vertices and edges) of G in such a way that no two adjacent elements (two vertices or two edges) and no two incident elements (a vertex and an edge) have the same colour.

By a *proper colouring* of, for example, vertices of G we mean an assignment of colours to vertices of G in such a way that adjacent vertices receive different colours. The colour of an element e of G will be denoted by $c(e)$. The notation $c(u, v)$ will be used for the colour of the point (u, v). We mention the well known result:

$$\gamma(G_1 \times G_2) = \max\{\gamma(G_1), \gamma(G_2)\}.$$

Main results

Let $\Delta(G)$ denote the maximum degree among the degrees of vertices of G. Concerning $\gamma'(G)$, Vizing [3] has shown that

$$\Delta(G) \leqslant \gamma'(G) \leqslant \Delta(G) + 1.$$

Since

$$\Delta(G_1 \times G_2) = \Delta(G_1) + \Delta(G_2),$$

we have

Corollary. $\Delta(G_1) + \Delta(G_2) \leqslant \gamma'(G_1 \times G_2) \leqslant \Delta(G_1) + \Delta(G_2) + 1.$

If the edge-chromatic number of G_i, $i = 1, 2$, equals its maximal degree, we shall show that $\gamma'(G_1 \times G_2)$ equals the maximal degree of $G_1 \times G_2$.

Theorem 1. *If $\gamma'(G_i) = \Delta(G_i)$, $i = 1, 2$, then $\gamma'(G_1 \times G_2) = \Delta(G_1) + \Delta(G_2)$.*

Proof. Clearly, we have

$$\gamma'(G_1) + \gamma'(G_2) \leqslant \gamma'(G_1 \times G_2).$$

The converse is true for every pair of graphs G_1 and G_2. To see this, colour the edges of each horizontal copy, properly, with colours $1, 2, \ldots, \gamma'(G_1)$ and each vertical copy, properly, with colours $\gamma'(G_1) + 1, \gamma'(G_1) + 2, \ldots, \gamma'(G_1) + \gamma'(G_2)$. □

Assuming that $\gamma'(G_i) = \Delta(G_i)$, $i = 1, 2$, one might think that $\gamma'(G_1 \times G_2) = \Delta(G_1) + \Delta(G_2) + 1$. Let $G_1 = G_2 = K_5 - x$, where K_n is the complete graph of order n, and $K_n - x$ denotes K_n minus one edge. Thus, $\gamma'(G_1) = \gamma'(G_2) = \Delta(G_1) + 1$. But $\gamma'(G_1 \times G_2)$ is shown to be $\Delta(G_1) + \Delta(G_2) = 8$. The graph $(K_5 - x) \times (K_5 - x)$ is the smallest graph with the above property.

Given two graphs G_1 and G_2, we have $\gamma(G_1) \leqslant \gamma''(G_2)$ or $\gamma(G_2) \leqslant \gamma''(G_1)$. Suppose that $\gamma(G_1) > \gamma''(G_2)$. Then

$$\gamma''(G_1) \geqslant \gamma(G_1) > \gamma''(G_2) \geqslant \gamma(G_2)$$

imply $\gamma(G_2) < \gamma''(G_1)$.

Theorem 2. *If $\gamma(G_1) \leqslant \gamma''(G_2)$, then we have*

$$\Delta(G_1) + \Delta(G_2) + 1 \leqslant \gamma''(G_1 \times G_2) \leqslant \gamma''(G_2) + \gamma'(G_1).$$

Proof. The first inequality is obvious. Colour the elements of G_{21} and the edges of each horizontal copy, properly, with colours $1, 2, \ldots, \gamma(G_1), \ldots, \gamma''(G_2)$ and colours $\gamma''(G_2)+1, \gamma''(G_2)+2, \ldots, \gamma''(G_2)+\gamma'(G_1)$, respectively. Suppose that $c(v_{11}, v_{21}) = 1$. Then, colour the vertices of G_{11} with colours $1, 2, \ldots, \gamma(G_1)$, properly, in such a way that the vertex (v_{11}, v_{21}) receives colour 1. Next, consider G_{2j}, $j = 2, 3, \ldots, p_1$ and let e be an element of G_{2j}. There is an element e' of G_{21} corresponding to e. Let $c(e) = c(v_{1j}, v_{21}) + c(e') - 1 \pmod{\gamma''(G_2)}$. Now, it is an easy matter to check that this colouring is a proper colouring of the elements of $G_1 \times G_2$, completing the proof. □

The bounds given in Theorem 2 cannot, in general, be improved, that is, for two positive integers m and n there exist two graphs G_1 and G_2 with $\gamma'(G_1) = m$, $\gamma''(G_2) = n$ and $\gamma''(G_1 \times G_2) = \gamma'(G_1) + \gamma''(G_2)$. Indeed, let $G_1 = K_{1,m}$ and $G_2 = K_{1,n-1}$, where $K_{m,n}$ denotes the complete bipartite graph of order $m+n$. Incidentally, for these graphs, $\Delta(G_1) + \Delta(G_2) + 1$ equals $\gamma''(G_1 \times G_2)$, too.

The second inequality in the theorem cannot be changed to an equality, as can be seen by considering $C_4 \times C_4$, where C_n, $n \geqslant 3$, denotes the cycle of length n.

If $\gamma(G_1) \leqslant \gamma''(G_2)$ and $\gamma(G_2) \leqslant \gamma''(G_1)$, then we have

$$\gamma''(G_1 \times G_2) \leqslant \min\{\gamma''(G_2) + \gamma'(G_1), \gamma''(G_1) + \gamma'(G_2)\}.$$

References

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