TWO SOLUTIONS FOR A NONLINEAR DIRICHLET PROBLEM WITH POSITIVE FORCING

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Abstract. Given a semilinear elliptic boundary value problem having the zero solution and where the nonlinearity crosses the first eigenvalue, we perturb it by a positive forcing term; we show the existence of two solutions under certain conditions that can be weakened in the onedimensional case.

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1. Introduction

Let Ω be a bounded, regular open set in $\mathbb{R}^N$. Consider the boundary value problem

$$\begin{aligned} &\Delta u + g(x,u) = h(x) \text{ in } \Omega \\ &u = 0 \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

where $h(x)$ is a *nonnegative* function in Ω and $g(x,u)$ is a nonlinear term that "crosses" the first eigenvalue λ_1 of $-\Delta$ in Ω with zero boundary condition. If we substitute the right-hand side of (1.1) for $t\varphi_1 + h_1(x)$ where t is a real parameter, φ_1 is the first (positive) eigenfunction and h_1 is a given function in Ω then (1.1) becomes a problem of Ambrosetti-Prodi type and it is well known that (see Ambrosetti and Prodi [1], Berger and Podolak [3], Kazdan and Warner [10] and De Figueiredo [5]), with a precise definition of what "crossing an eigenvalue" means, there exists a number t^* such that (1.1) is solvable if and only if $t \geqslant t^*$ and has two solutions if $t > t^*$. In the onedimensional case, with a parameter t multiplying $h(x)$ in (1.1), the existence of arbitrarily many solutions has been recently investigated by Zinner [17].

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Therefore it seems natural to ask under which conditions (1.1) possesses at least two solutions when h is a general positive function. Comparing both situations and noting that there is no parameter in (1.1) it becomes obvious that some additional assumption on g is needed to obtain such a result. In fact we make the simple, localizing hypothesis

$$g(x,0)=0,\ \forall x\in\Omega. \tag{1.2}$$

Before stating our first result let us introduce the following notation: by G we shall denote the primitive $G(x,u)=\int_0^u g(x,s)\,\mathrm{d}s$.

Theorem 1. *Let $g:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ be a locally Lipschitz continuous function such that* (1.2) *holds and in addition*

(1) *There exist $a,b>0$ such that $|g(x,u)|\leqslant a|u|+b$ in $\overline{\Omega}\times\mathbb{R}$.*

(2)
$$\limsup_{u\to-\infty}\frac{g(x,u)}{u}<\lambda_1<\beta(x)=\lim_{u\to+\infty}\frac{2G(x,u)}{u^2}$$
uniformly for x in $\overline{\Omega}$.

(3) *$g(x,u)\leqslant 0$ if $u\leqslant 0$.*

(4) *Setting β_1, $\beta_2\in L^\infty(\Omega)$ such that*
$$\liminf_{u\to+\infty}\frac{g(x,u)}{u}=\beta_1(x),\ \limsup_{u\to+\infty}\frac{g(x,u)}{u}=\beta_2(x)$$
uniformly in $\overline{\Omega}$, we have $\beta=\beta_1$ or $\beta=\beta_2$.

Then for any $h\in C^{0,\alpha}(\overline{\Omega})$ (with $0<\alpha<1$) such that $h\geqslant 0$ and $h\neq 0$ in Ω, problem (1.1) *has at least two solutions.*

This theorem is a consequence of lemmas that we state and prove in Section 3: first we study the existence of a *negative* solution and then we look for a second solution.

The case $N=1$ deserves special treatment since, as one might expect, weaker assumptions yield the same type of theorem. The simplest results are obtained by assuming that $g(x,u)=g(u)$ is independent of x. Setting $\Omega=(0,\pi)$ and $G(u)=\int_0^u g(s)\,\mathrm{d}s$ we can state

Theorem 2. *Let $g\colon\mathbb{R}\to\mathbb{R}$ be a continuous and locally Lipschitz function such that* (1.2) *holds (i.e. $g(0)=0$),*
$$\liminf_{u\to-\infty}\frac{2G(u)}{u^2}<1<\liminf_{u\to\infty}\frac{g(u)}{u},$$
and there exist $a,b>0$ such that $|g(u)|\leqslant a|u|+b,\forall u\in\mathbb{R}$.

Then, if $g(u)\leqslant 0$ for all $u\leqslant 0$, for any $h\in C([0,\pi])$ such that $h\geqslant 0$ and $h\neq 0$ in $[0,\pi]$ problem (1.1) *has at least two solutions.*

This and related results will be covered in the next section.

2. The onedimensional case

Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous and locally Lipschitz function such that $g(0) = 0$ and let $h\colon [0,\pi] \to \mathbb{R}$ be a continuous function such that $h \geqslant 0$ and $h \neq 0$ in $(0,\pi)$. Consider the two-point boundary value problem

$$\begin{gathered} u'' + g(u) = h(t) \\ u(0) = 0 = u(\pi). \end{gathered} \tag{2.1}$$

As was mentioned in the introduction we denote by G the primitive $G(u) = \int_0^u g(s)\,\mathrm{d}s$.

The following lemma is proved (under more general conditions) in Zanolin [15]. We extend h to $(-\infty,+\infty)$ as a nonnegative continuous function such that $\sup\limits_{\mathbb{R}} h = \sup\limits_{[0,\pi]} h$.

Lemma 3. *Assume that G is bounded below in $\mathbb{R}$ (i.e. $\inf G > -\infty$). Then any solution of the equation $u'' + g(u) = h(t)$ can be globally defined in $\mathbb{R}$.*

Remark. From this result we conclude that if $\inf G > -\infty$ then any maximal solution of the equation

$$u'' + g(u) = 0 \tag{2.2}$$

is global (i.e. is defined in $\mathbb{R}$).

As is well known, for each $\varepsilon \in \mathbb{R}$, the (unique) solution of the initial value autonomous problem

$$\begin{gathered} u'' + g(u) = 0 \\ u(0) = 0, \quad u'(0) = \varepsilon \end{gathered} \tag{2.3}$$

satisfies the equation $\frac{1}{2}{u'}^2 + G(u) = \frac{1}{2}\varepsilon^2$ in its interval of existence.

Next, we shall use the phase-plane method to show that under adequate assumptions there exist some large negative values ε for which the solution of the autonomous problem (2.3) cannot vanish in $(0,\pi)$.

Lemma 4. *Assume the following hypotheses:*

(G_1) $\lim\limits_{u\to-\infty} G(u) = +\infty$.

(G_2) $\liminf\limits_{u\to-\infty} \dfrac{2G(u)}{u^2} = \alpha < 1$.

Then there exists a sequence of negative real values (ε_n), *with* $\varepsilon_n \to -\infty$ *when* $n \to +\infty$, *such that for each* $n \in \mathbb{N}$ *the solution of* (2.3) *(with* $\varepsilon = \varepsilon_n$*) has no zeros in* $(0, \pi]$.

Proof. It is enough to consider the case $\inf G > -\infty$. Otherwise we consider the problem associated to the modified function $\tilde{g}$, where $\tilde{g}(u) = g(u)$ if $u < 0$ and $\tilde{g}(u) = 0$ if $u \geqslant 0$, noting that any solution of $u'' + \tilde{g}(u) = 0$ with initial conditions $u(0) = 0$ and $u'(0) = \varepsilon < 0$, which does not vanish in $(0, \pi]$, is a solution of (2.2) in $[0, \pi]$. In fact, if we set $\tilde{G}(u) = \int_0^u \tilde{g}(s)\,\mathrm{d}s$, then by (G_1), $\tilde{G}$ is bounded below in $\mathbb{R}$.

Assume then that $\inf G > -\infty$. Our hypotheses and Lemma 3 imply that, for each $\varepsilon \in \mathbb{R}$, we may refer to the solution of (2.3) as a function $u(t) = u(t, \varepsilon)$ defined and continuous in $\mathbb{R}$ that has continuous derivative with respect to the first variable.

Now we assume that there exists $R > 0$ such that

$$g(u) < 0 \quad \text{ if } u \leqslant -R.$$

Otherwise, by (G_1), there exists a sequence of negative real values u_n with $u_n \to -\infty$, such that $G'(u_n) = g(u_n) = 0$ and $G(u_n) = \max\limits_{u \in [u_n, 0]} G(u)$ for all $n \in \mathbb{N}$, and then by the phase-plane analysis of the autonomous system, associated to the equation (2.2), namely

$$u' = v, \quad v' = -g(u),$$

we obtain the assertion.

With this hypothesis it is easy to prove that there exists ε_0 such that if $\varepsilon \leqslant \varepsilon_0$ then the orbit of $u(t, \varepsilon)$ intersects the negative u-axis at a single point $(\delta, 0)$, where $\varepsilon \mapsto \delta(\varepsilon) < 0$ is a continuous function of $\varepsilon \leqslant \varepsilon_0$ which is uniquely defined by

$$G(\delta(\varepsilon)) = \frac{\varepsilon^2}{2} \quad \text{ and } \quad \delta(\varepsilon) < 0.$$

For each $\varepsilon \leqslant \varepsilon_0$, denote by Γ_ε the orbit of $u(\cdot, \varepsilon)$ and let t_ε be the minimal time needed for Γ_ε to intersect the u-axis at the point $(\delta(\varepsilon), 0)$. We have

$$t_\varepsilon = \int_\delta^0 \frac{\mathrm{d}u}{\sqrt{2(G(\delta) - G(u))}} \quad \text{ if } \varepsilon \leqslant \varepsilon_0.$$

Let $\varphi(u) = \frac{u^2}{2} - G(u)$. By (G_2), $\limsup\limits_{u \to -\infty} \varphi(u) = +\infty$. Then there exists a sequence of negative real values (δ_n) with $\delta_n \to -\infty$ when $n \to +\infty$, such that $\frac{1}{2}u^2 - G(u) < \frac{1}{2}\delta_n^2 - G(\delta_n)$ for $\delta_n < u \leqslant 0$, which means that

$$G(\delta_n) - G(u) < \frac{\delta_n^2 - u^2}{2} \quad \text{ for } \delta_n < u \leqslant 0.$$

Without loss of generality, we can suppose that, for each $n \in \mathbb{N}$, there exists only one $\varepsilon_n < 0$ such that $\delta(\varepsilon_n) = \delta_n$ ($\varepsilon_n = -\sqrt{2G(\delta_n)}$). We thus obtain a sequence (ε_n) of negative real values with $\varepsilon_n \to -\infty$, such that

$$t_{\varepsilon_n} = \int_{\delta_n}^{0} \frac{\mathrm{d}u}{\sqrt{2(G(\delta_n) - G(u))}} > \int_{\delta_n}^{0} \frac{\mathrm{d}u}{\sqrt{\delta_n^2 - u^2}} = \frac{\pi}{2}.$$

Finally, since for each $n \in \mathbb{N}$, the negative semi-period of the solution of (2.3) (with $\varepsilon = \varepsilon_n$), i.e. the minimal time needed for $(u(t), u'(t))$ to intersect the u'-axis in $(0, +\infty)$, is $2t_{\varepsilon_n} > \pi$, u has no zeros in $(0, \pi]$. □

Remark. With (G_1), for $\varepsilon < 0$, the solution u of (2.3) is global or takes a positive value in its interval of existence (in particular, for some $t > 0$, $u(t) = 0$).

Lemma 5. *Assuming that G is bounded below and satisfies* (G_2), *problem* (2.1) *has at least one nonpositive solution in* $(0, \pi)$.

Proof. Let $M = \|h\|_\infty$ and consider the auxiliary problem with initial conditions, with $\varepsilon \in \mathbb{R}$,

$$\begin{aligned} &u'' + g(u) = M \\ &u(0) = 0, \quad u'(0) = \varepsilon. \end{aligned} \tag{2.4}$$

Let $f(u) = g(u) - M$. Then $F(u) = \int_0^u f(s)\,\mathrm{d}s = G(u) - Mu$ and

$$\liminf_{u \to -\infty} \frac{2F(u)}{u^2} = \liminf_{u \to -\infty} \frac{2G(u)}{u^2} = \alpha < 1.$$

Since G is bounded below it follows that $\lim\limits_{u \to -\infty} F(u) = +\infty$. Lemma 4 implies that there exists $\varepsilon < 0$ such that the solution u of (2.4) cannot vanish in $(0, \pi]$. Therefore $u(t) < 0$ in $(0, \pi]$.

Since $u'' + g(u) = M \geqslant h(t)$ in $(0, \pi)$, $u(0) = 0$ and $u(\pi) \leqslant 0$, then u is a lower solution of the problem (2.1). On the other hand it is obvious that $w \equiv 0$ is an upper solution of the problem (2.1).

By the lower and upper solutions method (see e.g. Mawhin [13]) we conclude that (2.1) has at least one solution v, with $u \leqslant v \leqslant 0$ in $(0, \pi)$. □

Remark. If one of the hypotheses

(i) $g(u) \leqslant 0, \ \forall u \leqslant 0$,
(ii) $g(u) \geqslant 0, \ \forall u \leqslant 0$,
(iii) $\dfrac{g(u)}{u} < 1, \ \forall u < 0$,
(iv) $h > 0$ in $[0, \pi]$,
(v) there exists $\beta > 0$ such that $h(t) \geqslant \beta \sin t, \ \forall t \in [0, \pi]$

holds, then $v < 0$ in $(0,\pi)$, $v'(0) < 0$ and $v'(\pi) > 0$. This is a consequence of an elementary version of the maximum principle in cases (i), (ii) and (iii) and of elementary pointwise estimates in cases (iv) and (v).

For each $\varepsilon \in \mathbb{R}$ we denote by $u(\cdot,\varepsilon)$ the solution of the initial value problem

$$\begin{aligned} &u'' + g(u) = h(t) \\ &u(0) = 0, u'(0) = \varepsilon. \end{aligned} \tag{2.5}$$

Lemma 6. *Assume that G is bounded below and satisfies* (G_2) *and*

(G_3) $\liminf\limits_{u\to+\infty} \dfrac{g(u)}{u} > 1$.

Then, if there exists $S < 0$ such that $u \leqslant S$ implies $g(u) \leqslant 0$, then there exists a sequence (ε_n) with $\varepsilon_n \to +\infty$ such that, for each $n \in \mathbb{N}$ the solution $u(\cdot,\varepsilon)$ has exactly one root in $(0,\pi]$.

Proof. We divide the proof into two steps.

Step 1. We claim that, for ε sufficiently large, $u(\cdot,\varepsilon)$ has a first zero $T_\varepsilon \in (0,\pi)$ such that $u'(T_\varepsilon,\varepsilon) \to -\infty$ as $\varepsilon \to +\infty$.

In fact, let $\overline{\beta} > 1$ and let $R > 0$ be such that $g(u) - h(t) \geqslant \overline{\beta}u$ if $u \geqslant R$ and $t \in [0,\pi]$. Given $\tau > 0$, we can show by integrating the equation of problem (2.5) in $[0,\tau]$ that, if $\varepsilon > 0$ is large, then $u(\cdot,\varepsilon)$ reaches the value R for some $t_0 \in [0,\tau]$. Using a well known comparison argument with respect to the linear equation $u'' + \overline{\beta}u = 0$, it turns out that there exists $t_1 \leqslant t_0 + \dfrac{\pi}{\sqrt{\overline{\beta}}}$ such that $u(t_1,\varepsilon) = R$, and it is easy to see that $u'(t_1,\varepsilon) \to -\infty$ as $\varepsilon \to +\infty$. Then, as above, we conclude that, given $a > \tau + \dfrac{\pi}{\sqrt{\overline{\beta}}}$, $u(t,\varepsilon)$ has a zero $T_\varepsilon \leqslant a$ when ε is sufficiently large and, further, $u'(T_\varepsilon,\varepsilon) \to -\infty$ as $\varepsilon \to +\infty$.

Since we can choose τ and a such that $0 < \tau < \pi - \dfrac{\pi}{\sqrt{\overline{\beta}}}$ and $\tau + \dfrac{\pi}{\sqrt{\overline{\beta}}} < a < \pi$, hence for ε is sufficiently large, T_ε exists as claimed.

Step 2. Let us take the solution $u(t,\varepsilon)$ and suppose that the claim is false. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon > \varepsilon_0$, $u(\cdot,\varepsilon)$ has besides the root T_ε another root $S_\varepsilon \leqslant \pi$. We can suppose that ε_0 is so large that $A_\varepsilon := \min\limits_{[T_\varepsilon,S_\varepsilon]} u(\cdot,\varepsilon) < S$, and then there exist $T_\varepsilon \leqslant a_\varepsilon \leqslant b_\varepsilon \leqslant S_\varepsilon$ such that

$$u(a_\varepsilon,\varepsilon) = u(b_\varepsilon,\varepsilon) = A_\varepsilon,$$

$u'(\cdot,\varepsilon) < 0$ in $[T_\varepsilon,a_\varepsilon)$ and $u'(\cdot,\varepsilon) > 0$ in $(b_\varepsilon,S_\varepsilon]$. Integrating the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u'^2}{2} + G(u)\right) = h(t)u'$$

between $t \in (T_\varepsilon, a_\varepsilon)$ and a_ε and using the mean value theorem we obtain

$$\frac{u'(t)^2}{2} + G(u(t)) - G(A_\varepsilon) \leqslant M(u(t) - A_\varepsilon)$$

where $M = \max_{t\in[0,\pi]} h(t)$ so that, defining $\overline{G}(u) = G(u) - Mu$,

$$a_\varepsilon - T_\varepsilon = \int_{A_\varepsilon}^0 \left|\frac{\mathrm{d}t}{\mathrm{d}u}\right| \mathrm{d}u \geqslant \int_{A_\varepsilon}^0 \frac{\mathrm{d}u}{\sqrt{2(\overline{G}(A_\varepsilon) - \overline{G}(u))}}.$$

Similarly we obtain

$$S_\varepsilon - b_\varepsilon \geqslant \int_{A_\varepsilon}^0 \frac{\mathrm{d}u}{\sqrt{2(\overline{G}(A_\varepsilon) - \overline{G}(u))}}$$

so that

$$\pi > S_\varepsilon - T_\varepsilon \geqslant \sqrt{2}\int_{A_\varepsilon}^0 \frac{\mathrm{d}u}{\sqrt{\overline{G}(A_\varepsilon) - \overline{G}(u)}}. \tag{2.6}$$

Now it is easy to see that $A_\varepsilon \to -\infty$ as $\varepsilon \to +\infty$ (in fact $\frac{u'(T_\varepsilon)^2}{2} \leqslant \overline{G}(A_\varepsilon)$). On the other hand, since $\limsup_{u\to-\infty}\left(\frac{u^2}{2} - \overline{G}(u)\right) = +\infty$, there exists a sequence $A_n \to -\infty$ such that

$$2(\overline{G}(A_n) - \overline{G}(u)) < A_n^2 - u^2 \text{ if } A_n < u \leqslant 0.$$

Since A_ε is a continuous function of ε, it takes arbitrarily (negative) large values, in particular it assumes the values A_n for large n. But then, if ε is such that $A_\varepsilon = A_n$, we have

$$\sqrt{2}\int_{A_\varepsilon}^0 \frac{\mathrm{d}u}{\sqrt{\overline{G}(A_\varepsilon) - \overline{G}(u)}} > 2\int_{A_n}^0 \frac{\mathrm{d}u}{\sqrt{A_n^2 - u^2}} = \pi,$$

contradicting (2.6). Hence the lemma is proved. □

We are now in a position to state and prove our first multiplicity result.

Theorem 7. *Assume that G satisfies* (G_2), (G_3), *and the following growth assumption on g:*

(G_4) *There exist $a, b > 0$ such that $|g(u)| \leqslant a|u| + b$, $u \in \mathbb{R}$.*

Let one of the following conditions hold:

1. *$g(u) \leqslant 0$ for $u \leqslant 0$.*
2. *There exists $S < 0$ such that $g(u) \leqslant 0$ for $u \leqslant S$, and $h > 0$ in $[0, \pi]$.*

Then the problem (2.1) *has at least two solutions.*

P r o o f . The remark after Lemma 5 implies that we can take a negative solution of (2.1) v such that $v < 0$ in $(0, \pi)$, $v'(0) < 0$ and $v'(\pi) > 0$.

For each $\varepsilon \in \mathbb{R}$, denote by $m = m(\varepsilon)$ the number of zeros of $z(\cdot, \varepsilon) := u(\cdot, \varepsilon) - v(t)$, i.e. the number of intersections of the graphs of u and v in $(0, \pi)$. For $\varepsilon \neq v'(0)$, by the uniqueness theorem and the fact that $z(\cdot, \varepsilon)$ is nontrivial, the orbits of v and u intersect transversally and $m < +\infty$.

With our hypotheses $z(t, \varepsilon)$ and $z'(t, \varepsilon)$ depend continuously on ε and t, uniformly with respect to $t \in [0, \pi]$. It follows from the uniqueness theorem that the zeros of $z(\cdot, \varepsilon)$ (with $\varepsilon \neq v'(0)$, noting that $z(\cdot, v'(0)) \equiv 0$) are all simple and hence they depend continuously on ε. By the elementary implicit function theorem (see Kaper and Kwong [9], Lazer and McKenna [11], Dinca and Sanchez [6]), if $m(\varepsilon) \neq m(\varepsilon')$ for some $\varepsilon' > \varepsilon > v'(0)$, then there exists $\overline{\varepsilon} > v'(0)$ $(\varepsilon < \overline{\varepsilon} < \varepsilon')$ such that $u = u(\cdot, \overline{\varepsilon}) = v + z(\cdot, \overline{\varepsilon})$ is a solution of (2.1). If $m(\varepsilon) < m(\varepsilon')$ then

$$\overline{\varepsilon} = \inf\{\tilde{\varepsilon} > \varepsilon \colon z(t, \tilde{\varepsilon}) \text{ has at least } m(\varepsilon) + 1 \text{ zeros in } (0, \pi)\}.$$

If $m(\varepsilon) > m(\varepsilon')$ then

$$\overline{\varepsilon} = \sup\{\tilde{\varepsilon} > \varepsilon \colon z(t, \varepsilon'') \text{ has at least } m(\varepsilon) \text{ zeros in } (0, \pi), \forall \varepsilon \leqslant \varepsilon'' \leqslant \tilde{\varepsilon}\}.$$

Therefore v and u are distinct solutions of (2.1).

Now, it is enough to prove the assertion when $m(\varepsilon) = m(\varepsilon')$ for all $\varepsilon, \varepsilon' > v'(0)$. Assuming without loss of generality that $S < \min\limits_{[0,\pi]} v$ we can prove repeating the argument of Lemma 6 (*Step 1*) that there exists $\varepsilon_1 > v'(0)$ such that $m(\varepsilon) \geqslant 1$ for $\varepsilon \geqslant \varepsilon_1$. Hence, by Lemma 6, for some $\varepsilon > v'(0)$, $m(\varepsilon) \geqslant 1$ and $u(\cdot, \varepsilon)$ has exactly one root in $(0, \pi]$. Therefore, in case that $m(\varepsilon) = m$ for all $\varepsilon > v'(0)$, $m \geqslant 1$ and m is an odd number. Since $z(t, \varepsilon) \to 0$ uniformly in $[0, \pi]$ when $\varepsilon \to v'(0)$, we can choose $\varepsilon > v'(0)$ sufficiently small such that $u < 0$ in $(0, \pi]$. Then

$$u'' + g(u) = h(t) \text{ in } (0, \pi), \; u(0) = 0 \text{ and } u(\pi) = z(\pi) \leqslant 0,$$

and therefore u is a lower solution of (2.1). By the lower and upper solutions method, using the same argument as in Lemma 5, we conclude that (2.1) has a solution $v_1 \geqslant u$, nonpositive in $(0, \pi)$. Since u is strictly bigger than v in some neighbourhood of 0 ($u'(0) = 0$ and $v'(0) < 0$), v_1 is a solution of (2.1) distinct from v. □

Theorem 2 stated in the introduction follows from this theorem.

We observe that the case $g = g(t, u)$, for $t \in [0, \pi]$ and $u \in \mathbb{R}$, can be treated by variational methods (as we do in the next section).

Let Ω be a bounded domain in $\mathbb{R}^N$. Consider the Dirichlet boundary value problem (1.1), where $h \in C^{0,\alpha}(\overline{\Omega})$ (with $0 < \alpha < 1$) is a nonnegative and nonzero function in Ω, and $g\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function that satisfies the condition

$$g(x,0) = 0, \forall x \in \Omega \tag{3.1}$$

and the following growth hypothesis:

(G_5) There exist $a, b > 0$ such that $|g(x,u)| \leqslant a|u| + b$ in $\overline{\Omega} \times \mathbb{R}$.

Let $\lambda_1, \lambda_2, \dots, \lambda_i, \dots$ be the sequence of eigenvalues of the linear problem

$$\begin{aligned} \Delta u + \lambda u &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

with each λ_i ($i \in \mathbb{N}$) occuring in the sequence as often as its multiplicity. Let $\varphi_1, \varphi_2, \dots, \varphi_i, \dots$ be the corresponding sequence of eigenfunctions.

By the Krein-Rutman theorem, $0 < \lambda_1 < \lambda_2 \leqslant \dots \leqslant \lambda_i \leqslant \dots$ and we can assume that $\varphi_1 > 0$ in Ω and $\frac{\partial \varphi_1}{\partial n} < 0$ on $\partial\Omega$, where $n(x)$ is the outward pointing normal to $\partial\Omega$.

Lemma 8. *Assume that*

(G_6) $\displaystyle\limsup_{u \to -\infty} \frac{2G(x,u)}{u^2} < \lambda_1$ *uniformly in* $\overline{\Omega}$.

Then problem (1.1) *admits at least a nonpositive solution.*

P r o o f. Consider the modified problem

$$\begin{aligned} \Delta u + \tilde{g}(x,u) &= h(x) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{3.2}$$

where $\tilde{g}(x,u) = g(x,u)$ if $u < 0$ and $\tilde{g}(x,u) = 0$ if $u \geqslant 0$. By hypothesis (G_6), the functional $J \in C^1(H_0^1(\Omega), \mathbb{R})$ defined by

$$\tilde{J}(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} - \tilde{G}(x,u) + h(x)u \right) \mathrm{d}x$$

has an absolute minimum, where $\tilde{G}$ denotes the primitive

$$\tilde{G}(x,u) = \int_0^u \tilde{g}(x,s)\,\mathrm{d}s.$$

By well known regularity results the minimum is attained at a function $v(x)$ of class C^2 which is a solution of (3.2). An elementary form of the maximum principle shows that $v \leqslant 0$ in Ω. Therefore, v is a nonpositive solution of (1.1). $\square$

R e m a r k . If one of the following hypotheses holds:

(i) $g(x,u) \leqslant 0,\ \forall x \in \overline{\Omega}, \forall u \leqslant 0$,

(ii) $g(x,u) \geqslant 0,\ \forall x \in \overline{\Omega}, \forall u \leqslant 0$,

(iii) $\frac{g(x,u)}{u} \leqslant \lambda_1,\ \forall x \in \overline{\Omega}, \forall u \leqslant 0$ and inequality holds in a subset of $\overline{\Omega}$ with positive measure,

then $v < 0$ in Ω and $\frac{\partial v}{\partial n} > 0$ on $\partial\Omega$.

Therefore, assuming one of the above conditions together with the hypotheses of the last theorem, problem (1.1) has at least one *negative* solution.

Let $H = H_0^1(\Omega)$ and let $J\colon H \to \mathbb{R}$ be the functional defined by

$$J(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} - G(x,u) + h(x)u \right) \mathrm{d}x.$$

It follows that $J \in C^1(H, \mathbb{R})$ and

$$J'(u)v = \int_\Omega (\nabla u \cdot \nabla v - g(x,u)v + h(x)v) \, \mathrm{d}x$$

for $u, v \in H$, where we denote by $(\cdot,\cdot)$ the inner product in H and by $\nabla J(u)$ the gradient of J at a point $u \in H$. Actually the solutions of (1.1) are the critical points of J.

We introduce in H the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^{1/2}$.

Lemma 9. *Assume* (G$_5$) *and*

(G$_7$) $\lambda_1 < \beta(x) = \lim\limits_{u \to +\infty} \dfrac{2G(x,u)}{u^2}$ *uniformly in* $\overline{\Omega}$.

(G$_8$) *There exist* $\beta_1, \beta_2 \in L^\infty(\Omega)$ *such that*

$$\liminf_{u \to +\infty} \frac{g(x,u)}{u} = \beta_1(x), \ \limsup_{u \to +\infty} \frac{g(x,u)}{u} = \beta_2(x)$$

uniformly in $\overline{\Omega}$.

(G$_9$) $\beta = \beta_1$ or $\beta = \beta_2$.

(G$_{10}$) $\limsup\limits_{u \to -\infty} \dfrac{g(x,u)}{u} < \lambda_1$ *uniformly in* $\overline{\Omega}$.

If $g(x,u) \leqslant 0$ *for* $u \leqslant 0$ *then the functional* J *satisfies the Palais-Smale condition ((PS) for short) in* H.

R e m a r k . This lemma is reminiscent of another one by Marino, Micheletti and Pistoia [12, (1.6) Remark]. We sketch the proof for completeness.

Proof. Consider the case $\beta = \beta_1$. For the case $\beta = \beta_2$ the proof would be similar. Here we introduce $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ for $u \in \mathbb{R}$.

Let (u_n) be a sequence in H such that $J'(u_n) \to 0$ and $(J(u_n))$ is bounded. To prove that (u_n) possesses a convergent subsequence it is enough to show that (u_n) is bounded (see Rabinowitz [14]).

By the hypothesis

$$J'(u_n) \to 0 \tag{3.3}$$

there exists $M > 0$ such that $\|J'(u_n)\| \leqslant M$ for all $n \in \mathbb{N}$. Then

$$|J'(u_n)u_n^-| \leqslant M\|u_n^-\|, \ \forall n \in \mathbb{N}.$$

Since $g(x, u) \leqslant 0$ for $u \leqslant 0$ and $g(x, u_n)u_n^- = g(x, -u_n^-)u_n^-$, it follows from hypothesis (G_{10}) that

$$0 \leqslant -g(x, u_n)u_n^- \leqslant \alpha(u_n^-)^2 + C$$

for some $\alpha < \lambda_1$ and $C > 0$. By the Hölder and the Poincaré inequalities we obtain

$$\begin{aligned} |J'(u_n)u_n^-| &= \left| \int_\Omega (|\nabla u_n^-|^2 + g(x, u_n)u_n^- - h(x)u_n^-) \,\mathrm{d}x \right| \\ &\geqslant \left(1 - \frac{\alpha}{\lambda_1}\right) \|u_n^-\|^2 - \frac{\|h\|_2}{\sqrt{\lambda_1}} \|u_n^-\| - C. \end{aligned}$$

Hence (u_n^-) is bounded.

Assume by contradiction that (u_n) is not bounded or equivalently that (u_n^+) is not bounded. Then for some subsequence, $\|u_n^+\| \to +\infty$ (here and below, we keep the same index to denote subsequences).

Let $\tilde{u}_n = \frac{u_n^+}{\|u_n^+\|}$. There exists a subsequence $\tilde{u}_n \rightharpoonup \tilde{u}$ for some $\tilde{u} \in H$. From (3.3) we get $J'(u_n)v \to 0$ for all $v \in H$. In particular,

$$\frac{1}{\|u_n^+\|} J'(u_n)v \to 0, \ \forall v \in H. \tag{3.4}$$

Observe that, for $v \in H$,

$$\begin{aligned} \frac{1}{\|u_n^+\|} J'(u_n)v = &\int_\Omega \left(\nabla \tilde{u}_n \cdot \nabla v - \frac{g(x, u_n)}{\|u_n^+\|} v \right) \,\mathrm{d}x \\ &- \frac{1}{\|u_n^+\|} \int_\Omega \big(\nabla u_n^- \cdot \nabla v - h(x)v\big) \,\mathrm{d}x. \end{aligned}$$

Since (u_n^-) is bounded in H and $\|u_n^+\| \to +\infty$, (3.4) yields

$$\int_\Omega \left(\nabla \tilde{u}_n \cdot \nabla v - \frac{g(x, u_n)}{\|u_n^+\|} v \right) \,\mathrm{d}x \to 0 \tag{3.5}$$

for each $v \in H$. By (G_5) we get

$$\left|\frac{g(x,u_n)}{\|u_n^+\|}\right| \leqslant a\frac{|u_n|}{\|u_n^+\|} + \frac{b}{\|u_n^+\|} = a\left(\tilde{u}_n + \frac{u_n^-}{\|u_n^+\|}\right) + \frac{b}{\|u_n^+\|}.$$

Thus $\left(\frac{g(x,u_n)}{\|u_n^+\|}\right)$ is bounded in $L^2(\Omega)$ and for a subsequence, $\frac{g(x,u_n)}{\|u_n^+\|} \rightharpoonup g_0$ for some $g_0 \in L^2(\Omega)$. From (3.5) it follows that

$$\int_\Omega (\nabla\tilde{u}\cdot\nabla v - g_0 v)\,\mathrm{d}x = 0,\ \forall v \in H. \tag{3.6}$$

Since g satisfies the condition (3.1) we have

$$g(x,u_n) = g(x,u_n^+) + g(x,-u_n^-)$$

and it is easy to prove that $\frac{g(x,u_n^+)}{\|u_n^+\|} \rightharpoonup g_0$.

By standard argument based on assumption (G_5) (see e.g. Berestycki, Figueiredo [2] or Gossez, Omari [8]), g_0 can be written as

$$g_0(x) = m(x)\tilde{u}(x)$$

where the L^∞ function m satisfies

$$\beta(x) \leqslant m(x) \leqslant \beta_2(x) \text{ a.e. in } \Omega. \tag{3.7}$$

Consequently, by (3.6), $\tilde{u}$ is a solution of

$$\begin{aligned} \Delta\tilde{u} + m(x)\tilde{u} &= 0 \text{ in } \Omega\\ \tilde{u} &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.8}$$

The proof will be completed by the argument from Marino, Micheletti and Pistoia [12].

From the hypothesis (3.3) it follows that $\frac{1}{\|u_n^+\|^2}J'(u_n)u_n^+ \to 0$. Then we have

$$\int_\Omega m(x)\tilde{u}^2\,\mathrm{d}x = \int_\Omega g_0\tilde{u} = 1, \tag{3.9}$$

which yields by (3.6), taking $v = \tilde{u}$, $\|\tilde{u}\| = 1$.

Since $(J(u_n))$ is bounded we get

$$\frac{1}{\|u_n^+\|^2}J(u_n) \to 0. \tag{3.10}$$

Also we may assume that $\frac{G(x,-u_n^-)}{\|u_n^+\|^2} \to 0$ a.e. in Ω so that

$$\int_\Omega \frac{G(x,u_n)}{\|u_n^+\|^2} \to \int_\Omega \frac{\beta(x)}{2}\tilde{u}^2,$$

then (3.10) yields $\int_\Omega \beta(x)\tilde{u}^2 = 1$. From this equality, (3.9) and (3.7) we conclude that

$$m(x)\tilde{u}(x) = \beta(x)\tilde{u}(x) \text{ a.e. in } \Omega.$$

Thus, from (3.8), $\tilde{u}$ is a solution of

$$\begin{aligned} \Delta\tilde{u} + \beta(x)\tilde{u} &= 0 \text{ in } \Omega \\ \tilde{u} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

If $\tilde{u} \neq 0$ then by an elementary form of the maximum principle, $\tilde{u} > 0$ in Ω (since $\tilde{u} \geqslant 0$ in Ω), but then by the theory of positive operators (see Zeidler [16]) we obtain a contradiction with hypothesis (G_7). Therefore $\tilde{u} \equiv 0$, a contradiction. □

Applying the Poincaré inequality it is easy to prove the following result.

Lemma 10. *Assume hypothesis* (G_7). Then $J(t\varphi_1) \to -\infty$ when $t \to +\infty$.

Proof of Theorem 1. According to the remark following Lemma 8, it is easy to see that the negative solution v yields a local minimum of J with respect to the norm of $C_0^1(\overline{\Omega})$. A theorem of Brezis and Nirenberg ([4]) implies that in fact J attains a local minimum at v. By lemmas 9 and 10 we can invoke the mountain pass theorem to conclude. □

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