LOCALLY INNER DERIVATIONS OF STANDARD OPERATOR ALGEBRAS

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Abstract. It is proved that every locally inner derivation on a symmetric norm ideal of operators is an inner derivation.

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Introduction

Let X be a complex Banach space, let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on X and suppose that $\mathcal{A} \subset \mathcal{B}(X)$ is a subalgebra.

The linear operator $\delta\colon \mathcal{A} \to \mathcal{B}(X)$ is called a derivation if

$$\delta(TS) = T\delta(S) + \delta(T)S \qquad (T, S \in \mathcal{A}).$$

If there is an $S \in \mathcal{A}$ for which $\delta(T) = TS - ST$ $(T \in \mathcal{A})$, then we say that δ is an inner derivation.

There are two different types of operator algebras on which derivations are extensively studied. Namely, standard operator algebras, which are subalgebras of $\mathcal{B}(X)$ containing the ideal $\mathcal{F}(X)$ of finite rank operators, and von Neumann algebras. From their vast literature we refer only to [1, 2, 5–9] which are in closer relation to our present considerations. The main result concerning the structure of derivations on a

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standard operator algebra $\mathcal{A}$ states that every derivation $\delta\colon \mathcal{A} \to \mathcal{B}(X)$ is spatially implemented, i.e. it is of the form $\delta(T) = TA - AT$ with a fixed $A \in \mathcal{B}(X)$ [2, 9]. Moreover, there is a classical theorem due to Sakai [7] stating that every derivation on a von Neumann algebra into itself is inner.

The important concept of local derivations was introduced by Larson, Sourour and Kadison [6, 5] as follows. A linear operator $\theta\colon \mathcal{A} \to \mathcal{B}(X)$ is said to be a local derivation if for every $T \in \mathcal{A}$ there exists a derivation $\delta_T\colon \mathcal{A} \to \mathcal{B}(X)$ such that

$$\theta(T) = \delta_T(T).$$

It was proved in [6, Theorem 1.2] that in the case $\mathcal{A} = \mathcal{B}(X)$, every local derivation is a derivation (cf. [1]). This statement can be easily extended to the case when $\mathcal{A}$ is an arbitrary standard operator algebra. Indeed, if $I \in \mathcal{A}$, then this follows from the proof of [6, Theorem 1.2]. Otherwise consider the local derivation $T + \lambda I \mapsto \theta(T)$ on the algebra $\mathcal{A} \oplus \mathbb{C}I$.

In [5] Kadison studied similar questions for von Neumann algebras. He proved that every norm-continuous local derivation on a von Neumann algebra into itself is a derivation. Since the derivations of a von Neumann algebra are inner, one can reformulate this theorem in the following way. Every norm-continuous locally inner derivation on a von Neumann algebra into itself is an inner derivation. Here, a linear mapping $\theta\colon \mathcal{A} \to \mathcal{B}(X)$ defined on the subalgebra $\mathcal{A} \subset \mathcal{B}(X)$ is called a locally inner derivation if for every $T \in \mathcal{A}$ there exists an $S \in \mathcal{A}$ such that

$$\theta(T) = TS - ST.$$

Our aim is to treat this statement for standard operator algebras.

Throughout this paper H stands for a separable infinite dimensional complex Hilbert space. We shall need some concepts and theorems from the ideal theory of Calkin and Schatten that are summed up below (e.g. [10]).

Every non-trivial ideal of $\mathcal{B}(H)$ is self-adjoint, contains the ideal $\mathcal{F}(H)$ and is contained in the ideal of compact operators.

For a sequence $a = (a_n)_{n\in\mathbb{N}}$ of complex numbers converging to 0, a^* is the sequence of the absolute values of a_n's arranged in decreasing order with the appropriate multiplicities. A Calkin space s is a subspace of c_0 with the property that $a \in s$, $b \in c_0$ and $b_n^* \leqslant a_n^*$ $(n \in \mathbb{N})$ imply $b \in s$. Now, Calkin's theorem reads as follows (cf. [10, Theorem 2.5]).

If $\mathcal{I}$ is a proper ideal in $\mathcal{B}(H)$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ is a fixed orthonormal system in H, then

$$s(\mathcal{I}) = \left\{a = (a_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} a_n \varphi_n \otimes \varphi_n \in \mathcal{I}\right\}$$

is a Calkin space.

If s is a Calkin space, then the family $\mathcal{I}(s)$ of compact operators whose decreasing sequences of singular values (extended with 0's in the finite case) belong to s is a proper ideal of $\mathcal{B}(H)$.

Moreover, we have $\mathcal{I}(s(\mathcal{I})) = \mathcal{I}$ and $s(\mathcal{I}(s)) = s$.

Let f denote the subspace of c_0 consisting of all sequences with finitely many non-zero elements. A norm Φ on f is called symmetric if $\Phi(a) = \Phi(a^*)$ for every $a \in f$. Let

$$s_\Phi = \{a \in c_0 \colon \lim_{n\to\infty} \Phi(a_1, \ldots, a_n, 0, 0, \ldots) \text{ exists and is finite}\}.$$

Then s_Φ is a Calkin space onto which Φ can be extended as a Banach space norm in a natural way. The property of s_Φ that is most important for us can be found in [10, Theorem 1.16].

If $(\alpha_{nm})_{n,m\in\mathbb{N}}$ is a doubly substochastic matrix (i.e. $\sum\limits_{n=1}^{\infty} |\alpha_{nk}| \leqslant 1$, $\sum\limits_{m=1}^{\infty} |\alpha_{lm}| \leqslant 1$ $(k, l \in \mathbb{N})$), then $(\alpha a^T)^T \in s_\Phi$ for every $a \in s_\Phi$.

The ideal $\mathcal{I}(s_\Phi)$ is called a symmetric norm ideal on which there is a Banach space norm that is in a natural relation with Φ. Note that $\mathcal{F}(H) = \mathcal{I}(s_\Phi)$ cannot occur, which follows from the fact that there is no Banach space norm on f. The standard examples of symmetric norm ideals are the Schatten-von Neumann p-classes of compact operators.

Main results

In the proofs of our results we shall need the following observation. Let $\{e_n\}_{n\in\mathbb{N}}$ be a c.o.n.s. (complete orthonormal system) in H. Suppose that $(\mu_n)_{n\in\mathbb{N}}$ is a bounded sequence of non-zero complex numbers and $(\lambda_n)_{n\in\mathbb{N}} \in \ell_\infty$. If

$$A = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n \quad \text{and} \quad T = \sum_{n=1}^{\infty} \mu_n e_n \otimes e_{n+1},$$

then for arbitrary $S \in \mathcal{B}(H)$ the equation

$$TA - AT = TS - ST$$

implies

$$\sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)\mu_n e_n \otimes e_{n+1} = \sum_{n=1}^{\infty} \mu_n e_n \otimes S^* e_{n+1} - \sum_{n=1}^{\infty} \mu_n S e_n \otimes e_{n+1}.$$

Considering the operators on both sides at e_{n+1} and then taking inner products with e_n, we obtain that

$$(\lambda_{n+1} - \lambda_n)\mu_n = \mu_n \langle e_{n+1}, S^* e_{n+1}\rangle - \mu_n \langle S e_n, e_n\rangle,$$

which implies

$$\lambda_n - \lambda_1 = \langle S e_n, e_n\rangle - \langle S e_1, e_1\rangle \quad (n \in \mathbb{N}). \tag{1}$$

Our first assertion will ensure the existence of a standard operator algebra on which not every locally inner derivation is inner.

Theorem 1. *There are exactly three standard operator algebras on which every derivation is a locally inner derivation, namely $\mathcal{F}(H)$, $\mathcal{F}(H) \oplus \mathbb{C}I$ and $\mathcal{B}(H)$.*

Proof. We know that every derivation on a standard operator algebra is of the form $T \mapsto TA - AT$ with an appropriate $A \in \mathcal{B}(H)$. Hence, to prove that $\mathcal{F}(H)$, $\mathcal{F}(H) \oplus \mathbb{C}I$ and $\mathcal{B}(H)$ have the property in the statement, it is sufficient to consider only the case of $\mathcal{F}(H)$. But if $A \in \mathcal{B}(H)$ is arbitrary and $T \in \mathcal{F}(H)$, then for the orthogonal projection P onto the finite-dimensional subspace generated by the ranges of T and $(TA)^*$ we have

$$TA - AT = T(AP) - (AP)T.$$

Now, let $\mathcal{A}$ be a standard operator algebra on which every derivation is locally inner. We infer

$$TA - AT \in \mathcal{A} \qquad (T \in \mathcal{A},\ A \in \mathcal{B}(H)).$$

This means that $\mathcal{A}$ is a so-called Lie ideal. Then, by [3, Theorem 2] there is an (associative) ideal $\mathcal{I}$ in $\mathcal{B}(H)$ for which

$$[\mathcal{I}, \mathcal{B}(H)] \subset \mathcal{A} \subset \mathcal{I} + \mathbb{C}I,$$

where $[\mathcal{I}, \mathcal{B}(H)]$ denotes the subspace generated by the elements of $TA - AT$ ($T \in \mathcal{I}$, $A \in \mathcal{B}(H)$). Since for arbitrary $A \in \mathcal{B}(H)$ and $T \in \mathcal{I}$ we have $TA - AT \in \mathcal{A}$, thus there is an $S \in \mathcal{I}$ such that

$$(TA - AT)A - A(TA - AT) = (TA - AT)S - S(TA - AT). \tag{2}$$

We prove that $\mathcal{I} = \mathcal{F}(H)$ or $\mathcal{I} = \mathcal{B}(H)$. To this end, suppose that there is an element in $\mathcal{I}$ with an infinite dimensional range. Then the set $\{\mu_n\}_{n\in\mathbb{N}}$ of its singular values is certainly infinite. Let $(\lambda_n)_{n\in\mathbb{N}} \in \ell_\infty$ be a non-convergent sequence with pairwise different elements. Let A and T be just as in the first part of this section. We have $T \in \mathcal{I}$. If $S \in \mathcal{I}$ is such that (2) holds, then just as above we can arrive at

$$\lambda_n - \lambda_1 = \langle Se_n, e_n\rangle - \langle Se_1, e_1\rangle \quad (n \in \mathbb{N}).$$

However, in this case $\langle Se_n, e_n\rangle \not\to 0$, i.e. S cannot be compact. Since $\mathcal{I}$ contains a non-compact element, it follows that $\mathcal{I} = \mathcal{B}(H)$.

To complete the proof, observe that in the case $\mathcal{I} = \mathcal{F}(H)$ we have $\mathcal{A} = \mathcal{F}(H)$ or $\mathcal{A} = \mathcal{F}(H) \oplus \mathbb{C}I$. Moreover, by $[\mathcal{B}(H), \mathcal{B}(H)] = \mathcal{B}(H)$ (e.g. [3, Corollary 2, p. 187]), $\mathcal{I} = \mathcal{B}(H)$ implies $\mathcal{A} = \mathcal{B}(H)$. □

Turning back to our remark right before Theorem 1, consider the standard operator algebra $\mathcal{F}(H)$. If we show that there is a derivation on it which is not inner, then by the previous theorem this will mean the existence of a locally inner derivation which is not inner. In fact, since the commutant of $\mathcal{F}(H)$ is $\mathbb{C}I$, it is easy to see that a derivation $T \mapsto TA - AT$ on this algebra is inner if and only if A is the sum of a finite rank operator and a scalar multiple of the identity. Now, take an $A \in \mathcal{B}(H)$ which cannot be written in such a form to obtain the desired statement.

Theorem 2. *Let $\mathcal{I}$ be a symmetric norm ideal in $\mathcal{B}(H)$. Then every locally inner derivation on $\mathcal{I}$ is an inner derivation.*

P r o o f. Suppose that $\mathcal{I}$ is non-trivial. Then every locally inner derivation on $\mathcal{I}$ is a derivation. Let $A \in B(H)$ be such that for every $T \in \mathcal{I}$ there is an $S_T \in \mathcal{I}$ for which

$$TA - AT = TS_T - S_T T.$$

Considering this relation for T^* and taking adjoints we immediately have

$$TA^* - A^*T = TS^*_{T^*} - S^*_{T^*}T.$$

It follows from these equations that the self-adjoint operators $\operatorname{Re} A$ and $\operatorname{Im} A$ induce locally inner derivations on $\mathcal{I}$. Consequently, we may suppose that A is self-adjoint.

By the well-known Weyl-von Neumann theorem there are a c.o.n.s. $\{e_n\}_{n\in\mathbb{N}}$, a bounded sequence $(\nu_n)_{n\in\mathbb{N}}$ of real numbers and a compact self-adjoint operator K such that

$$A = \sum_{n=1}^{\infty} \nu_n e_n \otimes e_n + K.$$

Since $\mathcal{I} \neq \mathcal{F}(H)$, $\mathcal{I}$ must contain an operator with an infinite dimensional range. Let T be such as in the proof of Theorem 1. Since there is a compact operator S for which

$$T(A-K) - (A-K)T = TS - ST,$$

thus (1) implies that $(\nu_n)_{n\in\mathbb{N}}$ converges to a real ν. Let

$$A' = A - \nu I,$$

which is a compact self-adjoint operator inducing the same locally inner derivation as A does. Consequently, we may suppose even that A is compact.

But such an A can be written in the form

$$A = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n,$$

where $\lambda_n \to 0$ and $\{e_n\}_{n\in\mathbb{N}}$ is a c.o.n.s. Let $T \in \mathcal{I}$ be just as above and let $S \in \mathcal{I}$ be such that

$$TA - AT = TS - ST.$$

From (1) we then have

$$\lambda_n = \langle Se_n, e_n\rangle \quad (n \in \mathbb{N}).$$

If $S = \sum\limits_k a_k \psi_k \otimes \varphi_k$ is the canonical expansion of S, then

$$\langle Se_n, e_n\rangle = \sum_k a_k \langle e_n, \varphi_k\rangle \langle \psi_k, e_n\rangle.$$

Since

$$\sum_k |\langle e_n, \varphi_k\rangle|\,|\langle \psi_k, e_n\rangle| \leqslant \|e_n\|^2 = 1,$$
$$\sum_n |\langle e_n, \varphi_k\rangle|\,|\langle \psi_k, e_n\rangle| \leqslant \|\varphi_k\|\,\|\psi_k\| \leqslant 1,$$

by the introduction we have $A \in \mathcal{I}$. □

To conclude the paper we propose the following open

Question. Is there a non-trivial ideal of $\mathcal{B}(H)$ different from $\mathcal{F}(H)$ for which the conclusion of Theorem 2 fails?

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References

[1] *M. Brešar and P. Šemrl*: Mappings which preserve idempotents, local automorphisms and local derivations. Canad. J. Math. *45* (1993), 483–496.
[2] *P.R. Chernoff*: Representations, automorphisms and derivations of some operator algebras. J. Funct. Anal. *12* (1973), 275–289.
[3] *C.K. Fong, C.R. Miers and A.R. Sourour*: Lie and Jordan ideals of operators on Hilbert space. Proc. Amer. Math. Soc. *84* (1982), 516–520.
[4] *P.R. Halmos*: Hilbert Space Problem Book. D. Van Nostrand Company, Princeton, New York, 1967.
[5] *R.V. Kadison*: Local derivations. J. Algebra *130* (1990), 494–509.
[6] *D.R. Larson and A.R. Sourour*: Local derivations and local automorphisms of $B(X)$. Proc. Sympos. Pure Math. 51. Part 2, Providence, Rhode Island 1990, pp. 187–194.
[7] *S. Sakai*: Derivations of W*-algebras. Ann. Math. *83* (1966), 273–279.
[8] *P. Šemrl*: Additive derivations of some operator algebras. Illinois J. Math. *35* (1991), 234–240.
[9] *P. Šemrl*: Ring derivations on standard operator algebras. J. Funct. Anal. *112* (1993), 318–324.
[10] *B. Simon*: Trace Ideals and Their Applications. Cambridge University Press, Cambridge, 1979.

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