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ON CENTRALLY SYMMETRIC GRAPHS

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Dedicated to the memory of Eva Gedeonov

Abstract. In this note we extend results on the covering graphs of modular lattices (Zelinka) and semimodular lattices (Gedeonová, Duffus and Rival) to the covering graph of certain graded lattices.

Keywords: covering graph, symmetric graph, centrally symmetric lattice, semimodular lattice, graded lattice, strong lattice

MSC 1991: 06C10, 05C99

1. Introduction

A finite connected graph $G = (V, H)$ is a symmetric graph (S-graph, for short) if for every vertex $v \in V$ there exists a graded poset $(V, \leqslant_v)$ with least element v such that the covering graph of $(V, \leqslant_v)$ is isomorphic to the graph $G = (V, H)$. An S-graph is a centrally symmetric graph (CS-graph, for short) if $(V, \leqslant_v)$ is a graded lattice for each $v \in V$. The notion of an S-graph was introduced by A. Kotzig [7] and that of a CS-graph by D. Duffus and I. Rival [2].

If the covering graph of a lattice is a CS-graph or an S-graph, then the lattice will be called a CS-lattice or S-lattice, respectively. It is clear that every CS-lattice is an S-lattice while the converse does not hold in general (see E. Gedeonová [5]).

The direct product of n copies of the 2-element Boolean algebra is the Boolean algebra 2^n with n atoms whose covering graph is the highly symmetrical n-dimensional cube which is, of course, centrally symmetric. A nonmodular example is provided by the "benzine ring" K_4.

For an integer n, let K_{2n} denote the lattice determined by

$$0 = a_0 \prec a_1 \prec \ldots \prec a_n = 1 \qquad \text{and} \qquad 0 = b_0 \prec b_1 \prec \ldots \prec b_n = 1,$$

where $\prec$ denotes the covering relation. A. Kotzig [7] conjectured that a graph is centrally symmetric if and only if it is the covering graph of the direct product of lattices, each isomorphic to K_{2n} for some integer n. E. Gedeonová [4] provided a counterexample to this conjecture. However, in the modular case we have the following positive result due to B. Zelinka [12] (in what follows we denote by $C(L)$ the covering graph of a lattice L):

Theorem 1. *Let L be a finite modular lattice with n atoms. If $C(L)$ is centrally symmetric, then $L \cong 2^n$.*

This result was sharpened in

Theorem 2. *Let L be a finite semimodular lattice with n atoms. If $C(L)$ is centrally symmetric, then $L \cong 2^n$.*

Proofs were given by D. Duffus and I. Rival [2], Theorem 4.2, for CS-lattices (using properties of the distance function), and by E. Gedeonová [5], Theorem 11, for S-lattices.

It is the aim of the present note to prove still somewhat more, namely

Theorem 3. *Let L be a finite graded lattice with n atoms such that its dual lattice $L^\star$ is strong. If $C(L)$ is centrally symmetric, then $L \cong 2^n$.*

2. Some basic facts and proof of the theorem

Let us first recall some basic facts. For undefined notions we refer to standard books like P. Crawley and R. P. Dilworth [1] and G. Grätzer [6]. Let L always denote a finite lattice. The least and greatest elements of such a lattice are denoted by 0 and 1, respectively. By $c \prec d$ we mean that c is a lower cover of d (or, equivalently, that d is an upper cover of c). An atom (dual atom) is an upper cover of 0 (a lower cover of 1, respectively). A lattice is atomistic if each of its elements is a join of atoms. Dually atomistic lattices are defined in a dual way. If, for all $x, y \in L$, $x \wedge y \prec x$ implies $y \prec x \vee y$, then L is called (upper) semimodular. A lattice L is called lower semimodular if its dual $L^\star$ is upper semimodular. Any upper or lower semimodular lattice is graded.

The concept of a strong lattice is due to U. Faigle [3] and may be defined in the following way: denote by j a join-irreducible ($\neq 0$) and by j_+ its unique lower cover; a lattice L is said to be strong if, for all join-irreducibles j ($\neq 0$) and for all b, $j \leqslant b \vee j_+$ implies $j \leqslant b$. It is easy to see that every modular lattice is strong. An upper semimodular lattice may or may not be strong. In fact, for upper semimodular lattices the property "strong" is equivalent to "consistent" in the sense of J. P. S. Kung [8] and to "balanced" as defined by K. Reuter [9]. In contrast to this we have the following result due to U. Faigle (see M. Stern [10], Corollary 18.4):

(a) Every lower semimodular lattice is strong.

Dualizing this we get that, if L is an upper semimodular lattice, then its dual lattice $L^\star$ is strong. Hence what we do is to replace the condition "L is finite (upper) semimodular" of Theorem 2 by the property "L is graded and its dual $L^\star$ is strong" in Theorem 3. Let us observe that a lattice L which is graded and whose dual lattice is strong need not be upper semimodular: just take for L the dual of a nonmodular geometric lattice. Thus Theorem 3 is indeed a strengthening of Theorem 2. For our proof of Theorem 3 we shall further need the following properties and results:

(b) Every CS-lattice is self-dual (E. Gedeonová [4], Theorem 8 (iii)).

(c) The greatest element of a CS-lattice L is the (irredundant) join of all atoms of L (D. Duffus and I. Rival [2], Section 4).

(d) Let L be a lattice whose greatest element is a join of atoms. Then L is dually atomistic if and only if the dual lattice $L^\star$ is strong (M. Stern [11], Theorem 4).

(e) From the definition of a CS-graph $G = (V, H)$ it follows that to every vertex $v \in V$ there exists a unique vertex v' satisfying $\delta(v, v') = \mathrm{diam}\ (G)$ (here $\delta(x, y)$ denotes the distance from x to y and $\mathrm{diam}(G)$ the diameter of G).

(f) If L is a CS-lattice and p is an atom of L, then for every $z \in L$ either $p \leqslant z$ or $z \leqslant p'$ holds (E. Gedeonová [4], Theorem 8 (iv)).

(g) Let L be a CS-lattice. Then m is a dual atom if and only if m' is an atom (D. Duffus and I. Rival [2], Section 4).

(h) Let L be a CS-lattice, let b ($\in L$) be a join of atoms and $0 \prec p \leqslant b$. Then $b \wedge p' \prec b$ (E. Gedeonová [5], Lemma 2).

Now we give a proof of Theorem 3:

P r o o f. Let L be a lattice satisfying the assumptions of the theorem. We show that L is modular whence the assertion follows from Theorem 1. Since L is self-dual by (b), we first observe that strongness of $L^\star$ implies strongness of L. By (c) the greatest element is a join of atoms. This implies by (d) that L is dually atomistic. The self-duality of L implies that it is also atomistic. Next we show that L is lower semimodular. Since L is dually atomistic, it suffices to prove that for every $z \in L$ and for every dual atom m which is incomparable with z, the relation $z \wedge m \prec z$

holds. Now if $z \in L$ and if m is a dual atom incomparable with z, then $m \ngeqslant z$. Hence it follows by the dual of property (f) that $z \geqslant m'$. Condition (g) implies that m' is an atom. Moreover, z is a join of atoms since L is atomistic. Thus the assumptions of condition (h) are satisfied with $p = m'$ and $b = z$. Since $p' = m$, we conclude by (h) that $z \wedge m \prec z$, that is, L is lower semimodular. Now upper semimodularity follows since L is self-dual. Upper and lower semimodularity together yield that L is modular and thus Theorem 3 is reduced to Theorem 1. □

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