ON JÓNSSON'S THEOREM

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Abstract. A proof of Jónsson's theorem inspired by considering a natural topology on algebraic lattices is given.

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In [5] and [6] we have introduced the concept of compactness of a set of congruences and shown that this concept is central in the study of sheaf representation of semi–irreducible algebras (see, for example, [5, Corollary 1 of 3.3, Corollary 1 of 3.4 and 3.5] and [6, 8.3 and Corollary 1 of 8.3]). In this note we give a natural proof of Jónsson's famous theorem [3] using the concept of compactness of a set of congruences. The theorem states that every subdirectly irreducible algebra in the variety generated by a class $\mathscr{K}$ of similar algebras is in $HSP_u(\mathscr{K})$, provided $V(\mathscr{K})$ is congruence distributive. This theorem is the main impetus for the resurgence in the study of varieties of algebras, in particular lattice varieties.

For basic facts and notation on universal algebra we refer the reader to [2]. If $\mathscr{K}$ is a class of algebras then $I(\mathscr{K})$, $S(\mathscr{K})$, $P(\mathscr{K})$, $P_u(\mathscr{K})$ and $H(\mathscr{K})$ denote the classes of isomorphic images, subalgebras, direct products, ultraproducts and homomorphic images of members of $\mathscr{K}$, respectively, and $V(\mathscr{K})$ is the variety generated by $\mathscr{K}$. For any algebra A, $\mathbb{I}^A$ is the universal congruence on A and $\mathbb{O}^A$ is the trivial congruence on A. $\mathrm{Con}(A)$ denotes the congruence lattice of A. Given a set B, by an *ultrafilter* over B we understand a prime filter of the Boolean algebra $\{S\colon S \subseteq B\}$ (see [2]).

Let L be an algebraic lattice. Suppose B_1, B_2 are subsets of L. We say that B_1 *minorizes* B_2 when for each $x \in B_2$ there exists $y \in B_1$ such that $y \leqslant x$. A set $B \subseteq L$ is called *compact* if for every set C of compact elements of L which minorizes B there exists a finite set $C_0 \subseteq C$ which minorizes B. If we consider in L the topology with the basis

$$\{\{x \in L \colon x \geqslant c\} \colon c \text{ is a compact element of } L\},$$

then L becomes a topological lattice [1, p. 63], and the above defined notion of compactness coincides with the one in this topological lattice.

An element $x \in L$ is called *meet irreducible (prime)* if for every $z, w \in L$ we have that $x = z \wedge w$ ($x \geqslant z \wedge w$) implies $x = z$ or $x = w$ ($x \geqslant z$ or $x \geqslant w$). It is well known that if L is distributive, then meet irreducibles and meet primes coincide.

Lemma 1. *x is meet prime iff for every compact set $B \subseteq L$, $x \geqslant \bigwedge B \Rightarrow x \geqslant z$ for some $z \in B$.*

Proof. Let $B \subseteq L$ be a compact set such that $x \geqslant \bigwedge B$. Suppose there is no such z. Then for each $z \in B$ let c_z be such that $c_z \leqslant z$ and $c_z \nleqslant x$. The set $\{c_z \colon z \in B\}$ minorizes B. Therefore there exists a finite set $B_0 \subseteq B$ such that $\{c_z \colon z \in B_0\}$ minorizes B. Thus we have $\bigwedge\{c_z \colon z \in B_0\} \leqslant \bigwedge B \leqslant x$, which is absurd. □

Let $B \subseteq L$. For each ultrafilter $\mathscr{U}$ over B, let $I_{\mathscr{U}} = \{c \colon c \text{ is compact and } \{x \in B \colon x \geqslant c\} \in \mathscr{U}\}$. Note that $I_{\mathscr{U}}$ is an ideal in the semilattice of compact elements. Let $x_{\mathscr{U}} = \bigvee I_{\mathscr{U}}$ and define $B' = \{x_{\mathscr{U}} : \mathscr{U} \text{ is an ultrafilter over } B\}$.

Lemma 2. *Let c be a compact element of L. Then $c \in I_{\mathscr{U}}$ if and only if $c \leqslant x_{\mathscr{U}}$.*

Lemma 3. *$B \subseteq B'$, $B'' = B'$ and $B_1 \subseteq B_2$ implies $B_1' \subseteq B_2'$.*

Proof. Note that for every $x \in B$ we have $x = x_{\mathscr{U}}$, where $\mathscr{U} = \{S \subseteq B \colon x \in S\}$. Thus $B \subseteq B'$. Let $\mathscr{Y}$ be an ultrafilter over B'. Let

$$\mathscr{U}_1 = \big\{S \subseteq B \colon \{x_{\mathscr{U}} : \mathscr{U} \in U_L \text{ and } S \in \mathscr{U}\} \in \mathscr{Y}\big\},$$

where U_L denotes the set of all ultrafilters over B. It is easy to check that $\mathscr{U}_1 \in U_L$. Furthermore if c is compact, then

$$\begin{aligned} c \in I_{\mathscr{Y}} \quad & \text{iff } \{x \in B' : x \geqslant c\} \in \mathscr{Y} \\ & \text{iff } \{x_{\mathscr{U}} : \mathscr{U} \in U_L \text{ and } c \in I_{\mathscr{U}}\} \in \mathscr{Y} \\ & \text{iff } \big\{x_{\mathscr{U}} : \mathscr{U} \in U_L \text{ and } \{x \in B \colon x \geqslant c\} \in \mathscr{U}\big\} \in \mathscr{Y} \\ & \text{iff } \{x \in B \colon x \geqslant c\} \in \mathscr{U}_1 \\ & \text{iff } c \in I_{\mathscr{U}_1}. \end{aligned}$$

Thus we have proved that $B'' \subseteq B'$. Next suppose that $B_1 \subseteq B_2$. Let $\mathscr{U}_1$ be an ultrafilter over B_1. Let $\mathscr{U}_2$ be an ultrafilter over B_2 such that $\mathscr{U}_1 \subseteq \mathscr{U}_2$. It is easy to check that $I_{\mathscr{U}_1} = I_{\mathscr{U}_2}$. □

Lemma 4. *B' is compact.*

Proof. Suppose that C is a set of compact elements such that no finite subset of C minorizes B'. It is easy to check that the set $\{\{x \in B \colon c \not\leqslant x\} \colon c \in C\}$ can be extended to an ultrafilter $\mathscr{U}$ over B, which by Lemma 2 implies that C does not minorize $\{x_{\mathscr{U}}\}$. $\square$

Lemma 5. *Let A be an algebra. If $B \subseteq \mathrm{Con}(A)$, then $A/x_{\mathscr{U}} \in ISP_u(\{A/\theta \colon \theta \in B\})$.*

Proof. Let f be the map

$$\begin{aligned} &A \to \Pi\{A/\theta \colon \theta \in B\}/\mathscr{U}. \\ &a \to \langle a/\theta \colon \theta \in B\rangle /\mathscr{U} \end{aligned}$$

Note that $f(a) = f(b)$ iff $\{\theta \in B \colon (a,b) \in \theta\} \in \mathscr{U}$ iff $\{\theta \in B \colon \theta(a,b) \subseteq \theta\} \in \mathscr{U}$ iff $\theta(a,b) \subseteq x_{\mathscr{U}}$ iff $(a,b) \in x_{\mathscr{U}}$, where $\theta(a,b)$ is the principal congruence generated by the pair (a,b). $\square$

We recall that a class is universal (i.e., definable by a set of universal first-order sentences) iff it is of the form $ISP_u(\mathscr{K})$ for some class $\mathscr{K}$. In particular, we obtain that $ISP_uISP_u(\mathscr{K}) = ISP_u(\mathscr{K})$.

Theorem (Jónsson [3])**.** *Let $\mathscr{K}$ be a class such that $V(\mathscr{K})$ is a congruence distributive variety. Then the subdirectly irreducible elements of $V(\mathscr{K})$ are in $HSP_u(\mathscr{K})$.*

Proof. Let $A \in SP(\mathscr{K})$ and let $B = \{\theta \in \mathrm{Con}(A) \colon A/\theta \in ISP_u(\mathscr{K})\}$. Note that by Lemmas 3, 4 and 5, $B = B'$ is compact. Furthermore note that $\bigcap B = \mathbb{O}^A$. Suppose A/θ is subdirectly irreducible. Since $\mathrm{Con}(A)$ is distributive, θ is a meet prime element of $\mathrm{Con}(A)$ and therefore there exists $\delta \in B$ such that $\theta \supseteq \delta$ (Lemma 1). Thus $A/\theta \in HSP_u(\mathscr{K})$. Now, recall that $V(\mathscr{M}) = HSP(\mathscr{M})$. $\square$

In a similar way as above the generalizations of Jónsson's theorem for quasivarieties, established by D. Pigozzi in [4], can be proved.

References

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