ASYMPTOTIC BEHAVIOR OF T-PERIODIC SOLUTIONS OF SINGULARLY PERTURBED SECOND-ORDER DIFFERENTIAL EQUATION

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Abstract. We examine the asymptotic behavior of T-periodic solutions of the singularly perturbed differential equation $\mu y'' = f(t, y)$ as a small parameter μ tends to zero.

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1. Introduction

The problem of existence of a T-periodic solution for differential equations was examined by Mawhin [2, 3], Fučík [1, 2] and others applying various methods. We establish sufficient conditions for existence of T-periodic solutions for the singularly perturbed semilinear differential equation $\mu y'' = f(t, y)$ which converge to a solution of the reduced problem (RP, for short) $f(t, u) = 0$ as the small parameter μ tends to zero, using the method of upper and lower solutions. We will consider the second-order differential equation

$$\mu y'' = f(t, y) \tag{1}$$

where $f \in C^1(\mathbb{R}^2)$ is a T-periodic function in the variable t and μ is a small positive parameter. This is a singular perturbation problem because the order of the differential equation drops when μ becomes zero. We can think of this equation as the mathematical model of nonlinear dynamical systems with a high-speed back coupling.

Without loss of generality we can consider the interval $[0, T]$. Denote

$$D_\delta(u) = \{(t, y) \colon 0 \leqslant t \leqslant T, |y - u(t)| < d(t)\}, \tag{2}$$

where $d(t)$ is the positive continuous function on $[0, T]$ such that

$$d(t) = |u(0) - u(T)| + \delta \quad \text{for } 0 \leqslant t \leqslant \delta/2 \text{ and } T - \delta/2 \leqslant t \leqslant T$$

and

$$d(t) = \delta \quad \text{for } \delta \leqslant t \leqslant T - \delta,$$

δ is a small positive constant and $u \in C^2$ is a solution of RP.

2. Main result

The following theorem is the main result of this paper.

Theorem. *Let $f \in C^1(\mathbb{R}^2)$ be T-periodic. Let $u \in C^2(\mathbb{R})$ be a T-periodic function such that $f\bigl(t, u(t)\bigr) = 0$ on $\mathbb{R}$. Let $\delta > 0$ be such that*

$$\frac{\partial f(t, y)}{\partial y} \geqslant m > 0 \quad \text{for every } (t, y) \in D_\delta(u). \tag{3}$$

Then there exists μ_0 such that for each $\mu \in (0, \mu_0]$ the problem (1) has a unique T-periodic solution defined on $\mathbb{R}$ which converges uniformly to the solution of RP *$f(t, u) = 0$ as μ tends to zero.*

Example. Consider the problem $\mu y'' = y + \arctan y + \sin 2\pi t$. The function $f(t, y) = y + \arctan y + \sin 2\pi t$ is a T-periodic function for $T = 1$ and satisfies the condition (3) for $m = 1$. By virtue of Theorem, its unique 1-periodic solution tends uniformly to the solution of RP on $\mathbb{R}$.

3. Proof of Theorem

Theorem follows easily and immediately as a special case of the following lemma.

Lemma 1. *Consider the periodic boundary value problem*

$$(1')\qquad \begin{gathered}\mu y'' = f(t,y), \quad t \in [0,T]\\ y(0,\mu) - y(T,\mu) = 0, \quad y'(0,\mu) - y'(T,\mu) = 0.\end{gathered}$$

Let a function $f \in C^1\big(D_\delta(u)\big)$ satisfy the condition (3) where $D_\delta(u)$ is defined in (2). Then there exists μ_0 such that for each $\mu \in (0,\mu_0]$ the problem (1′) has a unique solution, satisfying the inequality

$$-v_{11} - v_2 - C\mu \leqslant y(t,\mu) - u(t) \leqslant v_{12} + v_2 + C\mu$$

for $u(0) \geqslant u(T)$ and

$$v_{12} - v_2 - C\mu \leqslant y(t,\mu) - u(t) \leqslant -v_{11} + v_2 + C\mu$$

for $u(0) \leqslant u(T)$, where

$$\begin{aligned} v_{11}(t,\mu) &= \big(u(0)-u(T)\big)\frac{\exp[-(m/\mu)^{1/2}(t-T)]}{\exp[(m/\mu)^{1/2}T]-1},\\ v_{12}(t,\mu) &= \big(u(0)-u(T)\big)\frac{\exp[(m/\mu)^{1/2}t]}{\exp[(m/\mu)^{1/2}T]-1},\\ v_2(t,\mu) &= |u'(0)-u'(T)|\frac{\exp[-(m/\mu)^{1/2}t)]+\exp[-(m/\mu)^{1/2}(T-t)]}{2(m/\mu)^{1/2}(1-\exp[-(m/\mu)^{1/2}T])} \end{aligned}$$

and $C \geqslant \max\{|u''(t)|m^{-1} : t \in [0,T]\}$ is a positive constant.

P r o o f. We apply the method of upper and lower solutions. As usual, we say that $\alpha \in C_2([0,T])$ is a lower solution for (1′) if $\alpha(0,\mu) - \alpha(T,\mu) = 0$, $\alpha'(0,\mu) - \alpha'(T,\mu) \geqslant 0$, and $\mu\alpha''(t,\mu) \geqslant f\big(t,\alpha(t,\mu)\big)$ for every $t \in [0,T]$. An upper solution $\beta \in C^2([0,T])$ satisfies $\beta(0,\mu) - \beta(T,\mu) = 0$, $\beta'(0,\mu) - \beta'(T,\mu) \leqslant 0$ and $\mu\beta''(t,\mu) \leqslant f\big(t,\beta(t,\mu)\big)$ for every $t \in [0,T]$. The proof is based upon the following lemma.

Lemma 2 (compare with [4], Theorem 3)**.** *Let $f \in C([0,T]\times\mathbb{R})$. If α, β are respectively lower and upper solutions for (1′) such that $\alpha \leqslant \beta$ on $[0,T]$, then there exists a solution y of (1′) with $\alpha \leqslant y \leqslant \beta$ on $[0,T]$.*

For $u(0) \geqslant u(T)$ we define the lower solutions by

$$\alpha(t,\mu) = u(t) - v_{11} - v_2 - \Gamma$$

and the upper solutions by

$$\beta(t,\mu) = u(t) + v_{12} + v_2 + \Gamma$$

(in the case $u(0) \leqslant u(T)$ we proceed analogously).

Here $\Gamma(\mu) = \mu\tau/m$, where τ is a constant which will be defined below. One can easily check that the functions α, β satisfy the boundary conditions required for the lower and upper solutions of $(1')$ and $\alpha \leqslant \beta$ on $[0,T]$. Now we show that $\mu\alpha''(t,\mu) \geqslant f\big(t,\alpha(t,\mu)\big)$ and $\mu\beta''(t,\mu) \leqslant f\big(t,\beta(t,\mu)\big)$ on $[0,T]$. By Taylor theorem we obtain

$$\begin{aligned} f\big(t,\alpha(t,\mu)\big) &= f\big(t,\alpha(t,\mu)\big) - f\big(t,u(t)\big) \\ &= \frac{\partial f(t,\theta(t,\mu))}{\partial y}\big(v_{11}(t,\mu) + v_2(t,\mu) + \Gamma(\mu)\big), \end{aligned}$$

where $\big(t,\theta(t,\mu)\big)$ is a point between $\big(t,\alpha(t,\mu)\big)$ and $\big(t,u(t)\big)$, $\big(t,\theta(t,\mu)\big) \in D_\delta(u)$ for sufficiently small μ, for instance if $\mu \in (0,\mu_0]$. Then

$$\mu\alpha''(t,\mu) - f\big(t,\alpha(t,\mu)\big) \geqslant \mu u'' - \mu v_{11}'' - \mu v_2'' + m(v_{11} + v_2 + \Gamma) \geqslant -\mu|u''| + \mu\tau$$

(because $\mu v_{11}'' = m v_{11}$ and $\mu v_2'' = m v_2$ on $[0,T]$) for every $t \in [0,T]$. If we choose a constant τ such that $\tau \geqslant |u''(t)|$, $t \in [0,T]$ then $\mu\alpha''(t,\mu) \geqslant f\big(t,\alpha(t,\mu)\big)$ in $[0,T]$. The inequality for β can be proved similarly. The existence of solutions of $(1')$ satisfying the just stated inequalities follows from the above considerations. Since f is increasing in the variable y the solution of $(1')$ is unique. □

R e m a r k. We note (on the basis of Lemma 1) that if $f(0,y) \neq f(T,y)$ (i.e. $u(0) \neq u(T)$) then there are initial and endpoint nonuniformities (i.e. the solution y of $(1')$ tends uniformly to a solution u of RP on every compact set $K \subset (0,T)$, but $|y'(0,\mu)|$ $\big(=|y'(T,\mu)|\big) \to \infty$ as $\mu \to 0$).

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