ATOMARY TOLERANCES ON FINITE ALGEBRAS

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Abstract. A tolerance on an algebra is defined similarly to a congruence, only the requirement of transitivity is omitted. The paper studies a special type of tolerance, namely atomary tolerances. They exist on every finite algebra.

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Let $\mathbf{A} = \langle A, \mathscr{F} \rangle$ be a universal algebra. A reflexive and symmetric binary relation T on A is called a tolerance on $\mathbf{A}$, if for any operation $f \in \mathscr{F}$ and for any elements $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, where n is the arity of f, the inclusions $(x_i, y_i) \in T$ for $i = 1, \ldots, n$ imply $\big(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)\big) \in T$. If, moreover, T is transitive, T is a congruence on A [2].

Among tolerances on $\mathbf{A}$ there is always a universal binary relation U on A and an identity relation Δ on A. The tolerance on $\mathbf{A}$ form a lattice $\mathrm{Tol}(\mathbf{A})$ in which U is the greatest and Δ the least element. The meet in this lattice is equal to the set intersection.

Now let x, y be two elements of A. There exists at least one tolerance on $\mathbf{A}$ which contains the pair (x, y), namely U. Therefore there exists the intersection of all tolerances on $\mathbf{A}$ containing the pair (x, y); it is again a tolerance on $\mathbf{A}$ and we denote it by $T(x, y)$. If $x = y$, then obviously $T(x, y) = \Delta$.

This definition enables us to introduce a quasiordering TQ on the set A_2 of all elements $(x, y) \in A \times A$ such that $x \neq y$. We write $\big((a, b), (c, d)\big) \in TQ$ if and only if $(a, b) \in T(c, d)$. The reflexivity and transitivity of TQ is clear.

In a usual way we assign an ordering TO to TQ. Let E_{TQ} be the equivalence on A_2 defined so that $\big((a, b), (c, d)\big) \in E_{TQ}$ if and only if $\big((a, b), (c, d)\big) \in TQ$ and $\big((c, d), (a, b)\big) \in TQ$. For each $(x, y) \in A_2$ we denote by $E(x, y)$ the equivalence class of E_{TQ} which contains (x, y). Now TO is an ordering on the set of all equivalence classes of E_{TQ}; we put $\big(E(a,b), E(c,d)\big) \in TO$ if and only if $\big((a,b),(c,d)\big) \in TQ$.

Proposition 1. *Let (x, y) be such an element of A_2 that $E(x, y)$ is a minimal element in the ordering TO. Then $T(x, y) = E(x, y)$ and $T(x, y) = T(u, v)$ for each $(u, v) \in T(x, y)$, $u \neq v$.*

Proof can be easily done by the reader. □

Definition 1. A tolerance $T(x, y)$, where (x, y) is such a pair from A_2 that $E(x, y)$ is a minimal element in the ordering TO, is called an atomary tolerance on A.

In the sequel we will consider finite algebras.

Proposition 2. *Let $\mathbf{A} = \langle A, \mathscr{F} \rangle$ be a finite algebra, let T be a tolerance on $\mathbf{A}$, $T \neq \Delta$. Then there exists an atomary tolerance contained in T.*

Proof. As $T \neq \Delta$, there exists a pair $(x, y) \in T$ such that $x \neq y$. Obviously $T(x, y) \subseteq T$. Now consider a descending chain in TO starting with $E(x, y)$. As A is finite, it has a minimal element $E(u, v)$ for some u, v; this is an atomary tolerance $T(u, v) = E(u, v)$. We have $\big(E(u,v), E(x,y)\big) \in TO$, which implies $\big((u,v),(x,y)\big) \in TQ$ and $(u, v) \in T(x, y) \subseteq T$. As $(u, v) \in T$, also $T(u, v) \subseteq T$. □

Theorem 1. *Let $\mathbf{A} = \langle A, \mathscr{F} \rangle$ be a finite algebra. Then $\mathrm{Tol}(\mathbf{A})$ is a Boolean algebra if and only if the following two conditions are true:*

(i) *For each pair $(x, y) \in A_2$ the tolerance $T(x, y)$ is the join in $\mathrm{Tol}(\mathbf{A})$ of all atomary tolerances which are contained in it.*

(ii) *No atomary tolerance on A is contained in the join in $\mathrm{Tol}(\mathbf{A})$ of all the others.*

Proof. Let (i) and (ii) hold. Let $T \in \mathrm{Tol}(\mathbf{A}) - \{\Delta\}$. Then $T = \bigcup\limits_{(x,y)\in T} T(x, y)$. By (i) each $T(x, y)$ for $(x, y) \in T$ is the join of all atomary tolerances contained in it and thus T is the join of all atomary tolerances contained in T. Now let $\mathscr{S}_1$, $\mathscr{S}_2$ be two different sets of atomary tolerances on A and suppose that the join of all tolerances from $\mathscr{S}_1$ is equal to the join of all atomary tolerances from $\mathscr{S}_2$. As $\mathscr{S}_1 \neq \mathscr{S}_2$, at least one of the sets $\mathscr{S}_1 - \mathscr{S}_2$, $\mathscr{S}_2 - \mathscr{S}_1$ is non-empty; without loss of generality let $\mathscr{S}_1 - \mathscr{S}_2 \neq \emptyset$. Let $T_0 \in \mathscr{S}_1 - \mathscr{S}_2$. Then T_0 is contained in the join of all tolerances from $\mathscr{S}_2$ and therefore also in the join of all atomary tolerances different from T_0, which contradicts (ii). Hence each tolerance $T \in \mathrm{Tol}(\mathbf{A}) - \{\Delta\}$ is determined uniquely as the join of atomary tolerances. Evidently also any join of atomary tolerances is a tolerance. Therefore $\mathrm{Tol}(\mathbf{A})$ is isomorphic to the lattice of all subsets of the set of all atomary tolerances on A; this lattice is a Boolean algebra and $\mathrm{Tol}(\mathbf{A})$ is a Boolean algebra, too.

Now suppose that $\mathrm{Tol}(\mathbf{A})$ is a Boolean algebra. Each tolerance different from Δ contains an atomary tolerance as its subset and each atomary tolerance contains no other tolerance than Δ and itself. Therefore the set of atoms of $\mathrm{Tol}(\mathbf{A})$ is the set of all atomary tolerances on A. Every element of $\mathrm{Tol}(\mathbf{A})$ is uniquely determined as the join of atomary tolerances; in particular, this holds for tolerances $T(x,y)$, which implies (i). Every atomary tolerance is the complement of the join of all the others, which implies (ii). □

As the first example we will consider rectangular bands. A band is a semigroup in which all elements are idempotent. If, moreover, $xyx = x$ for any elements x, y, the band is called rectangular. It is well-known that every rectangular band is isomorphic to the direct product of a semigroup of left zeros with a semigroup of right zeros [1].

Therefore let B be a finite rectangular band. Let B be the direct product $L \times R$, where L is a semigroup of left zeros and R a semigroup of right zeros. Let x, y be two different elements of B; we may express them in the above mentioned direct product by $x = (x_1, x_2)$, $y = (y_1, y_2)$.

Lemma 1. *The tolerance $T(x,y)$ on B consists of the pairs $\big((x_1,x_2),(y_1,y_2)\big)$, $\big((y_1,y_2),(x_1,x_2)\big)$, $\big((x_1,y_2),(y_1,x_2)\big)$, $\big((y_1,x_2),(x_1,y_2)\big)$, of all pairs $\big((x_1,u_2),(y_1,u_2)\big)$, $\big((y_1,u_2),(x_1,u_2)\big)$ for $u_2 \in R$, of all pairs $\big((u_1,x_2),(u_1,y_2)\big)$, $\big((u_1,y_2),(u_1,x_2)\big)$ for $u_1 \in L$ and of all pairs from Δ.*

P r o o f can be easily done by the reader. □

Lemma 2. *The tolerance $T(x,y)$ on B is atomary if and only if either $x_1 = y_2$ and $x_2 \neq y_2$, or $x_1 \neq y_2$ and $x_2 = y_2$.*

P r o o f. If $x_1 = y_1$ and $x_2 \neq y_2$, then $T(x,y)$ consists of all pairs $\big((u_1,x_2),(u_1,y_2)\big)$, $\big((u_1,y_2),(u_1,x_2)\big)$ for $u_1 \in L$ and of all pairs from Δ. Evidently none of its proper subsets except Δ is a tolerance on B, therefore $T(x,y)$ is atomary. Similarly, if $x_1 \neq y_1$ and $x_2 = y_2$, then $T(x,y)$ consists of all pairs $\big((x_1,u_2),(y_1,u_2)\big)$, $\big((y_1,u_2),(x_1,u_2)\big)$ for all $u_2 \in R$ and of all pairs from Δ and it is also atomary. If $x_1 \neq y_1$ and $x_2 \neq y_2$, then $T(x,y) = T\big((x_1,x_2),(x_1,y_2)\big) \vee T\big((x_1,x_2),(y_1,x_2)\big)$ and therefore it is the join of two atomary tolerances and is not itself atomary. □

Theorem 2. *The lattice $\mathrm{Tol}(B)$ of all tolerances on a finite rectangular band B is a Boolean algebra.*

P r o o f. Consider a tolerance $T(x,y)$ for an arbitrary pair of distinct elements x, y of B, let $x = (x_1, x_2)$, $y = (y_1, y_2)$. If $x_1 = y_1$ or $x_2 = y_2$, then $T(x,y)$ is an atomary tolerance by Lemma 1. If $x_1 \neq y_1$ and $x_2 \neq y_2$, then, as was shown in the proof of Lemma 2, $T(x,y) = T\big((x_1,x_2),(x_1,y_2)\big) \vee T\big((x_1,x_2),(y_1,x_2)\big)$. It follows from Lemmas 1 and 2 that the tolerances on the right-hand side are exactly all atomary tolerances contained in $T(x,y)$, therefore (i) holds. From these lemmas it is also clear that no atomary tolerance on B is contained in the join of the others, therefore (ii) holds. By Theorem 1 the assertion is true. $\square$

Now we turn our attention to the algebras $\mathbf{A}$ with the property that the join in $\mathrm{Tol}(\mathbf{A})$ is equal to the set union. Evidently, such algebras always satisfy (ii). Among them there are all unary algebras.

Let us start with monounary algebras, i.e. algebras with one unary operation. Let $\mathbf{A} = \langle A, f\rangle$ be such an algebra. The unary operation f is a mapping of the support A into itself.

Theorem 3. *Let $\mathbf{A} = \langle A, f\rangle$ be a finite monounary algebra. The lattice* $\mathrm{Tol}(\mathbf{A})$ *is a Boolean algebra if and only if f is either surjective, or constant.*

P r o o f. Let us define f^n for every non-negative integer n. By f^0 we denote the identical mapping on A and for $n \geqslant 1$ let $f^n(x) = f\big(f^{n-1}(x)\big)$ for each $x \in A$. Now let x, y be two distinct elements of A. The tolerance $T(x,y)$ consists of all pairs $\big(f^n(x), f^n(y)\big)$, $\big(f^n(y), f^n(x)\big)$ for all non-negative integers n and from all elements of Δ.

If f is a surjection of a finite set onto itself, it is a permutation of A. Then each of the elements x, y is in some cycle of the permutation f. Let the orders of these cycles be p, q. Let $(u,v) \in T(x,y)$. Then there exists a non-negative integer $m \leqslant pq$ such that either $f^m(x) = u$, $f^m(y) = v$, or $f^m(x) = v$, $f^m(y) = u$. In the former case $f^{pq-m}(u) = f^{pq}(x) = x$, $f^{pq-m}(v) = f^{pq}(y) = y$, in the latter $f^{pq-m}(u) = f^{pq}(y) = y$, $f^{pq-m}(v) = f^{pq}(x) = x$. In both these cases we have $(x,y) \in T(u,v)$ and thus $T(x,y) \subseteq T(u,v)$ and clearly $T(x,y) = T(u,v)$. Each $T(x,y)$ is an atomary tolerance and (i) holds. As was mentioned above, also (ii) holds and $\mathrm{Tol}(\mathbf{A})$ is a Boolean algebra.

Now suppose that f is constant. Then for any two distinct elements x, y the tolerance $T(x,y) = \{(x,y),(y,x)\} \cup \Delta$; it is evidently an atomary tolerance. Again $\mathrm{Tol}(\mathbf{A})$ is a Boolean algebra.

Now suppose that f is neither surjective nor constant. There exists an element $a \in A$ such that $a \neq f(x)$ for all $x \in A$. Further, there exists $b \in A$ such that $f(b) \neq f(a)$. Consider the tolerance $T(a,b)$. Let $u = f(a)$, $v = f(b)$. Obviously $T(u,v) \subseteq T(a,b)$ and $a \neq u$, $a \neq v$. The tolerance $T(u,v)$ consists of the pairs

$\big(f^n(u), f^n(v)\big)$, $\big(f^n(v), f^n(u)\big)$ for all non-negative integers n. As $a \neq f^n(u)$, $a \neq f^n(v)$ for all n, we have $(a,b) \notin T(u,v)$ and thus $T(u,v)$ is a proper subset of $T(a,b)$. The tolerance $T(a,b)$ is not atomary. It cannot be the join of atomary tolerances contained in it, because such a join would be the union and (a,b) would belong to some atomary tolerance, which is impossible. Therefore in this case $\mathrm{Tol}(\mathbf{A})$ is not a Boolean algebra. $\square$

Now we consider unary algebras in general. Let $\mathbf{A} = \langle A, \mathscr{F} \rangle$ be such an algebra. The set $\mathscr{F}$ of operations on A consists of mappings of A into itself. It generates a semigroup $S(\mathscr{F})$.

Theorem 4. *Let $\mathbf{A} = \langle A, \mathscr{F} \rangle$ be a finite unary algebra such that $S(\mathscr{F})$ is a group. Then $\mathrm{Tol}(\mathbf{A})$ is a Boolean algebra.*

Proof. Let x, y be two distinct elements of A. The tolerance $T(x,y)$ consists of all pairs $\big(g(x), g(y)\big)$, $\big(g(y), g(x)\big)$, where $g \in S(\mathscr{F})$. If $(u,v) \in T(x,y)$, we have either $u = g(x)$, $v = g(y)$ or $u = g(y)$, $v = g(x)$ for some $g \in S(\mathscr{F})$. As $S(\mathscr{F})$ is a group, the inverse mapping $g^{-1} \in S(\mathscr{F})$ exists. Then either $x = g^{-1}(u)$, $y = g^{-1}(v)$ or $x = g^{-1}(v)$, $y = g^{-1}(u)$, and $(x,y) \in T(u,v)$. The rest of the proof is the same as in the proof of Theorem 3. $\square$

The condition is not necessary. For example, if $\mathscr{F}$ consists of constant mappings, then $S(\mathscr{F})$ is not a group, but every reflexive and symmetric binary relation on A is a tolerance on $\mathbf{A}$ and therefore $\mathrm{Tol}(\mathbf{A})$ is a Boolean algebra.

References

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