ON THE UNIQUENESS OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONINCREASING RIGHT-HAND SIDES

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Dedicated to Professor Zygfryd Kucharski on the occasion of his 50th birthday

Abstract. It is proved that nonincreasing and satisfying the Volterra condition right-hand side of a functional differential equation does not guarantee the uniqueness of solutions.

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Suppose that $I = [0, a]$, B is a Banach space, $f: I \times B \to B$, $g: I \times C(I, B) \to B$ are continuous functions satisfying the Volterra condition (it means that $g(t, x) = g(t, y)$ if $x(s) = y(s)$ for $s \in [0, t]$), where $C(I, B)$ denotes the Banach space of all functions from I into B. It is well known that the Cauchy problems

$$
\begin{aligned}
x'(t) &= f(t, x(t)) \\
x(0) &= x_0
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
x'(t) &= g(t, x) \\
x(0) &= x_0
\end{aligned}
\tag{2}
$$

have many fundamental properties in common. For instance, the Peano Theorem and the Picard Theorem are valid for both of them (see [3]). However, there are some differences. For example, graphs of each two solutions of (1) are tangent at any common point, but this need not be true for every two solutions of problem (2) (see [3]). In this note we construct an example which illustrates another difference.

In the case of a Hilbert space B with a scalar product $\langle\cdot,\cdot\rangle$ and a norm $|\cdot|_*$, some generalizations of Kamke type conditions for (1) of the form

$$\operatorname{Re}\langle v-u, f(t,v)-f(t,u)\rangle \leqslant w(t,|v-u|_*)$$

were considered in literature. For some classes of functions w these conditions guarantee the existence and uniqueness of a solution of (1) (see, for instance, [1], [2], [4] and [5]). The strongest condition of the above type is

$$\operatorname{Re}\langle v-u, f(t,v)-f(t,u)\rangle \leqslant 0.$$

If B is the one-dimensional Euclidean space $\mathbb{R}$, the above condition means that f is nonincreasing with respect to the second variable. The example we construct shows that even in the case $B=\mathbb{R}$ the condition "$g(t,\cdot)$ is a nonincreasing function for any $t\in I$" is not sufficient for the uniqueness of solutions of (2).

First we prove

Lemma. *Suppose that $y_1, y_2, z_1, z_2 \in C = C(I,\mathbb{R})$, $z_1(0)=z_2(0)$ and for any $t\in(0,a]$ we have*

$$l_1(t)=\sup_{s\in[0,t]}\big(y_1(s)-y_2(s)\big)>0,\quad l_2(t)=\sup_{s\in[0,t]}\big(y_2(s)-y_1(s)\big)>0.$$

Then there exists a continuous function $g\colon I\times C\to\mathbb{R}$ such that

1. $z_i(t)=g(t,y_i)$ *for* $i=1,2$, $t\in I$;
2. g *satisfies the Volterra condition;*
3. $g(t,\cdot)$ *is a nonincreasing function;*
4. g *is bounded.*

P r o o f. Let us define an operator $r\colon C\to \mathrm{C}$ by the formula

$$(rx)(t)=\begin{cases} m(t), & \text{if } x(t)<m(t),\\ x(t), & \text{if } m(t)\leqslant x(t)\leqslant M(t),\\ M(t), & \text{if } x(t)>M(t),\end{cases}$$

where $m(t)=\min\{y_1(t),y_2(t)\}$, $M(t)=\max\{y_1(t),y_2(t)\}$.

Our function $g : I \times C \to \mathbb{R}$ is defined by

$$g(t,x) = \begin{cases} z_1(0), & \text{if } t = 0, \\ l_1(t)^{-1} \sup\limits_{s\in[0,t]} \big(y_1(s) - (rx)(s)\big)\big(z_2(t) - z_1(t)\big) + z_1(t), & \\ & \text{if } t > 0 \text{ and } z_1(t) \leqslant z_2(t), \\ l_2(t)^{-1} \sup\limits_{s\in[0,t]} \big(y_2(s) - (rx)(s)\big)\big(z_1(t) - z_2(t)\big) + z_2(t), & \\ & \text{if } t > 0 \text{ and } z_1(t) > z_2(t). \end{cases}$$

Since $ry_i = y_i$, $i = 1, 2$, condition 1 holds true. It is easy to verify that conditions 2 and 3 are also satisfied. Condition 4 holds true because

$$(3) \qquad 0 \leqslant l_i(t)^{-1} \sup_{s\in[0,t]} \big(y_i(s) - (rx)(s)\big) \leqslant 1$$

for $t \in (0, a]$, and

$$\min\big\{z_1(t), z_2(t)\big\} \leqslant g(t,x) \leqslant \max\big\{z_1(t), z_2(t)\big\}$$

for $t \in I$, $x \in C$.

We prove that the function g is continuous. For $t \in (0, a]$, $x, y \in C$ we get

$$\Big| \sup_{s\in[0,t]} (y_i(s) - (rx)(s)) - \sup_{s\in[0,t]} (y_i(s) - (ry)(s))\Big|$$

$$\leqslant \sup_{s\in[0,t]} |(rx)(s) - (ry)(s)| \leqslant \sup_{s\in[0,t]} |x(s) - y(s)| \leqslant \|x - y\|,$$

where $\|\cdot\|$ denotes the norm of the uniform convergence. Hence

$$|g(t,x) - g(t,y)| \leqslant l(t)\|x - y\|,$$

where $l(t) = \max\{l_1(t)^{-1}, l_2(t)^{-1}\}|z_1(t) - z_2(t)|$. It means that g is a continuous function on $(0, a] \times C$, since $g(\cdot, x)$ is a continuous function on $(0, a]$ for each $x \in C$. Let us verify the continuity of g at any point of $\{0\} \times C$. Suppose that $(t_n, x_n) \to (0, x_0)$, $n \to \infty$, for some $x_0 \in C$. Then we get from (3)

$$|g(t_n, x_n) - z_1(t_n)| \leqslant |z_2(t_n) - z_1(t_n)|.$$

Since $z_1(0) = z_2(0)$ we obtain $\lim\limits_{n\to\infty} g(t_n, x_n) = z_1(0) = g(0, x_0)$. We conclude that the function g is continuous on $I \times C$ and the proof is complete. $\square$

The main result is presented in

Theorem. *There exists a continuous function g satisfying conditions 2–4 of Lemma, such that for any $a > 0$ the problem (2) has at least two different solutions on $[0, a]$.*

P r o o f. Suppose that y_1 and y_2 satisfy the assumptions of Lemma and are continuously differentiable on I, and $y_1'(0) = y_2'(0)$ (we can take, for instance, $y_1(t) = x_0$, $y_2(t) = x_0 + t^3 \sin t^{-1}$). Assume that g is a function satisfying the assertion of Lemma for $z_i = y_i'$, $i = 1, 2$. It follows from condition 1 that the Cauchy problem (2) has two different solutions y_1 and y_2 . $\square$

R e m a r k. It follows from the above proof that the graphs of two different solutions of the problem (2) may have infinitely many common points on any finite interval $(0, b)$, $b > 0$ and need not be tangent at any point.

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