A METHOD FOR DETERMINING CONSTANTS IN THE LINEAR COMBINATION OF EXPONENTIALS

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Abstract. Shifting a numerically given function $b_1 \exp a_1 t + \ldots + b_n \exp a_n t$ we obtain a fundamental matrix of the linear differential system $\dot{y} = Ay$ with a constant matrix A. Using the fundamental matrix we calculate A, calculating the eigenvalues of A we obtain $a_1, \ldots, a_n$ and using the least square method we determine $b_1, \ldots, b_n$.

Keywords: fundamental matrix, linear differential system, shifted exponentials, eigenvalues, the least square method

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Let $n \geqslant 1$ denote an integer, $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ real numbers, $a_i \neq a_j$ if $i \neq j$, $b_i \neq 0$ for $i = 1, \ldots, n$,

$$f(t) = b_1 \exp a_1 t + \ldots + b_n \exp a_n t$$

for real t. Let $h_1, \ldots, h_n; k_1, \ldots, k_n$ denote real numbers, $h_1 = k_1 = 0$, $h_i \neq h_j$ and $k_i \neq k_j$ if $i \neq j$; $i, j = 1, \ldots, n$. Define the $n \times n$-matrix valued function

$$Y(t) = \begin{bmatrix} f(t - h_1 - k_1) & \ldots & f(t - h_1 - k_n) \\ \ldots & & \\ f(t - h_n - k_1) & \ldots & f(t - h_n - k_n) \end{bmatrix} \quad \text{for real } t.$$

Theorem. *Y is a fundamental matrix of the linear differential system $\dot{y} = Ay$ with a constant $n \times n$-matrix A, and $a_1, \ldots, a_n$ are the eigenvalues of A.*

P r o o f. Let us set $y_i = \exp(-a_i)$; $i = 1, \ldots, n$,

$$E_1 = E(h_1, \ldots, h_n) \equiv \begin{bmatrix} y_1^{h_1} & \ldots & y_n^{h_1} \\ \ldots & & \\ y_1^{h_n} & \ldots & y_n^{h_n} \end{bmatrix}, \quad E_2 = E(k_1, \ldots, k_n),$$

$$G = \mathrm{diag}[a_1, \ldots, a_n], \quad B = \mathrm{diag}[b_1 \exp a_1 t, \ldots, b_n \exp a_n t].$$

Using induction we shall prove that E_1 is regular or, equivalently, the function

$$\varphi(y) = c_1 y^{h_1} + \ldots + c_n y^{h_n}$$

has at most $n-1$ positive roots for arbitrary $c_1, \ldots, c_n$ excluding $c_1 = \ldots = c_n = 0$ and arbitrary $h_1, \ldots, h_n$ satisfying our assumptions. This is clear for $n = 1$. Let $n > 1$, let our assertion be true for $n-1$ and let us suppose φ has n positive roots. Hence, the derivative φ' has $n-1$ positive roots which, using $h_1 = 0$, contradicts the induction hypothesis. Similarly, E_2 is regular. Using our notation we obtain $Y = E_1 B E_2^T$, $\dot{Y} = E_1 G B E_2^T$. Hence Y is regular and $A \equiv \dot{Y} Y^{-1} = E_1 G E_1^{-1}$ is constant, which proves our theorem. □

Let $p \geqslant 2n$ be an integer, $t_0, h > 0$ real numbers, $f_i = f(t_0 - (i-1)h)$ for $i = 1, \ldots, p$. Let $n, h, f_1, \ldots, f_p$ be known, while $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ are to be determined. We put $h_i = k_i = (i-1)h$ for $i = 1, \ldots, n$. (However, there exist many methods for choosing h_i, k_i.) Now, we may calculate $Y(t)$ for

$$t \in M \equiv \{t_0 - (i-1)h \colon i = 1, \ldots, q\},$$

where $q = p - 2n + 2$, $q \geqslant 2$. We will determine $\dot{Y}(t)$ numerically for some fixed $t \in M$ and put

$$A = \dot{Y}(t) Y(t)^{-1}.$$

Concerning numerical errors, we would probably obtain better results putting

$$A = \frac{1}{m}(\dot{Y}(t_1) Y(t_1)^{-1} + \ldots + \dot{Y}(t_m) Y(t_m)^{-1}),$$

where $m > 1$ is an integer, $t_1 \ldots, t_m \in M$, $t_i \neq t_j$ if $i \neq j$. We obtain $a_1, \ldots, a_n$ calculating the eigenvalues of A.

Alternatively, the formula

$$A = \frac{1}{t_2 - t_1} \ln Y(t_2) Y(t_1)^{-1}$$

can be used. Let $g_1, \ldots, g_n$ denote the eigenvalues of the matrix $Y(t_2) Y(t_1)^{-1}$ for some fixed $t_1, t_2 \in M$, $t_1 \neq t_2$. Hence, the values $a_1, \ldots, a_n$ coincide with the values

$$\frac{1}{t_2 - t_1} \ln g_i; \quad i = 1, \ldots, n.$$

Now, $b_1, \ldots, b_n$ may be determined using the least square method.

References

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