Σ-HAMILTONIAN AND Σ-REGULAR ALGEBRAIC STRUCTURES

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(Received March 23, 1995)

Abstract. The concept of a Σ-closed subset was introduced in [1] for an algebraic structure $\mathscr{A} = (A, F, R)$ of type τ and a set Σ of open formulas of the first order language $L(\tau)$. The set $C_\Sigma(\mathscr{A})$ of all Σ-closed subsets of $\mathscr{A}$ forms a complete lattice whose properties were investigated in [1] and [2]. An algebraic structure $\mathscr{A}$ is called Σ-*hamiltonian*, if every non-empty Σ-closed subset of $\mathscr{A}$ is a class (block) of some congruence on $\mathscr{A}$; $\mathscr{A}$ is called Σ-*regular*, if $\theta = \Phi$ for every two θ, $\Phi \in \operatorname{Con} \mathscr{A}$ whenever they have a congruence class $B \in C_\Sigma(\mathscr{A})$ in common. This paper contains some results connected with Σ-regularity and Σ-hamiltonian property of algebraic structures.

Keywords: algebraic structure, closure system, Σ-closed subset, Σ-hamiltonian and Σ-regular algebraic structure, Σ-transferable congruence

MSC 1991: 08A05, 04A05

The concept of an algebraic structure was introduced in [6] and [9]. A *type* of a structure is a pair $\tau = \langle \{n_i\,;\, i \in I\}, \{m_j\,;\, j \in J\}\rangle$, where n_i and m_j are non-negative integers. A *structure* $\mathscr{A}$ of type τ is a triplet (A, F, R), where $A \neq \emptyset$ is a set and $F = \{f_i, i \in I\}$, $R = \{\varrho_j\,;\, j \in J\}$ are such that for each $i \in I$, $j \in J$, f_i is an n_i-ary operation on A and ϱ_j is an m_j-ary relation on A. Denote by $L(\tau)$ a first order language containing operational and relational symbols of type τ, see [6] for some details. If $R = \emptyset$, the structure $(A, F, \emptyset)$ is denoted by (A, F) and called an *algebra*. If $F = \emptyset$, the structure $(A, \emptyset, R)$ is denoted by (A, R) and called a *relational system*. A relational system (A, R) is called *binary* if each $\varrho_j \in R$ is binary; moreover (A, R) is said to be *antisymmetrical* if each $\varrho_j \in R$ is an antisymmetrical relation.

Let us introduce the following concepts: for each $\gamma \in \Gamma$, where Γ is an index set, let $G_\gamma(x_1, \ldots, x_{k_\gamma}, y_1, \ldots, y_{s_\gamma}, z, p)$ be an open formula containing individual variables $x_1, \ldots, x_{k_\gamma}$, $y_1, \ldots, y_{s_\gamma}$, z and a symbol p of an n_i-ary term of type τ; for each $\lambda \in \Lambda$, where Λ is an index set such that $\Gamma \cap \Lambda = \emptyset$, let $G_\lambda(x_i, \ldots, x_{k_\lambda}, y_1, \ldots, y_{s_\lambda}, z, \varrho_j)$ be an open formula containing individual variables $x_1, \ldots, x_{k_\lambda}$, $y_1, \ldots, y_{s_\lambda}$, z and a symbol ϱ_j of an m_j-ary relation. Put $\Sigma = \{G_\gamma\,;\,\gamma\in\Gamma\}\cup\{G_\lambda\,;\,\lambda\in\Lambda\}$. The set $\Sigma = \{G_\gamma,\gamma\in\Gamma\}\cup\{G_\lambda,\lambda\in\Lambda\}$ of formulas of a language $L(\tau)$ is called *limited* if there exists a non-negative integer n, such that $n = \max(\{k_\gamma,\gamma\in\Gamma\}\cup\{k_\lambda,\lambda\in\Lambda\}\cup\{s_\gamma,\gamma\in\Gamma\}\cup\{s_\lambda,\lambda\in\Lambda\})$.

Let $\mathscr{A} = (A,F,R)$ be a structure of type τ and let $B\subseteq A$.

Definition 1. A subset B of A is said to be Σ-*closed* if for each $\gamma\in\Gamma$, $\lambda\in\Lambda$ and every $b_1,\dots,b_{k_\gamma}$, $b'_1,\dots,b'_{k_\lambda}\in B$, $a_1,\dots,a_{s_\gamma}$, $a'_1,\dots,a'_{s_\lambda}$, $c,c'\in A$, if $G_\gamma(b_1,\dots,b_{k_\gamma},a_1,\dots,a_{s_\gamma},c,p)$ is satisfied in $\mathscr{A}$ then $c\in B$ and if $G_\lambda(b'_1,\dots,b'_{k_\lambda}$, $a'_1,\dots,a'_{s_\lambda},c',\varrho_j)$ is satisfied in $\mathscr{A}$ then $c'\in B$.

Denote by $C_\Sigma(\mathscr{A})$ the set of all Σ-closed subsets of $\mathscr{A}$.

Since the concept of Σ-closed subsets is defined by the set of universal formulas, $B = \cap\{B_\delta\,;\,\delta\in\Delta\}$ is also a Σ-closed subset of $\mathscr{A}$, provided B_δ has this property for each $\delta\in\Delta$. Thus we have

Lemma 1. *Let $\mathscr{A} = (A,F,R)$ be a structure of type τ and let Σ be a set of open formulas of the language $L(\tau)$. Then the set $C_\Sigma(\mathscr{A})$ of all Σ-closed subsets of $\mathscr{A}$ forms a complete lattice with respect to set inclusion with the greatest element A.*

Corollary 1. *For any $\mathscr{A}$, Σ and $M\subseteq A$ there exists the least Σ-closed subset $C_{\mathscr{A}}(M)$ containing M.*

If $M = \{a_1,\dots,a_n\}$ then we will write briefly $C_{\mathscr{A}}(M) = C_{\mathscr{A}}(a_1,\dots,a_n)$.

If the set Σ is implicitly known, we will use only the lattice $C_\Sigma(\mathscr{A})$ to specify the closure system; we will use the more familiar notation of $C_\Sigma(\mathscr{A})$ provided it was introduced before, see the following examples.

Examples.

(1) Let $\mathscr{A} = (A,\leqslant)$ be an ordered set. Put $\Gamma=\emptyset$, $\Lambda=\{1\}$, $k_1 = 2$, $s_1 = 0$ and $\Sigma=\{G_1\}$, where $G_1(x_1,x_2,z,\leqslant)$ is the formula ($x_1\leqslant z$ and $z\leqslant x_2$). Then the Σ-closed subsets of $\mathscr{A}$ are just the convex subsets of $(A,\leqslant)$.

(2) Let $\mathscr{A} = (A,F)$ be an algebra, $F=\{f_i\,;\,i\in I\}$. Let $\Lambda=\emptyset$, $\Gamma=I$, $k_i=n_i$, $s_i = 0$ for $i\in I$. Put $\Sigma=\{G_i\,;\,i\in I\}$, where $G_i(x_i,\dots,x_{n_i},z,f_i)$ is the formula $\big(f_i(x_1,\dots,x_{n_i}) = z\big)$. Then the Σ-closed subsets of $\mathscr{A}$ are subalgebras of $\mathscr{A} = (A,F)$, and $C_\Sigma(\mathscr{A}) = \operatorname{Sub}\mathscr{A}$.

(3) Let $\mathscr{R} = (R,+,.,0)$ be a ring, $\Lambda=\emptyset$, $\Gamma=\{1,2,3\}$, $k_1=2$, $k_2=k_3=1$, $s_1=0$, $s_2=s_3=1$ and $\Sigma=\{G_1,G_2,G_3\}$, where G_1 is a formula $(x_1-x_2=z)$, G_2 is the formula $(x_1\cdot y_1 = z)$ and G_3 is the formula $(y_1\cdot x_1 = z)$. Then the Σ-closed subsets of $\mathscr{R}$ are ideals of $\mathscr{R}$ and $C_\Sigma(\mathscr{R}) = \operatorname{Id}\mathscr{R}$, the lattice of all ideals of $\mathscr{R}$. Analogously we can introduce the left or right ideals of $\mathscr{R}$.

(4) Similarly, if $\mathscr{L} = (L, \vee, \wedge)$ is a lattice, $\Lambda = \emptyset$, $\Gamma = \{1, 2\}$, $k_1 = 2$, $k_2 = 1$, $s_1 = 0$, $s_2 = 1$, $\Sigma = \{G_1, G_2\}$, where G_1 is the formula $(x_1 \vee x_2 = z)$ and G_2 is the formula $(x_1 \wedge y_2 = z)$, then the Σ-closed subsets are lattice ideals, i.e. $C_\Sigma(\mathscr{L}) = \operatorname{Id} \mathscr{L}$.

(5) Let $\mathscr{L} = (L, \vee, \wedge)$ be a lattice, $\Gamma = \{1, 2\}$, $\Lambda = \{1'\}$, $k_1 = k_2 = k_{1'} = 2$, $s_1 = s_2 = s_{1'} = 0$, $\Sigma = \{G_1, G_2, G_{1'}\}$, where G_1 is the formula $(x_1 \vee x_2 = z)$, G_2 the formula $(x_1 \wedge x_2 = z)$ and $G_{1'}$ is the formula $(x_1 \wedge z = x_1$ and $x_2 \vee z = x_2)$. Then the Σ-closed subsets form the convex sublattices of $\mathscr{L}$.

(6) Let $\mathscr{G} = (G, ., ^{-1}, e)$ be a group, let $p(x, y)$ be the term $p(x, y) = yxy^{-1}$ and $\Sigma = \{G_1, G_2, G_3, G_4\}$, where $G(x_1, x_2, z, .)$ is the formula $(x_1 \cdot x_2 = z)$, $G_2(x_1, z, ^{-1})$ is the formula $(x_1^{-1} = z)$, $G_3(z, e)$ is the formula $(e = z)$ and $G_4(x_1, y_1, z, p)$ is the formula $\big(p(x_1, y_1) = z\big)$. Then $C_\Sigma(\mathscr{G})$ is the lattice of normal subgroups of $\mathscr{G}$. This lattice will be denoted by $N(\mathscr{G})$.

(7) Example (1) can be generalized as follows: For a binary relational system $\mathscr{A} = (A, R)$ with $R = \{\varrho_j\,;\, j \in J\}$ we call $C_\Sigma(\mathscr{A})$ the lattice of convex subsets if $\Sigma = \{G_j\,;\, j \in J\}$ and every $G_j(x_1, x_2, z)$ is the formula $(x_1 \varrho_j z$ and $z \varrho_j x_2)$; we denote $C_\Sigma(\mathscr{A})$ by $\operatorname{Conv} \mathscr{A}$.

(8) Example (5) can be generalized as follows: An algebraic structure $\mathscr{A} = (A, F, R)$ is called a binary algebraic structure if a relational system (A, R) is binary. Let $\mathscr{A}$ be a binary algebraic structure, $\mathscr{A}_1 = (A, F)$, $\mathscr{A}_2 = (A, R)$, $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 = \{G_\gamma\,;\, \gamma \in \Gamma\}$ and $\Sigma_2 = \{G_\lambda\,;\, \lambda \in \Lambda\}$. The lattice $C_\Sigma(\mathscr{A})$ is called the lattice of convex subalgebras of $\mathscr{A}$ if $C_{\Sigma_1}(\mathscr{A}_1) = \operatorname{Sub} \mathscr{A}_1$ and $C_{\Sigma_2}(\mathscr{A}_2) = \operatorname{Conv} \mathscr{A}_2$; $C_\Sigma(\mathscr{A})$ is denoted by $C \operatorname{Sub} \mathscr{A}$.

The concept of the Hamiltonian group is well-known in the group theory. A group is Hamiltonian if each of its subgroups is normal. This concept was generalized for algebras in [8]: an algebra $\mathscr{A}$ is Hamiltonian if each of its subalgebras is a class of some congruence on $\mathscr{A}$. Hamiltonian algebras were characterized in [7].

An important concept of universal algebra is that of a regular algebra, i.e. an algebra $\mathscr{A}$ such that any two congruences on $\mathscr{A}$ coincide whenever they have a congruence class in common.

In this paper we generalize the concept of the Hamiltonian algebra by the concept of the Σ-hamiltonian algebraic structure and the concept of the regular algebra by the concept of the Σ-regular algebraic structure. Furthermore, we will formulate some conditions for Σ-regularity and Σ-hamiltonian property of the algebraic structures and we also show the relation between these concepts.

Definition 2. Let $\mathscr{A} = (A, F, R)$ be an algebraic structure of type τ and let Σ be a set of open formulas of the language $L(\tau)$. The structure $\mathscr{A}$ is called Σ-*hamiltonian* if each non-empty Σ-closed subset of $\mathscr{A}$ is a class of some congruence on $\mathscr{A}$.

E x a m p l e s .

(9) If $\mathscr{G} = (G, ., ^{-1}, e)$ is an Abelian group and $C_\Sigma(\mathscr{G}) = \operatorname{Sub}\mathscr{G}$, then $\mathscr{G}$ is a Σ-hamiltonian algebraic structure.

(10) If $\mathscr{G} = (G, ., ^{-1}, e)$ is a group and $C_\Sigma(\mathscr{A}) = N(\mathscr{G})$, then $\mathscr{G}$ is a Σ-hamiltonian structure.

(11) Let $\mathscr{R} = (R, +, ., 0)$ be a ring and $C_\Sigma(\mathscr{R}) = \operatorname{Id}\mathscr{R}$. Then $\mathscr{R}$ is Σ-hamiltonian.

(12) Let $\mathscr{D} = (D, \vee, \wedge, 0)$ be a distributive lattice and $C_\Sigma(\mathscr{D}) = \operatorname{Id}\mathscr{D}$. Then $\mathscr{D}$ is Σ-hamiltonian.

Theorem 1. *Let $\mathscr{A} = (A, F, R)$ be an algebraic structure and $C_\Sigma(\mathscr{A})$ a set of its Σ-closed subsets. Then the following conditions are equivalent:*

(1) *$\mathscr{A}$ is Σ-hamiltonian;*

(2) *for each $(n+1)$-ary term q and for every a, b, $a_1, \dots, a_n \in A$ we have $q(b, a_1, \dots, a_n) \in C_\mathscr{A}\big(a, b, q(a, a_1, \dots, a_n)\big)$.*

P r o o f. (1) $\Rightarrow$ (2): Let $\mathscr{A}$ be a Σ-hamiltonian structure, $B \in C_\Sigma(\mathscr{A})$ and let B be generated by elements a, b, $q(a, a_1, \dots, a_n) \in A$, i.e. $B = C_\mathscr{A}\big(a, b, q(a, a_1, \dots, a_n)\big)$. Then B is a congruence class of some $\theta \in \operatorname{Con}\mathscr{A}$, i.e. it is a class of congruence $\theta(B)$ which is generated by the relation $B \times B$. However, $a, b \in B$, then $\langle a, b\rangle \in \theta$, hence $\langle q(a, a_1, \dots, a_n), q(b, a_1, \dots, a_n)\rangle \in \theta$, i.e. $q(b, a_1, \dots, a_n)$ and $q(a, a_1, \dots, a_n)$ belong to the same class, thus $q(b, a_1, \dots, a_n) \in B$.

(2) $\Rightarrow$ (1): Let $B \in C_\Sigma(\mathscr{A})$ and suppose that (2) holds and B is not a class of any congruence $\theta \in \operatorname{Con}\mathscr{A}$. Then there exist $a, b \in B$ such that $q(a, a_1, \dots, a_n) \in B$ but $q(b, a_1, \dots, a_n) \notin B$ for some $(n+1)$-ary term q and $a_1, \dots, a_n \in A$. Thus a, b, $q(a, a_1, \dots, a_n) \in B$ implies $C_\mathscr{A}\big(a, b, q(a, a_1, \dots, a_n)\big) \subseteq B$, and $q(b, a_1, \dots, a_n) \in C_\mathscr{A}\big(a, b, q(a, a_1, \dots, a_n)\big) \subseteq B$ according to (2), a contradiction. Hence (2) implies (1). $\square$

Theorem 2. *Let $\mathscr{A} = (A, F, R)$ be an algebraic structure and $0 \in A$. Let $C_\Sigma(\mathscr{A})$ be a system of Σ-closed subsets of $\mathscr{A}$ such that $0 \in B$ for every $B \in C_\Sigma(\mathscr{A})$ and, furthermore, for every $a, b \in B$ there exists $d \in B$ such that $\theta(0, a) \vee \theta(0, b) = \theta(0, d)$. Then the condition*

$(*)$ *$C_\mathscr{A}(a)$ is a class of the congruence $\theta(0, a)$ for each $a \in A$*

implies that $\mathscr{A}$ is Σ-hamiltonian.

P r o o f. Let $B \in C_\Sigma(\mathscr{A})$. Then $B = \vee\{C_\mathscr{A}(x);\, x \in B\}$ in the lattice $\big(C_\Sigma(\mathscr{A}), \subseteq\big)$. Put $\theta = \vee\{\theta(0, x);\, x \in B\}$ in the lattice $(\operatorname{Con}\mathscr{A}, \subseteq)$. Then:

(a) $\langle a,0\rangle \in \theta(0,a)$ and $\langle 0,b\rangle \in \theta(0,b)$ for every $a,b \in B$, hence $\langle a,b\rangle \in \theta(0,a)\cdot\theta(0,b) \subseteq \theta(0,a)\vee\theta(0,b) \subseteq \theta$. Thus $B\times B \subseteq \theta$, i.e. there exists a class C of the congruence θ such that $B \subseteq C$.

(b) Suppose that B is not a class of the congruence θ. Then there exist $d \in B$, $c \notin B$ such that $\langle c,d\rangle \in \theta$. Since the lattice $\operatorname{Con}\mathscr{A}$ is algebraic, there exist elements $b_1,\dots,b_n \in B$ such that $\langle c,0\rangle \in \theta(0,b_1)\vee\theta(0,b_2)\vee\dots\vee\theta(0,b_n)$. By the assumption there exists $h \in B$ such that $\langle c,0\rangle \in \theta(0,h)$. Hence $c \in C_{\mathscr{A}}(h) \subseteq B$ by $(*)$, a contradiction. Thus B is a class of θ. $\square$

Example. Let $\mathscr{D} = (D,\vee,\wedge,0)$ be a distributive lattice with zero 0 and $C_\Sigma(\mathscr{D}) = C\operatorname{Sub}\mathscr{D}$ (0 means a nullary operation). Then the assumption and condition $(*)$ of Theorem 2 are fulfilled, see e.g. Theorem 1 in [5], thus $\mathscr{D}$ is a Σ-hamiltonian structure.

Definition 3. Let $\mathscr{A} = (A,F,R)$ be an algebraic structure of type τ, let Σ be a set of open formulas of the language $L(\tau)$ and let $C_\Sigma(\mathscr{A})$ be the closure system. The structure $\mathscr{A}$ is called *Σ-regular* if $\theta = \Phi$ for θ, $\Phi \in \operatorname{Con}\mathscr{A}$ whenever they have a congruence class $B \in C_\Sigma(\mathscr{A})$ in common; $\mathscr{A}$ is called *strongly Σ-regular* if every $B \in C_\Sigma(\mathscr{A})$ is a class of exactly one congruence on $\mathscr{A}$.

The following proposition is evident:

Lemma 2. *If an algebraic structure is strongly Σ-regular, then it is also Σ-regular.*

Definition 4. We say that an algebraic structure $\mathscr{A} = (A,F,R)$ has *Σ-transferable* congruences, if for every $a,b,c \in A$ and $[c]_{\theta(a,b)} \in C_\Sigma(\mathscr{A})$ there exist elements $d_1,\dots,d_n \in [c]_{\theta(a,b)}$ such that $\theta(a,b) = \theta(c,d_1,\dots,d_n)$.

Theorem 3. *Let $\mathscr{A} = (A,F,R)$ be an algebraic structure of type τ and let Σ be a set of open formulas of the language $L(\tau)$. Then the following conditions are equivalent:*

(i) *$\mathscr{A}$ is Σ-regular;*

(ii) *$\mathscr{A}$ has Σ-transferable congruences.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathscr{A}$ be Σ-regular, $a,b \in A$ and $[c]_{\theta(a,b)} \in C_\Sigma(\mathscr{A})$. Then, by the Σ-regularity we have $\theta(a,b) = \theta([c]_{\theta(a,b)}) = \theta(\{c\}\times[c]_{\theta(a,b)})$. Since the lattice $\operatorname{Con}\mathscr{A}$ is algebraic, i.e. compactly generated, there exists a finite subset $F \subseteq [c]_{\theta(a,b)}$ such that $\theta(a,b) = \theta(\{c\}\times F)$. If $F = \{d_1,\dots,d_n\}$ then $\theta(a,b) = \theta(c,d_1,\dots,d_n)$, i.e. the structure $\mathscr{A}$ has Σ-transferable congruences.

(ii) $\Rightarrow$ (i): Let $\theta_1,\theta_2 \in \operatorname{Con}\mathscr{A}$ and let $B \in C_\Sigma(\mathscr{A})$ be their common congruence class. Then B is also a class of the congruence $\theta_1\cap\theta_2$. Thus we can suppose without loss of generality that $\theta_1 \subseteq \theta_2$. Further suppose $\langle a,b\rangle \in \theta_2$ and $c \in B$. By the

Σ-transferability we obtain the existence of elements $d_1, \ldots, d_n \in [c]_{\theta(a,b)} \subseteq B$ with $\theta(a,b) = \theta(c, d_1, \ldots, d_n)$, i.e. $\langle c, d_1\rangle \in B \times B$. Hence $\langle c, d_i\rangle \in \theta_1$ for $i = 1, \ldots, n$, thus $\theta(a,b) = \theta(c,d_1) \vee \ldots \vee \theta(c,d_n) \subseteq \theta_1$. Then $\langle a,b\rangle \in \theta_1$, i.e. $\theta_2 \subseteq \theta_1$. So we have $\theta_1 = \theta_2$ and $\mathscr{A}$ is Σ-regular. $\square$

It is evident that every strongly Σ-regular algebraic structure is also Σ-hamiltonian. Hence every strongly Σ-regular structure is Σ-regular and Σ-hamiltonian by Lemma 2. Conversely, if a structure $\mathscr{A}$ is Σ-hamiltonian and Σ-regular, then by the first property, every $B \in C_\Sigma(\mathscr{A})$ is a class of at least one congruence on $\mathscr{A}$ and, by Σ-regularity, B is a class of at most one congruence on $\mathscr{A}$. Thus we have

Theorem 4. *An algebraic structure is strongly Σ-regular if and only if it is Σ-regular and Σ-hamiltonian.*

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