A CHARACTERIZATION OF FINITE STONE PSEUDOCOMPLEMENTED ORDERED SETS

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Abstract. A distributive pseudocomplemented set S [2] is called Stone if for all $a \in S$ the condition $LU(a^*, a^{**}) = S$ holds. It is shown that in a finite case S is Stone iff the join of all distinct minimal prime ideals of S is equal to S.

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First of all, let us recall some basic notions.

If $(S, \leqslant)$ is an ordered set and $X \subseteq S$, let $U(X) = \{y \in S;\, y \geqslant x \text{ for all } x \in X\}$ and $L(X) = \{y \in S;\, y \leqslant x \text{ for all } x \in X\}$. A subset $I \subseteq S$ is called an ideal (filter) if $LU(a, b) \subseteq I$ $(UL(a, b) \subseteq I)$ whenever $a, b \in I$.

An ideal (filter) I is called a u-ideal (l-filter) if I is an up (down) directed set. An ideal (filter) I is called prime if $L(a, b) \subseteq I$ $(U(a, b) \subseteq I)$ implies $a \in I$ or $b \in I$. It is well-known that the set of all ideals $\mathrm{Id}(S)$ forms an algebraic lattice.

The set S is called

distributive if $\forall a, b, c \in S\colon L\big(U(a, b), c\big) = LU\big(L(a, c), L(b, c)\big)$, [4];

complemented if $\forall a \in S\, \exists a' \in S\colon LU(a, a') = UL(a, a') = S$, [3];

boolean if it is both distributive and complemented;

w-boolean if $\forall x, y, z \in S\colon L\big(z, U(x, y)\big) \subseteq LU\big(x, L(y, z)\big)$ and S is complemented, [2].

In [3] it was shown that the notions of boolean and w-boolean sets coincide. In [2], the concept of a pseudocomplement was introduced and studied. The set S with the least element 0 and the greatest one 1 is called pseudocomplemented if for every $a \in S$ there exists the greatest element a^* with $L(a, a^*) = \{0\}$. Then the element a^*

is called a pseudocomplement of a. In [2] it was shown that the set $\mathcal{B}(S) = \{x \in S;\ x = x^{**}\}$ is a w-boolean ordered set and $\mathcal{D}(S) = \{x \in S;\ x^* = 0\}$ is a filter in S (see [2]).

In Fig. 1, a distributive and pseudocomplemented set is visualized for which the set $\{a, b, 0\}$ is an ideal but not a u-ideal.

A distributive and pseudocomplemented ordered set is called Stone if $LU(a^*, a^{**}) = S$ holds for every $a \in S$. There exists a Stone set which is not a lattice, e.g. every boolean ordered set which is not a lattice.

In [1] Grätzer and E.T. Schmidt proved that a distributive and pseudocomplemented lattice S is Stone iff $P \vee Q = S$ holds for every two minimal prime ideals P, Q, $P \neq Q$. The aim of this note is to show an analogous characterization in the case of finite Stone sets.

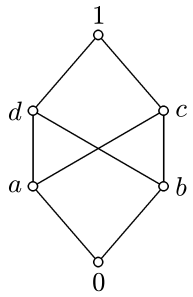

Fig. 1

Theorem. *Let S be a finite distributive and pseudocomplemented ordered set. Then S is Stone iff $P \vee Q = S$ holds for every two different minimal prime ideals P, Q of S.*

P r o o f. Let S be a Stone set, P, Q minimal prime ideals, $P \neq Q$. Then there exists $a \in P \setminus Q$ and $L(a, a^*) \subseteq Q$. Since Q is prime, $a^* \in Q$. Let us show that $S \setminus P$ is a maximal element in the set of all l-filters of S. To this aim, let a, $b \in S \setminus P$, i.e. a, $b \notin P$. If there exists $z \in UL(a, b)$, $z \in P$, then $z \geqslant y$ for every $y \in L(a, b)$ and since P is an ideal, $L(a, b) \subseteq P$. Since P is prime, we have $a \in P$ or $b \in P$, a contradiction, so $S \setminus P$ is a filter. Further, $L(a, b) \nsubseteq P$, so there exists $z \in L(a, b)$, $z \notin P$ and $S \setminus P$ is an l-filter.

Now, since S is finite, each l-filter has to be contained in some maximal l-filter. Since S is finite, each maximal l-filter in a finite set has to have the least element q, so it is in the form $U(q)$ where $q \succ 0$. We conclude that $S \setminus P \subseteq U(q)$ for some $q \succ 0$. Now, if $S \setminus P \neq U(q)$, we have $S \setminus U(q) \subseteq P$ and $S \setminus U(q) \neq P$.

We shall show that $U(q)$ is a prime filter. To this end, let $U(a, b) \subseteq U(q)$, i.e. $L(q) \subseteq LU(a, b)$. By distributivity we have

$$L(q) = L\big(q, U(a, b)\big) = LU\big(L(q, a), L(q, b)\big).$$

Obviously, $L(q, a) \neq \{0\}$ or $L(q, b) \neq \{0\}$, since in the opposite case $L(q) = L(0)$ which implies $q = 0$, a contradiction. However, q covers 0 and, therefore, $L(a, q) = L(q)$ or $L(b, q) = L(q)$, i.e. $a \geqslant q$ or $b \geqslant q$, $U(a) \subseteq U(q)$ or $U(b) \subseteq U(q)$. Now, because $U(q)$ is a prime filter, $S \setminus U(q)$ is an ideal. Moreover, we shall show that

$S \setminus U(q)$ is prime: if a, $b \notin S \setminus U(q)$, i.e. a, $b \in U(q)$, then $q \in L(a,b)$, $q \in U(q)$, so we have $L(a,b) \nsubseteq S \setminus U(q)$ and $S \setminus U(q)$ is prime.

Further, we have proved that $S \setminus U(q)$ is a prime ideal for which $S \setminus U(q) \subsetneq P$, a contradiction with minimality of P. Consequently, $S \setminus P$ is a maximal l-filter, so $S \setminus P = U(q)$ for an element $q \succ 0$. If $a \notin S \setminus P$, then $a \ngeqslant q$, so $L(a,q) = \{0\}$. Then $UL(a,q) = S \subseteq S \setminus P \vee U(a)$, $S \setminus P \vee U(a) = S$. Since $L(a,q) = \{0\}$, we have $a^* \geqslant q \in S \setminus P$, $a^* \notin P$. However, then $a^* \in Q \setminus P$. Similarly, $a^{**} \in P \setminus Q$, hence $S = LU(a^* \vee a^{**}) \subseteq P \vee Q$, $P \vee Q = S$.

Conversely, let $U(a^*, a^{**}) \neq \{1\}$ for an element $a \in S$. Since S is finite, there exists $q \in U(a^*, a^{**})$ such that $1 \succ q$. The ideal $L(q)$ is maximal u-ideal, so it is a prime ideal and $S \setminus L(q)$ is an l-filter. Hence we have $S \setminus L(q) = U(b)$ for some $b \in S$. It is evident that $S \setminus L(q) \vee U(a^*) = U(b) \vee U(a^*) = UL(a^*, b)$. We can show that $UL(a^*, b) \neq S$. If not, then $0 \in UL(a^*, b)$, so $L(a^*, b) = \{0\}$ and $a^{**} \geqslant b$. But then $a^{**} \in S \setminus L(q)$, $a^{**} \nleqslant q$, a contradiction.

So we have proved that

$$S \setminus L(q) \vee U(a^*) \neq S$$

and, analogously,

$$S \setminus L(q) \vee U(a^{**}) \neq S.$$

Further, we can prove that the filter $S \setminus L(q) \vee U(a^*) = UL(a^*, b)$ is contained in some maximal l-filter $U(z)$, $z \succ 0$: since $UL(a^*, b) \neq S$, there exists $z \in S$ such that $z \succ 0$ and $L(z) \subseteq L(a^*, b)$. Consequently, we have $U(z) \supseteq UL(a^*, b)$ and $U(z)$ is a maximal l-filter. Analogously, there exists $y \in S$ such that $y \succ 0$ and $S \setminus L(q) \vee U(a^{**}) \subseteq U(y)$. Let us put $P = S \setminus U(z)$, $Q = S \setminus U(y)$. Because $U(z)$, $U(y)$ are prime l-filters, both P, Q are prime ideals. We show that P, Q are moreover minimal ones: if not, then there exists a prime ideal $R \subseteq P$, $R \neq P$. But then $S \setminus R \supseteq S \setminus P = U(z)$. It can be proved that $S \setminus R$ is a filter and $s \in U(z)$ for some $s \in S \setminus R$. Since $z \succ 0$, $L(s,z) = \{0\}$ and $UL(s,z) = S \subseteq S \setminus R$, we have $R = \emptyset$, a contradiction. So P, Q are minimal prime ideals. Moreover, $a^* \in U(z)$, $a^{**} \in U(y)$, $a^* \notin P$, $a^{**} \notin Q$. This means $P \neq Q$. Finally, $S \setminus L(q) \subseteq U(z)$, $U(y)$, so we have $L(q) \supseteq S \setminus U(z) = P$, $L(q) \supseteq S \setminus U(y) = Q$, hence $P \vee Q \subseteq L(q)$, $P \vee Q \neq S$. $\square$

Example 1. Let us consider an ordered set whose diagram is in Fig. 1. This set is distributive and pseudocomplemented, moreover, $a^* = b$, $b^* = a$, $0^* = 1$, $c^* = d^* = 1^* = 0$. Nonetheless, it is not Stone, since $LU(a^*, a^{**}) = LU(b,a) = \{a, b, 0\} \neq S$. The set of all prime ideals is equal to $\{L(a), L(b), L(c), L(d), S\}$ and there are just two minimal ones, namely $L(a)$, $L(b)$. However, $L(a) \vee L(b) = LU(a,b) \neq S$.

Example 2. The set depicted in Fig. 2 is a Stone set with $a^* = y$, $b^* = x$, $x^* = y$, $y^* = x$, $c^* = d^* = 1^* = 0$, $0^* = 1$. It is neither a lattice nor a boolean ordered set and has just two minimal prime ideals: $L(x)$, $L(y)$. Their join is equal to $L(x) \vee L(y) = LU(x, y) = S$.

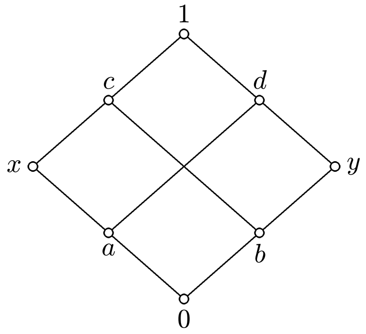

Fig. 2

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