A METRIC ON A SYSTEM OF ORDERED SETS

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Abstract. In [3] a metric on a system of isomorphism classes of ordered sets was defined. In this paper we define another metric on the same system and investigate some of its properties. Our approach is motivated by a problem from practice.

Keywords: partially ordered set, metric

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The following practical problem can be considered as a motivation. The usual superiority in an office (organisation) can be represented in a natural way as a partial ordering on a finite set of working positions and it yields a "hierarchy of employees" working on these positions. Formally, let an ordered set $(P, \leqslant_1)$ represent a superiority on a set P of working positions (i.e. $p_1 \leqslant_1 p_2$ for $p_1, p_2 \in P$ if the position p_1 is directed by the position p_2). Further, let $E(P)$ be a set of employees working in the given office on the positions from P (thus $|E(P)| = |P|$) and let $e_1(p_1) \leqslant_1 e_2(p_2)$ $(e_1(p_1), e_2(p_2) \in E(P))$ iff $p_1 \leqslant_1 p_2$. Now assume that a reorganization yields a new ordering represented by $\leqslant_2$. One wishes to move the employees to positions in the new hierarchy in such a way that the previous hierarchy of employees $(E(P), \leqslant_1)$ agreed "as much as possible" with the new one $(E(P), \leqslant_2)$. Hence, we wish the cardinality of the set

$$\{(e_1, e_2) \in E(P)^2 ;\, e_1 \neq e_2 \text{ and } e_1 \leqslant_1 e_2 \Leftrightarrow e_1 \leqslant_2 e_2\}$$

to be maximal.

Throughout this paper all ordered sets are assumed to be finite. If R is a binary relation we will often write aRb instead of $[a, b] \in R$. We will write $a \parallel b(R)$ if neither aRb nor bRa holds. As usual, we denote the cardinality of a set A by $|A|$.

Let R be an ordering (i.e. a reflexive, antisymmetric and transitive binary relation) on a set P and let S be an ordering on M, where $|P| = |M|$. Let $F(P,M)$ denote the set of all bijections of P onto M. For any bijection $f \in F(P,M)$ we denote by d_f the number defined by

$$d_f = |f(R) \setminus S| + |S \setminus f(R)| \tag{1}$$

where $f(R) = \{[f(a), f(b)]\,;\ [a,b] \in R\}$.

It is obvious that

$$d_f = |R| + |S| - 2|f(R) \cap S|. \tag{1'}$$

Lemma 1. *For any ordered sets (P,R) and (M,S) with $|P| = |M|$ and any bijections f, $g \in F(P,M)$ the following conditions are satisfied:*

$$d_f = d_g \quad \text{iff} \quad |f(R) \cap S| = |g(R) \cap S|, \tag{2}$$

$$d_f < d_g \quad \text{iff} \quad |f(R) \cap S| > |g(R) \cap S|, \tag{2'}$$

$$|f(R) \cap S| = |R \cap f^{-1}(S)|. \tag{3}$$

P r o o f. (2) and (2′) follows from (1′). Further, $[a,b] \in f(R) \cap S$ if and only if

$$[f^{-1}(a), f^{-1}(b)] \in R \cap f^{-1}(S),$$

which proves (3). □

Definition. Let (P,R) and (M,S) be ordered sets with $|P| = |M|$. The number $d((P,R),(M,S))$ given by the equality

$$d((P,R),(M,S)) = \min\{d_f\,;\ f \in F(P,M)\} \tag{4}$$

will be called the distance of the ordered sets (P,R) and (M,S).

R e m a r k 1. We will often write $d(R,S)$ instead of $d((P,R),(M,S))$.

R e m a r k 2. The bijection f for which $d(R,S) = d_f$ can be regarded as "the most isotone" mapping of (P,R) onto (M,S). If f is an isotone bijection of an ordered set (P,R) onto an ordered set (M,S), then $d(R,S) = d_f$.

Theorem 1. *Let $\mathscr{S}$ be an arbitrary system of ordered sets having the same cardinality. The function d on the system $\mathscr{S}$ given by (4) is a pseudometric.*

P r o o f. It follows from Lemma 1 that $d(R,S)=d(S,R)$ for any ordered sets (P,R) and (M,S).

Let (P,R), (M,S) and (Q,T) be ordered sets with $|P|=|M|=|Q|$, and let

$$d(R,S)=d_f,\quad d(S,T)=d_g,\quad d(R,T)=d_h.$$

The inequality

$$d_f+d_g\geqslant d_h \tag{a}$$

is equivalent to the inequality

$$|R|+|S|-2|f(R)\cap S|+|S|+|T|-2|g(S)\cap T|\geqslant|R|+|T|-2|h(R)\cap T|,$$

i.e. to

$$|S|+|h(R)\cap T|\geqslant|f(R)\cap S|+|g(S)\cap T|. \tag{b}$$

From the minimality of d_h we have $d_{f\circ g}\geqslant d_h$; this implies

$$|g(f(R))\cap T|\leqslant|h(R)\cap T|\qquad\text{by Lemma 1,}$$

therefore to prove (a) it is sufficient to show that

$$|S|+|g(f(R))\cap T|\geqslant|f(R)\cap S|+|g(S)\cap T|. \tag{c}$$

Without loss of generality we can assume that $S\cap T=\emptyset$. Then we can write (c) in the form

$$|S\cup(g(f(R))\cap T)|\geqslant|(f(R)\cap S)\cup(g(S)\cap T)|. \tag{d}$$

Now, we will show that there exists an injective mapping

$$\varphi\colon(f(R)\cap S)\cup(g(S)\cap T)\longrightarrow S\cup(g(f(R))\cap T).$$

We will distinguish two possibilities:

1. If $[a,b]\in f(R)\cap S$, then we put $\varphi(a,b)=[a,b]$.

2. Let $[a,b]\in g(S)\cap T$, i.e. there exist elements x, y such that $g(x)=a$, $g(y)=b$, $[x,y]\in S$. If $[x,y]\notin f(R)$ we put $\varphi(a,b)=[x,y]$, otherwise $\varphi(a,b)=[a,b]$.

It is not hard to verify that φ is injective, which completes the proof. $\square$

R e m a r k. In the previous proof we have not used the fact that the relations R, S and T were reflexive, antisymmetric and transitive. Consequently, the function d is a pseudometric on any system $\mathscr{S}$ that contains relation structures of the same finite type which have base sets (universes) of the same cardinality. In this case

$$d((P;R_1,\ldots,R_n),(M;S_1,\ldots,S_n)) = d((P,R_1),(M,S_1))+\ldots+d((P,R_n),(M,S_n)).$$

Corollary. *Let $\mathscr{S}$ be a system of classes of isomorphic ordered sets. The function δ defined on $\mathscr{S}$ by*

$$\delta(\mathscr{P},\mathscr{M}) = d((P,R),(M,S))$$

for any elements $\mathscr{P},\mathscr{M} \in \mathscr{S}$, $(P,R) \in \mathscr{P}$, $(M,S) \in \mathscr{M}$, is a metric on $\mathscr{S}$.

R e m a r k. We identify the functions δ and d throughout this paper.

Lemma 2. *Let (P,R), (M,S) be ordered sets, $|P| = |M|$ and $d(R,S) = d_f$. If aRb, $a \neq b$ for some elements $a,b \in P$, then $f(b)Sf(a)$ does not hold.*

P r o o f. Suppose to the contrary that there exist elements $a,b \in P$ such that aRb, $a \neq b$ and $f(b)Sf(a)$. It is sufficient to show that there exists a bijection $g\colon P \to M$ such that $d_g < d_f$.

Let the map $g\colon P \to M$ be defined as follows:

$$g(x) = f(x) \quad \text{for all} \quad x \in P \setminus \{a,b\},$$
$$g(a) = f(b), \quad g(b) = f(a).$$

We will prove that $|f(R) \cap S| < |g(R) \cap S|$. Let $[u,v] \in f(R) \cap S$. We distinguish the following cases:

1. $u = f(x) = g(x) \notin \{f(a),f(b)\}$, $v = f(y) = g(y) \notin \{f(a),f(b)\}$. This yields $[x,y] \in R$, consequently $[u,v] \in g(R) \cap S$.

2. $u = f(x) = g(x) \notin \{f(a),f(b)\}$, $v = f(a) = g(b)$. Then $[x,a] \in R$ and $[x,b] \in R$ (as $[a,b] \in R$), hence $[u,v] \in g(R) \cap S$.

3. $u = f(b) = g(a)$, $v = f(x) = g(x) \notin \{f(a),f(b)\}$. This again leads to $[u,v] \in g(R) \cap S$.

4. $u = f(x) = g(x) \notin \{f(a),f(b)\}$, $v = f(b) = g(a)$. Then either

a) $[x,b] \in R$ and $[x,a] \in R$

or

b) $[x,b] \in R$ and $[x,a] \notin R$.

5. $u = f(a) = g(b)$, $v = f(x) = g(x) \notin \{f(a),f(b)\}$. Then either

a) $[a,x]\in R$ and $[b,x]\in R$
or
b) $[a,x]\in R$ and $[b,x]\notin R$.

The cases 4a and 5a immediately give $[u,v]\in g(R)\cap S$.

On the other hand, in the case 4b we have $[r,s]=[g(x),g(b)]\in g(R)\cap S$ and $[r,s]=[f(x),f(a)]\notin f(R)\cap S$ and in the case 5b we get $[r,s]=[g(a),g(x)]\in g(R)\cap S$ and $[r,s]=[f(b),f(x)]\notin f(R)\cap S$. We proved that $|f(R)\cap S|\leqslant|g(R)\cap S|$. Observe that $[g(a),g(b)]=[f(b),f(a)]\in g(R)\cap S$ and $[g(a),g(b)]=[f(b),f(a)]\notin f(R)\cap S$. The proof is complete. □

Theorem 2. *Let (P,R), (M,S) be ordered sets such that $|P|=|M|$. If $d(R,S)=d_f$ and m is the least (greatest) element of the ordered set (P,R), then $f(m)$ is a minimal (maximal) element of the ordered set (M,S).*

P r o o f. It follows from Lemma 2. □

The following example shows that in Theorem 2, the words "least" and "greatest" cannot be replaced by the words "minimal" and "maximal", respectively.

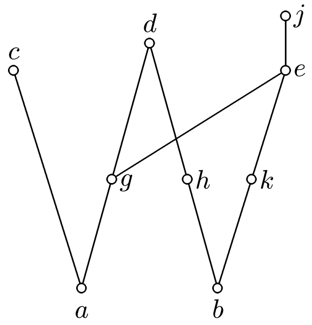

Fig. 1

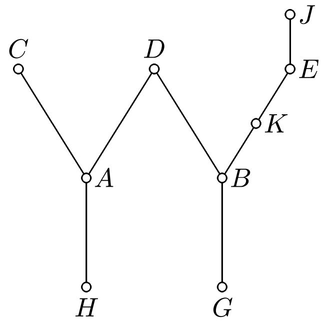

Fig. 2

E x a m p l e. Let (P,R) and (M,S) be the ordered sets drawn in Figures 1 and 2, respectively. The mapping $f\colon P\to M$ is given as follows: $f(a)=A$, $f(b)=B$, $f(c)=C$, $f(d)=D$, $f(e)=E$, $f(g)=G$, $f(j)=J$, $f(h)=H$, $f(k)=K$.

Then $d(R,S)=d_f=8$ but neither $f(a)$ nor $f(b)$ is a minimal element of (M,S)

P r o b l e m. Let (P,R) and (M,S) be arbitrary ordered sets with $|P|=|M|$. Does there exist a mapping $g\colon P\to M$ and minimal elements $x\in P$, $y\in M$ such that $d(R,S)=d_g$ and $g(x)=y$?

Lemma 3. *Let (P_1,R_1) and (P_2,R_2) be ordered sets and let $(P_1\times P_2,R)$ be their direct product. Then $|R|=|R_1|\cdot|R_2|$.*

P r o o f. It follows immediately from the definition of a direct product. □

Theorem 3. *Let (P,R) be the direct product of ordered sets (P_1,R_1), (P_2,R_2) and let (M,S) be the direct product of ordered sets (M_1,S_1), (M_2,S_2). If $d(R_1,S_1) = d_f$ and $d(R_2,S_2) = d_g$, then*

$$d(R,S) \leqslant \frac{1}{2}\cdot(|R_1|\cdot|R_2| + |S_1|\cdot|S_2| - |S_1|\cdot|R_2| - |R_1|\cdot|S_2| + d_f\cdot(|R_2|+|S_2|) + d_g\cdot(|R_1|+|S_1|) - d_f\cdot d_g). \tag{5}$$

P r o o f. We will denote by $f\times g$ the mapping $P_1\times P_2 \to M_1\times M_2$ for which $(f\times g)(x,y) = [f(x),g(y)]$. First, we show that

$$|(f\times g)(R)\cap S| = |f(R_1)\cap S_1|\cdot|g(R_2)\cap S_2|. \tag{e}$$

Indeed, $[a,b]\in f(R_1)$ and $[c,d]\in g(R_2)$ if and only if there exist elements $x,y\in P_1$, $u,v\in P_2$ such that $f(x)=a$, $f(y)=b$, $g(u)=c$, $g(v)=d$, $[x,y]\in R_1$ and $[u,v]\in R_2$, which is satisfied if and only if $[(f\times g)(x,u),(f\times g)(y,v)] = [[a,c],[b,d]]\in(f\times g)(R)$. Evidently, $[a,b]\in S_1$ and $[c,d]\in S_2$ if and only if $[[a,c],[b,d]]\in S$. Therefore, $[a,b]\in f(R_1)\cap S_1$ and $[c,d]\in f(R_2)\cap S_2$ if and only if $[[a,c],[b,d]]\in(f\times g)(R)\cap S$. Consequently, this implies (e). Now, it follows from $d_{f\times g} = |R|+|S|-2\cdot|(f\times g)(R)\cap S|$ (by Lemma 3 and (1′)) that

$$d_{f\times g} = \frac{1}{2}\cdot(|R_1|\cdot|R_2| + |S_1|\cdot|S_2| - |S_1|\cdot|R_2| - |R_1|\cdot|S_2| + d_f\cdot(|R_2|+|S_2|) + d_g\cdot(|R_1|+|S_1|) - d_f\cdot d_g).$$

The last equality implies (5), which completes the proof. □

E x a m p l e. Let (P,R) and (M,S) be ordered sets drawn in Figures 3 and 4, respectively. Let the mapping $f\colon P\to M$ be given as follows: $f(a)=A$, $f(b)=B$, $f(c)=C$. It can be verified that $d(R,S)=d_f=2$, $d_{f\times f}((P,R)\times(P,R),(M,S)\times(M,S)) = 18$ but $d((P,R)\times(P,R),(M,S)\times(M,S)) = 12$ (the last equality was verified by a computer).

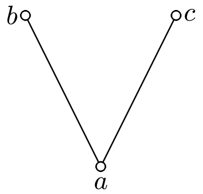

Fig. 3

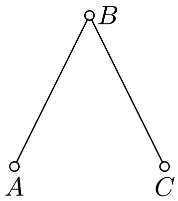

Fig. 4

Definition. Let (P, R) and (M, S) be ordered sets with $|P| = |M|$ and let f be a bijection of P onto M. The number d_f given by (1) will be called an f-distance of ordered sets (P, R) and (M, S). If $d_f = d(R, S)$ we will say that f is an optimal mapping between ordered sets (P, R) and (M, S).

Lemma 4. *Let f be an optimal mapping of an ordered set (P, R) onto (M, S). If an element b covers an element a in the ordering R and if $f(a) \parallel f(b)(S)$, then f is an optimal mapping of the ordered set $(P, R\setminus\{[a, b]\})$ onto (M, S) and $d(R\setminus\{[a, b]\}, S) = d(R, S) - 1$.*

P r o o f. It is obvious that the f-distance of the ordered sets $(P, R \setminus \{[a, b]\})$ and (M, S) is $d(R, S) - 1$. Now, we prove that $d(R \setminus \{[a, b]\}, S) \geqslant d(R, S) - 1$. Assume to the contrary that there exists a bijection $g\colon P \to M$ such that the g-distance of $(P, R \setminus \{[a, b]\}$ and (M, S) is at most $d(R, S) - 2$. Then the g-distance of (P, R) and (M, S) is at most $d(R, S) - 1$, which is a contradiction. $\square$

Let (P, R) be an ordered set. If $\mathscr{C}$ is a family of linear orderings (chains) whose intersection is the relation R, then $\mathscr{C}$ is said to be a realizer of R. The dimension of an ordered set (P, R) was defined as the minimal cardinality of a realizer of R (by B. Dushnik and E. W. Miller). As usual, we will denote the dimension of an ordered set (P, R) by $\dim(P, R)$.

Lemma 5. *Let (P, R) and (M, S) be ordered sets of the same cardinality such that $d(R, S) = 1$. Then*

$$|\dim(P, R) - \dim(M, S)| \leqslant 1.$$

P r o o f. Without loss of generality we can suppose that $P = M$ and $R = S \setminus \{[a, b]\}$, where a is an element covered by b in the ordering S. Therefore,

$$\dim(M, S) \in \{\dim(P, R),\ \dim(P, R) - 1,\ \dim(P, R) + 1\}.$$

$\square$

Theorem 4. *Let (P, R), (M, S) be any ordered sets with $|P| = |M|$. Then*

$$d(R, S) \geqslant |\dim(P, R) - \dim(M, S)|. \tag{6}$$

P r o o f. Let $d(R, S) = d_f$, $f(R) \setminus S = \{[a_1, b_1], \ldots, [a_r, b_r]\}$ and $S \setminus f(R) = \{[a_{r+1}, b_{r+1}], \ldots, [a_{r+s}, b_{r+s}]\}$. Because $f(R)$ and S are orderings on

M, the relation $f(R) \cap S$ is an ordering on M, too. Without loss of generality one can suppose that b_1 covers a_1 in the ordering $f(R)$ and that for every $i \in \{2, 3, \ldots, r\}$ the element b_i covers a_i in the ordering $f(R) \setminus \{[a_1, b_1], \ldots, [a_{i-1}, b_{i-1}]\}$. This guarantees that $f(R)$, $f(R) \setminus \{[a_1, b_1]\}$, $\ldots$, $f(R) \setminus \{[a_1, b_1], \ldots, [a_r, b_r]\} = f(R) \cap S$ are orderings on M. Hence, it is readily seen (Lemmas 4 and 5) that

$$\begin{aligned} &d(f(R), f(R) \setminus \{[a_1, b_1]\}) = 1 \geqslant |\dim(M, f(R)) - \dim(M, f(R) \setminus \{[a_1, b_1]\})|, \\ &d(f(R) \setminus \{[a_1, b_1]\}, f(R) \setminus \{[a_1, b_1], [a_2, b_2]\}) \\ &\qquad = 1 \geqslant |\dim(M, f(R) \setminus \{[a_1, b_1]\}) - \dim(f(R) \setminus \{[a_1, b_1], [a_2, b_2]\})|, \end{aligned}$$

etc. This implies that

$$\text{(j)} \quad d((M, f(R)), (M, f(R) \cap S)) = r \geqslant |\dim(M, f(R)) - \dim(M, f(R) \cap S)|.$$

Similarly, one can obtain

$$\text{(k)} \quad d((M, S), (M, f(R) \cap S)) = s \geqslant |\dim(M, f(R) \cap S) - \dim(M, S)|.$$

Consequently,

$$d((M, f(R)), (M, S)) = r + s \geqslant |\dim(M, f(R)) - \dim(M, S)|.$$

Since the ordered set $(M, f(R))$ is isomorphic to the ordered set (P, R), the proof is complete. $\square$

Remark. Let m, k be natural numbers with $n \geqslant 3$, $0 \leqslant k \leqslant n-2$. We show that there exist ordered sets (P, R) and (M, S) satisfying the conditions $d(R, S) = k$, $\dim(P, R) = n$, $\dim(M, S) = n - k$. The statement is obvious for $k = 0$ and $k = 1$. Let $k \geqslant 2$. For the ordered set (P, R) we can take the standard ordered set which has the cardinality $2n$ and the dimension n (see Fig. 5). Furthermore, we can put $M = P$ and $S = R \cup \{[0, 1], [2, 3], \ldots, [2k-2, 2k-1]\}$ (see Fig. 6).

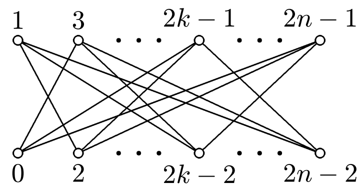

Fig. 5

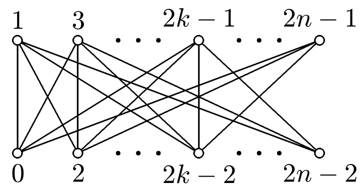

Fig. 6

P r o b l e m. Let (P,R) be an ordered set with $\dim(P,R)=n$ and let (C,S) be a linear ordered set (chain) with $|P|=|C|$. What is the minimal and maximal distance $d(R,S)$?

We recall that for an ordered set (P,R), the Möbius number of (P,R) is given as follows:

$$\mu(P,R) = -1 + \sum_{i=0}^{n(P)} (-1)^i c_i(P),$$

where $n(P)$ is the length of the longest chain of (P,R) and $c_i(P)$ is the cardinality of all chains whose length is i.

The analogue to (6) for the Möbius numbers does not hold. For instance, if (P,R) and (M,S) are the ordered sets drawn on Figs. 7 and 8, then $d(R,S)=1$ but the Möbius numbers of (P,R) and (M,S) are $\mu(P,R)=0$, $\mu(M,S)=1-n$.

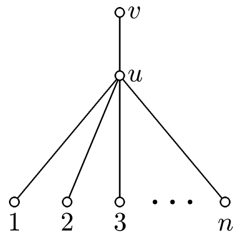

Fig. 7

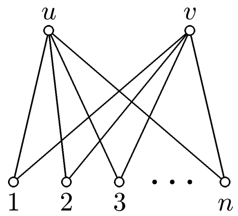

Fig. 8

References

[1] *D. Kelly, W. T. Trotter*: Dimension Theory for Ordered Sets. Ordered Sets, Proc. NATO Adv. Study Inst., Banf, Aug. 28–Sept. 12, 1981. Dordrecht, 1982, pp. 171–211.

[2] *L. Budach, B. Graw, Ch. Meinel, S. Waack*: Algebraic and Topological Properties of Finite Partially Ordered Sets. Teubner, Leipzig, 1988.

[3] *B. Zelinka*: Distances between partially ordered sets. Math. Bohem. *118* (1993), 167–170.

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