ON THE MATRICES OF CENTRAL LINEAR MAPPINGS

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Abstract. We show that a central linear mapping of a projectively embedded Euclidean n-space onto a projectively embedded Euclidean m-space is decomposable into a central projection followed by a similarity if, and only if, the least singular value of a certain matrix has multiplicity $\geqslant 2m-n+1$. This matrix is arising, by a simple manipulation, from a matrix describing the given mapping in terms of homogeneous Cartesian coordinates.

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1. Introduction

A linear mapping between projectively embedded Euclidean spaces is called *central*, if its exceptional subspace is not at infinity. Such a linear mapping is in general not decomposable into a central projection followed by a similarity. Necessary and sufficient conditions for the existence of such a decomposition have been given in [4] for arbitrary finite dimensions; cf. also [1], [2], [3]. However, those results do not seem to be immediately applicable on a *central axonometry*, i.e., a central linear mapping given via an axonometric figure. On the other hand, in a series of recent papers [5], [6], [7] this problem of decomposition has been discussed for central axonometries of the Euclidean 3-space onto the Euclidean plane from an elementary point of view[1].

Loosely speaking, the concept of central axonometry is a geometric equivalent to the algebraic concept of a *coordinate matrix* for a linear mapping of the underlying vector spaces. However, from the results in [2] and [4] it is also not immediate whether or not a given matrix describes (in terms of homogeneous Cartesian coordinates) a mapping that permits the above-mentioned factorization. The aim of this communication is to give a criterion for this.

[1] A lot of further references can be found in the quoted papers.

Let $\mathbf{I}$, $\mathbf{J}$ be finite-dimensional Euclidean vector spaces. Given a linear mapping $f\colon \mathbf{I} \to \mathbf{J}$ denote by $f^{\mathrm{ad}}\colon \mathbf{J} \to \mathbf{I}$ its adjoint mapping. Then $f^{\mathrm{ad}} \circ f$ is self-adjoint with eigenvalues

$$v_1 \geqslant \cdots \geqslant v_r > v_{r+1} = \cdots = v_n = 0.$$

Here r equals the rank of f and $n = \dim \mathbf{I}$. Moreover, each eigenvalue is written down repeatedly according to its multiplicity[2]. The positive real numbers $\sqrt{v_1}, \ldots, \sqrt{v_r}$ are frequently called the *singular values* of f. The multiplicity of a singular value of f is defined via the multiplicity of the corresponding eigenvalue of $f^{\mathrm{ad}} \circ f$. It is immediate from the singular value decomposition that f and f^{ad} share the same singular values (counted with their multiplicities). See, e.g., [8].

These results hold true, mutatis mutandis, when replacing f by any real matrix, say A, and f^{ad} by the transpose matrix A^{T}.

2. Decompositions

When discussing central linear mappings it will be convenient to consider Euclidean spaces embedded in projective spaces. Thus let $\mathbf{V}$ be an $(n+1)$-dimensional real vector space ($3 \leqslant n < \infty$) and $\mathbf{I}$ one of its hyperplanes. Assume, furthermore, that $\mathbf{I}$ is equipped with a positive definite inner product $(\cdot)$ so that $\mathbf{I}$ is a Euclidean vector space. In the projective space on $\mathbf{V}$, denoted by $\mathcal{P}(\mathbf{V})$, we consider the projective hyperplane $\mathcal{P}(\mathbf{I})$ as the hyperplane at infinity. The absolute polarity in $\mathcal{P}(\mathbf{I})$ is determined by the inner product on $\mathbf{I}$. Hence $\mathcal{P}(\mathbf{V}) \setminus \mathcal{P}(\mathbf{I})$ is a projectively embedded Euclidean space[3]. Similarly, let $\mathcal{P}(\mathbf{W}) \setminus \mathcal{P}(\mathbf{J})$ be an m-dimensional projectively embedded Euclidean space ($2 \leqslant m < n < \infty$). Given a linear mapping

$$f\colon \mathbf{V} \to \mathbf{W} \tag{1}$$

of vector spaces then the associate (projective) linear mapping

$$\varphi\colon \mathcal{P}(\mathbf{V}) \setminus \mathcal{P}(\ker f) \to \mathcal{P}(\mathbf{W}),\ \mathbb{R}\mathbf{x} \mapsto \mathbb{R}(f(\mathbf{x})) \tag{2}$$

has $\mathcal{P}(\ker f)$ as its exceptional subspace. In the sequel we shall assume that

$$\ker f \not\subset \mathbf{I} \quad \text{and} \quad f(\mathbf{V}) = \mathbf{W}, \tag{3}$$

[2] For a self-adjoint mapping the algebraic and geometric multiplicities of an eigenvalue are identical. Hence we may unambiguously use the term 'multiplicity'.

[3] We do not endow this space with a unit segment.

or, in other words, that φ is central and surjective[4]. Obviously, (3) is equivalent to

$$f(\mathbf{I}) = \mathbf{W}. \tag{4}$$

We recall some results [2], [4]: If $\mathbf{T}$ is any complementary subspace of $\ker f$ in $\mathbf{V}$, then denote by

$$\psi_{\mathbf{T}} \colon \mathcal{P}(\mathbf{V}) \setminus \mathcal{P}(\ker f) \to \mathcal{P}(\mathbf{T}) \tag{5}$$

the projection with the exceptional subspace $\mathcal{P}(\ker f)$ onto $\mathcal{P}(\mathbf{T})$. The restricted mapping

$$\varphi_{\mathbf{T}} \colon = \varphi \mid \mathcal{P}(\mathbf{T}) \colon \mathcal{P}(\mathbf{T}) \to \mathcal{P}(\mathbf{W}) \tag{6}$$

is a collineation and

$$\varphi = \varphi_{\mathbf{T}} \circ \psi_{\mathbf{T}}; \tag{7}$$

every decomposition of φ into a projection and a collineation is of this form. In the Euclidean vector space $\mathbf{I}$ we have the distinguished subspace

$$\mathbf{E} \colon = f^{-1}(\mathbf{J}) \cap \mathbf{I}. \tag{8}$$

Write

$$f_{\mathbf{E}} \colon \mathbf{E} \to \mathbf{J},\ \mathbf{x} \mapsto f(\mathbf{x}); \tag{9}$$

this $f_{\mathbf{E}}$ is well-defined and surjective, since $\mathbf{E} \subset f^{-1}(\mathbf{J})$ and $\ker f \not\subset \mathbf{E}$. The subspace $\mathbf{T}$ can be chosen with $\varphi_{\mathbf{T}}$ being a similarity if, and only if, the least singular value of $f_{\mathbf{E}}$ has multiplicity[5] $\geqslant 2m - n + 1$.

Next, we assume that $\mathcal{P}(\mathbf{T}) \not\subset \mathcal{P}(\mathbf{I})$ is orthogonal to $\mathcal{P}(\ker f)$. This means that $(\mathbf{T} \cap \mathbf{I})^{\perp} \subset \ker f \cap \mathbf{I}$ or $(\mathbf{T} \cap \mathbf{I})^{\perp} \supset \ker f \cap \mathbf{I}$. Hence $\psi_{\mathbf{T}}$ is an *orthogonal central projection*[6]. It is easily seen from [2] that φ permits a decomposition into an orthogonal central projection followed by a similarity if, and only if, all singular values of $f_{\mathbf{E}}$ are equal.

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[4] This assumption of surjectivity is made 'without loss of generality' in most papers on this subject. It will, however, be essential several times in this paper.

[5] In [2, Satz 10] this multiplicity is printed incorrectly as $2m - n - 1$.

[6] The central projections used in elementary descriptive geometry are trivial examples of orthogonal central projections.

Finally, we are going to show that the crucial properties of $f_{\mathbf{E}}$ can be read off from another mapping: Denote by

$$p\colon \mathbf{I} \to \mathbf{E} \tag{10}$$

the orthogonal projection with the kernel $\mathbf{E}^{\perp} \subset \mathbf{I}$. Then

$$(f_{\mathbf{E}} \circ p) \circ (f_{\mathbf{E}} \circ p)^{\mathrm{ad}} = f_{\mathbf{E}} \circ p \circ p^{\mathrm{ad}} \circ (f_{\mathbf{E}})^{\mathrm{ad}} = f_{\mathbf{E}} \circ (f_{\mathbf{E}})^{\mathrm{ad}}, \tag{11}$$

since p^{ad} is the natural embedding $\mathbf{E} \to \mathbf{I}$. Thus, by (11) and the results stated in Section 1, $f_{\mathbf{E}}$ and $(f_{\mathbf{E}} \circ p)^{\mathrm{ad}}$ have the same singular values (counted with their multiplicities). Hence, by the surjectivity of $f_{\mathbf{E}}$ and (11), all singular values of $f_{\mathbf{E}}$ are equal if, and only if, there exists a real number $v > 0$ such that

$$(f_{\mathbf{E}} \circ p) \circ (f_{\mathbf{E}} \circ p)^{\mathrm{ad}} = v\,\mathrm{id}_{\mathbf{J}}. \tag{12}$$

We shall use this in the next section.

3. A matrix characterization

Introducing homogeneous Cartesian coordinates in $\mathcal{P}(\mathbf{V})$ is equivalent to choosing a basis $\{\mathbf{b}_0, \ldots, \mathbf{b}_n\}$ of $\mathbf{V}$ such that $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbf{I}$ is an orthonormal system. The origin is given by $\mathbb{R}\mathbf{b}_0$ and the unit points are $\mathbb{R}(\mathbf{b}_0 + \mathbf{b}_1), \ldots, \mathbb{R}(\mathbf{b}_0 + \mathbf{b}_n)$. In the same manner we are introducing homogeneous Cartesian coordinates in $\mathcal{P}(\mathbf{W})$ via a basis $\{\mathbf{c}_0, \ldots, \mathbf{c}_m\}$.

Theorem 1. *Suppose that $f\colon \mathbf{V} \to \mathbf{W}$ is inducing a surjective central linear mapping φ according to formula* (2)*. Let*

$$A = \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} \tag{13}$$

be the coordinate matrix of f with respect to bases of $\mathbf{V}$ and $\mathbf{W}$ that are yielding homogeneous Cartesian coordinates. Write

$$\mathbf{a}_i := (a_{i1}, \ldots, a_{in}) \in \mathbb{R}^n \text{ for all } i = 0, \ldots, m \tag{14}$$

and

$$\widetilde{A} := \begin{pmatrix} \mathbf{a}_1 - \frac{\mathbf{a}_0 \cdot \mathbf{a}_1}{\mathbf{a}_0 \cdot \mathbf{a}_0}\mathbf{a}_0 \\ \vdots \\ \mathbf{a}_m - \frac{\mathbf{a}_0 \cdot \mathbf{a}_m}{\mathbf{a}_0 \cdot \mathbf{a}_0}\mathbf{a}_0 \end{pmatrix}. \tag{15}$$

Then the following assertions hold true:

1. φ *is decomposable into a central projection followed by a similarity if, and only if, the least singular value of the matrix* $\widetilde{A}$ *has multiplicity* $\geqslant 2m-n+1$.
2. φ *is decomposable into an orthogonal central projection followed by a similarity if, and only if, there exists a real number* $v>0$ *such that*

$$\widetilde{A}\widetilde{A}^{\mathrm{T}} = \operatorname{diag}(v,\dots,v). \tag{16}$$

Proof. We read off from the top row of A that

$$a_{00}x_0 + \dots + a_{0n}x_n = 0$$

is an equation of $f^{-1}(\mathbf{J}) \neq \mathbf{I}$ so that $\mathbf{a}_0 \cdot \mathbf{a}_0 \neq 0$. Write $\widetilde{f}\colon \mathbf{I} \to \mathbf{J}$ for the linear mapping whose coordinate matrix with respect to $\{\mathbf{b}_1,\dots,\mathbf{b}_n\}$ and $\{\mathbf{c}_1,\dots,\mathbf{c}_m\}$ equals $\widetilde{A}$. A straightforward calculation shows that

$$\widetilde{f}(\mathbf{x}) = f(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{E}$$

and

$$\widetilde{f}(a_{01}\mathbf{b}_1 + \dots + a_{0n}\mathbf{b}_n) = 0,$$

i.e., $\mathbf{E}^{\perp} \subset \ker \widetilde{f}$. Thus $\widetilde{f}$ equals the mapping $f_{\mathbf{E}} \circ p$ discussed above. Now the proof is completed by translating formulae (11) and (12) into the language of matrices. $\square$

We remark that (3) and the linear independence of $\mathbf{a}_1,\dots,\mathbf{a}_m$ are equivalent conditions.

In contrast to the results in [5], [6], [7], the φ-image of the origin $\mathbb{R}\mathbf{b}_0$ does not appear in our characterization. On the other hand, we have

$$f(\mathbf{E}^{\perp}) = \mathbb{R}((\mathbf{a}_0 \cdot \mathbf{a}_0)\mathbf{c}_0 + \dots + (\mathbf{a}_0 \cdot \mathbf{a}_m)\mathbf{c}_m).$$

In projective terms this 1-dimensional subspace of $\mathbf{W}$ gives the *principal point* of the mapping φ. Exactly if the principal point of φ equals the origin $\mathbb{R}\mathbf{c}_0$, then $\widetilde{A}$ arises from A merely by deleting the top row and the leading column.

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