EXISTENCE OF QUASICONTINUOUS SELECTIONS FOR THE SPACE $2^{\mathbb{R}}$

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Abstract. The paper presents new quasicontinuous selection theorem for continuous multifunctions $F\colon X \longrightarrow \mathbb{R}$ with closed values, X being an arbitrary topological space. It is known that for $2^{\mathbb{R}}$ with the Vietoris topology there is no continuous selection. The result presented here enables us to show that there exists a quasicontinuous and upper$\langle$lower$\rangle$-semicontinuous selection for this space. Moreover, one can construct a selection whose set of points of discontinuity is nowhere dense.

Keywords: continuous multifunction, selection, quasicontinuity

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1. Introduction

Up to now, papers dealing with the problem of existence of quasicontinuous selections have considered multifunctions with compact values in metric spaces (see e.g. [2, 6, 7, 8]). Another classical condition in selection theory is the convexity of values ([9, 10]).

In this paper we present a quasicontinuous selection theorem for continuous multifunctions $F\colon X \longrightarrow \mathbb{R}$ with closed values, X being an arbitrary topological space. It is shown that the graph of F can be constructed as the union of graphs of quasicontinuous and upper-semicontinuous selections of F. Moreover, the sets of points of discontinuity of these selections are nowhere dense. Our result enables us to complete the work of [1] concerning the hyperspace $2^{\mathbb{R}}$.

2. Preliminaries

By $2^{\mathbb{R}}$ we mean the class of all nonempty closed subsets of $\mathbb{R}$ equipped with the Vietoris topology (for definition of basic notions: Vietoris topology, hyperspace, multifunction, selection, l.s.c., u.s.c., Hausdorff continuous multifunction etc. see e.g. [4] and [11]).

Let X and Y be two topological spaces. A multifunction F from X to Y is called continuous, if it is l.s.c. and u.s.c. (lower and upper semicontinuous).

Let us denote $F^{-}(A;B) = \{x;\ F(x) \cap A \neq \emptyset \text{ and } F(x) \subset B\}$. Of course, for $B, A \subset Y$ open and F continuous, the set $F^{-}(A;B)$ is an open subset of X.

Let B be a subset of a topological space X. In what follows $\operatorname{int} B$ and $\operatorname{cl} B$ denote the interior and the closure of the set B, respectively. There are several equivalent definitions of quasicontinuity, we will use the following one: A function $f\colon X \longrightarrow Y$ is said to be quasicontinuous at $x \in X$ if and only if for any open set V such that $f(x) \in V$ and any open set U such that $x \in U$, there exists a nonempty open set $W \subset U$ such that $f(W) \subset V$ ([5.12, 13]).

3. Results

In the next theorem the space X is an arbitrary topological space, which is quite rare in the selection theory. Nevertheless, the fact that $Y = \mathbb{R}$ permits us to give a constructive proof of the assertion.

Theorem 1. *Let X be an arbitrary topological space. Let $F\colon X \longrightarrow \mathbb{R}$ be a continuous multifunction with closed values. Then F has a quasicontinuous and upper-semicontinuous selection h such that its set of points of discontinuity is a nowhere dense set.*

P r o o f. Let us define $g(x) = \min\{|y|;\ y \in F(x)\}$ for every x from X. We denote $A = \{x \in X;\ g(x) \in F(x) \text{ and } -g(x) \in F(x)\}$. The set A is closed. We will prove it by proving that $X - A$ is open.

Let $b \in X - A$. Let us consider the case $g(b) \in F(b)$, the other $(-g(b) \in F(b))$ being analogous. In this case $-g(b)$ is not an element of $F(b)$ and since the set $F(b)$ is closed, there exists $\delta > 0$ such that

(i) $$F(b) \subset U = (-\infty, -g(b) - \delta) \cup (g(b) - \delta, +\infty)$$

and

(ii) $$g(b) - \delta > 0.$$

Let us denote $V = (g(b)-\delta, g(b)+\delta)$. Since F is continuous and (i) and (ii) hold, the set $W = F^{-}(V;U)$ is an open neighborhood of the point b. Of course $W \subset X - A$. So the set $X - A$ is open.

Let us denote $B = A - \operatorname{cl}(\operatorname{int} A)$. For every element x of X one of the following assertions is true:

(1) $x \in X - A$ and $g(x) \in F(x)$;

(2) $x \in X - A$ and $-g(x) \in F(x)$;

(3) $x \in \operatorname{cl}(\operatorname{int} A)$;

(4) $x \in B$ and for every open neighborhood $O(x)$ of the point x there exists a point $t \in O(x)$ such that $g(t) \in F(t)$ and $-g(t) \notin F(t)$ hold;

(5) $x \in B$ and there exists an open neighborhood $O(x)$ of the point x such that for every element t of $O(x)$, $-g(t) \in F(t)$ holds.

Let us define a function $h\colon X \longrightarrow \mathbb{R}$ as follows:

$$
\begin{aligned}
&h(x) = 0 \text{ if } 0 \in F(x),\\
&h(x) = g(x) \text{ if (1) or (3) or (4) is true and } 0 \notin F(x),\\
&h(x) = -g(x) \text{ if (2) or (5) holds and } 0 \notin F(x).
\end{aligned}
$$

It is easy to see that h is a selection of F. We will prove that h is quasicontinuous at every x in X.

First, let $x \in X$ be such that $h(x) = 0$. Let V be an open neighborhood of $h(x)$. Then there exists $\varepsilon > 0$ such that $U = (-\varepsilon, \varepsilon) \subset V$. The set $W = F^{-}(U;\mathbb{R})$ is an open neighborhood of x and for every element w of W $\{-g(w), g(w)\} \subset U$ holds. So $h(w) \in V$, $\forall w \in W$ is true, and the function f is continuous at the point x.

If $0 \notin F(x)$, we distinguish five cases:

(I) Let $x \in X$ and let (1) hold. Let $O \subset \mathbb{R}$ be an open set such that $h(x) \in O$. Then there exists $\delta > 0$ such that

$$F(x) \subset G = (-\infty, -h(x)-\delta) \cup (h(x)-\delta, +\infty)$$

and

$$h(x) - \delta > 0 \quad \text{and} \quad H = (h(x)-\delta, h(x)+\delta) \subset O$$

is true.

Hence x is an element of the set $C = F^{-}(H;G)$ and since F is continuous, the set C is open. It is easy to verify from the definition of C that (1) holds for every $t \in C$ and this implies $h(C) \subset H \subset O$. So the function h is continuous at the point x.

(II) Quite analogously, if (2) is satisfied for an x from X, then h is continuous at the point x.

(III) Let $x \in X$ and let (3) hold. Let $O \subset \mathbb{R}$, $G \subset X$ be two open sets such that $x \in G$ and $h(x) \in O$. Then there exists $\delta > 0$ such that following holds:

$$h(x) - \delta > 0,$$
$$V = (h(x) - \delta, h(x) + \delta) \subset O,$$
$$F(x) \subset U = (-\infty, -h(x) + \delta) \cup (h(x) - \delta, +\infty).$$

Let us denote $W = G \cap F^-(V; U)$. Since W is an open neighborhood of x and $x \in \mathrm{cl}(\operatorname{int} A)$, the set $P = W \cap \operatorname{int} A$ is nonempty open, $P \subset G$. For every p from P we have $p \in F^-(V; U)$, hence $h(p) \in V \subset O$. This proves the quasicontinuity of h at x. Moreover, if $x \in \operatorname{int} A$, then $x \in P$ and we see that h is continuous at the point x. If x is not from $\operatorname{int} A$, it is still true that for every $\varepsilon > 0$ and for every $v \in F^-((h(x) - \varepsilon, h(x) + \varepsilon); \mathbb{R})$ the inequality

$$h(v) \leqslant h(x) + \varepsilon$$

holds; so, h is upper-semicontinuous at x.

(IV) Let $x \in X$ and let (4) hold. Let $O \subset \mathbb{R}$, $G \subset X$ be two open sets such that $x \in G$ and $h(x) \in O$. Then there exists $\delta > 0$ such that

$$h(x) - \delta > 0, \quad V = (h(x) - \delta, h(x) + \delta) \subset O$$

and

$$F(x) \subset U = (-\infty, -h(x) + \delta) \cup (h(x) - \delta, +\infty)$$

hold. Let us denote $W = G \cap F^-(V; U)$. W is an open neighborhood of the point x. From the validity of (4) we obtain that there exists $t \in W$ such that (1) is true for t and $h(t) = g(t)$. Since $t \in W, h(t) \in V$ holds. By (I) the function h is continuous at the point t; so, there exists an open neighborhood H of t such that $h(s) \in V$ for every $s \in H$. Let us denote $P = H \cap W$. The set P is an open subset of G and $h(p) \in V$ for every $p \in P$. This proves the quasicontinuity of h at the point x. The proof of the upper-semicontinuity of h at the point x is left to the reader.

(V) Let $x \in X$ and let (5) hold. Let $O \subset \mathbb{R}$ be an open set such that $h(x) \in O$. Then there exists $\delta > 0$ such that

$$h(x) + \delta < 0, \qquad V = (h(x) - \delta, h(x) + \delta) \subset O$$

and

$$F(x) \subset U = (-\infty, h(x) + \delta) \cup (-h(x) - \delta, +\infty)$$

hold. Let us denote $W = (F^{-}(V;U) \cap O(x)) - \mathrm{cl}(\mathrm{int}\, A)$ where $O(x)$ is the set mentioned in (5). Then W is an open neighborhood of the point x and for every $w \in W$ either (2) or (5) is true. Therefore $h(w) = -g(w) \in V \subset O$ holds for every $w \in W$. This implies the continuity of the function h at x.

To complete the proof, it suffices now to show that the set of points of discontinuity of h is nowhere dense. But it is easy to see that this set is a subset of the set

$$B \cup (\mathrm{cl}(\mathrm{int}\, A) - \mathrm{int}\, A) = (A - \mathrm{cl}(\mathrm{int}\, A)) \cup (\mathrm{cl}(\mathrm{int}\, A) - \mathrm{int}\, A).$$

Since A is closed, this set is the union of two nowhere dense sets. $\square$

Now we present two examples relevant to Theorem 1.

Example 1 ([2]). We show that the assumption "F is u.s.c." in Theorem 1 cannot be omitted.

Let $X = \{a, b, c\}$, let (X, T) be a topological space with the topology $T = \{\emptyset\} \cup \big\{\{a\}, \{c, a\}, \{b, a\}, X\big\}$. Define $F\colon X \longrightarrow \mathbb{R}$ as follows:

$$F(a) = \{1, 2\}, \quad F(b) = \{1\}, \quad F(c) = \{2\}.$$

F is a l.s.c. multifunction with compact values and F has no quasicontinuous selection.

Example 2. Let $X = \mathbb{N} = \{1, 2, \ldots\}$ be a topological space with the topology $T = \{A;\ A \subset \mathbb{N},\ \mathbb{N} - A \text{ is a finite set}\} \cup \{\mathbb{N}, \emptyset\}$. Let us define a multifunction $F\colon X \longrightarrow \mathbb{R}$ as follows:

$$F(k) = \mathbb{N} - \{1, 2, \ldots, k\}.$$

The multifunction F is u.s.c., it has closed values, but it is not l.s.c. It is easy see that it has no quasicontinuous selection, because all quasicontinuous functions from (X, T) to $\mathbb{R}$ are constant ones.

Reading the proof of Theorem 1 we see that for every x in X, $0 \in F(x)$ implied $h(x) = 0$. This fact will be used in the proof of the following assertion:

Theorem 2. *Let X be an arbitrary topological space. Let $G\colon X \longrightarrow \mathbb{R}$ be a continuous multifunction with closed values. Let (x, y) be an element of the graph of G. Then there exists a quasicontinuous and upper-semicontinuous selection $g\colon X \longrightarrow Y$ such that $g(x) = y$, g is continuous at x and the set of points of discontinuity of g is nowhere dense.*

Proof. Let us define a multifunction $F\colon X \longrightarrow \mathbb{R}$ as follows: $F(t) = G(t) - y$ for $t \in X$. Then $F\colon X \longrightarrow \mathbb{R}$ is a continuous multifunction with closed values and according to Theorem 1 there exists a quasicontinuous and upper-semicontinuous selection h of F. Since 0 is an element of $F(x)$, $h(x) = 0$ holds and h is continuous at x. Let us define a function $g\colon X \longrightarrow \mathbb{R}$ in the following way: $g(t) = h(t) + y$ for $t \in X$. The function g is quasicontinuous, upper-semicontinuous and it is a selection of G.

Moreover, $g(x) = h(x) + y = y$ holds. $\square$

It is well known that there is no continuous selection for the hyperspace of nonempty closed subsets of $\mathbb{R}$ with the Vietoris topology ([1]). However, Theorem 2 gives us the following result:

Corollary 1. *Let I be the "identity multifunction" from $2^{\mathbb{R}}$ to $\mathbb{R}$, such that $I(A) = A$ holds for every $A \in 2^{\mathbb{R}}$. Then for every point (x, y) of the graph of I there exists a quasicontinuous and upper-semicontinuous selection f of I such that $f(x) = y$ and the set of points of discontinuity of g is nowhere dense.*

Remark 1. Theorem 1 and Corollary 1 also imply (under the same conditions) the existence of a quasicontinuous selection which is lower-semicontinuous. It suffices to consider a multifunction $G = -F$. Then there exists an upper-semicontinuous (and quasicontinuous) selection g of G. Then $h = -g$ is the lower-semicontinuous selection of F we wanted.

Remark 2. It is easy to check that Theorem 1 and Theorem 2 are true also if the assumption "F is a continuous multifunction" is replaced by the assumption "F is Hausdorff continuous". In this case Corollary 1 can be reformulated: $2^{\mathbb{R}}$ can be replaced by the hyperspace of nonempty closed subsets of $\mathbb{R}$ with the topology derived from Hausdorff metric.

Another example relevant to our results, an example of a continuous and Hausdorff continuous multifunction $F\colon [-1, 0] \longrightarrow \mathbb{R}$ with closed values which has no continuous selection can be found in [3].

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