# JAROSLAV KURZWEIL SEPTUAGENARIAN

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On May 7, 1996 Professor Jaroslav Kurzweil reaches seventy years of age. Ten years ago his jubilee was mentioned among other in the Czechoslovak Mathematical Journal[1] and in Časopis pro pěstování matematiky.[2] This note should be considered a continuation of these articles. The list of publications of J. Kurzweil is continued here by starting the enumeration of his papers where it ended in 1986.

Jaroslav Kurzweil has been continuing his work in differential equations and related fields during the period of the last ten years. We start with explaining this part of the work of J. Kurzweil.

A very important contribution of J. Kurzweil is concerned with the concept of the multiplicative integral. In the papers [80] and [83] the Kurzweil approach to Perron integration is applied for defining the product integral $\prod\limits_a^b V(t,\,\mathrm{d}t)$ for an $n\times n$-matrix valued function $V\colon [a,b]\times\mathcal{J}\to L(\mathbb{R}^n)$, where $\mathcal{J}$ is the set of all compact subintervals of the interval $[a,b]\subset\mathbb{R}$ and $L(\mathbb{R}^n)$ denotes the set of all $n\times n$-matrices. The Perron-product integral is defined as follows:

Given a positive function $\delta\colon [a,b]\to(0,+\infty)$, called a *gauge*, assume that

$$D=\{(t_i,J_i);\, t_i\in J_i=[x_{i-1},x_i]\subset\mathcal{J}, i=1,\dots,k\}$$

is a *tagged partition* of $[a,b]$, i.e.

$$x_0=a<x_1<x_2<\ldots<x_k=b,$$

which is $\delta$-fine, i.e.

$$J_i\subset(t_i-\delta(t_i),t_i+\delta(t_i)).$$

For a given function $V\colon [a,b]\times\mathcal{J}\to L(\mathbb{R}^n)$ and a tagged partition $D=\{(t_i,J_i), i=1,\dots,k\}$ we denote by

$$P(V,D)=V(t_k,J_k)V(t_{k-1},J_{k-1})\ldots V(t_1,J_1)$$

[1] Czechoslovak Mathematical Journal *36* (1986), pp. 147–166.
[2] Časopis pro pěstování matematiky *111* (1986), pp. 91–111 (In Czech).

the ordered product of matrices $V(t_i, J_i)$.

The function $V$ is called Perron product integrable if there is a regular $Q \in L(\mathbb{R}^n)$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ on $[a, b]$ such that

$$\|P(V, D) - Q\| < \varepsilon$$

for every $\delta$-fine partition $D$ of $[a, b]$. $Q \in L(\mathbb{R}^n)$ is the Perron product integral $\prod\limits_a^b V(t, \mathrm{d}t)$.

Typical representatives of functions $V$ are $V(\tau, [\alpha, \beta]) = I + A(\tau)(\beta - \alpha)$ or $V(\tau, [\alpha, \beta]) = \exp^{A(\tau)(\beta-\alpha)}$ where $A\colon [a, b] \to L(\mathbb{R}^n)$ is a given matrix valued function and $[\alpha, \beta] \subset [a, b]$ is an interval. The relations to the linear system of ordinary differential equations of the form $\dot{x} = A(t)x$ are studied using the *indefinite* Perron product integral $\prod\limits_a^t V(s, \mathrm{d}s)$.

In [84] the system

$$\dot{x} = A(t)x$$

with a continuous $n \times n$-matrix valued function $A(t)$, $t \in \mathbb{R}$ is studied provided $A(t) + A^*(t) = 0$ and $A$ is uniformly quasiperiodic with at most $r + 1$ frequencies.

The problem is the following: Given $A$ and $\eta > 0$, does there exist a matrix valued function $C$ that both $C$ and the matrix solution $X_C(t)$ of

$$\dot{x} = C(t)x, \quad X_C(0) = I$$

are uniformly quasiperiodic with at most $r + 1$ frequencies and $\|A(t) - C(t)\| \leqslant \eta$ for $t \in \mathbb{R}$?

The answer to this question is affirmative for such couples $(n, r)$ that the manifold $SO(n)$ of orthonormal $n \times n$-matrices with determinant equal to 1 has the estimation property of homotopies of order $1, 2, \ldots, r$.

The ordinary differential equation

$$\dot{x} = f(x, t)$$

is considered in [89] in the integral form

$$x(t) = x(a) + \int_a^t f(x(s), s)\,\mathrm{d}s$$

with the Perron integral on the right hand side. R. Henstock[3] gave an existence result for this equation under some conditions on the right hand side $f$. In [89] it is

[3] R. Henstock: *Lectures on the Theory of Integration.* World Scientific Singapore, 1988.

shown that Henstock's conditions are satisfied if and only if $f(x,t) = g(t) + h(x,t)$ where $g$ is Perron integrable and for $h$ the well known Carathéodory conditions hold.

In [87] and [92] the linear difference equation

$$(1) \qquad x(n+1) = A(n)x(n), \quad n \in N = \{0, 1, \ldots\}$$

is studied for the case that $A(n)$ is a $k \times k$ invertible matrix function for $n \in N$. In the first paper [87] it is shown that if the difference equation has an exponential dichotomy then it is topologically equivalent to the system

$$(2) \qquad x_i(n+1) = e_i x_i(n), \quad n \in N \text{ and } i = 1, 2, \ldots, k$$

where $e_i = \frac{1}{e}$ or $e_i = e$ and that the difference equation is structurally stable if and only if it has an exponential dichotomy. In [92] these results are completed by showing that the system (1) is topologically equivalent to the system (2) if and only if the matrix functions $A(n)$ and $A^{-1}(n)$ are bounded on $N$. Moreover it is proved in [92] that if two linear difference equations with invertible coefficient matrices are topologically equivalent and one of them has bounded coefficient matrix together with its inverse, then the coefficient matrix of the other system has the same property.

In the papers [82], [85], [86] Kurzweil studied certain convergence phenomena in ordinary differential equations; these papers amend former results on continuous dependence of solutions of ODE's on a parameter which date back to the fifties, cf. [12]–[14]. A model equation for these results is

$$\dot{x} = \sum_{i=1}^{r} f_i(x) k^\sigma \cos(kt + \theta_i).$$

It is shown that for $\sigma = \frac{1}{2}$ the solutions $x_k$ tend to the solution of a "limit equation" which involves the Lie brackets of the functions on the righthand sides. For the above model equation it has the form

$$\dot{x} = \tfrac{1}{2} \sum_{i<j} [f_i, f_j](x) \sin(\theta_j - \theta_i),$$

where the Lie bracket is given by $[f,g] = Dgf - Dfg = D_f g - D_g f$. The case which leads to a limit equation involving iterated Lie brackets was studied in [85].

The papers [78], [88] and [91] elaborate the basic idea from [76], namely the idea of an integral defined via partitions of unity (hence the name PU-integral). Let us recall the definition. If $f$ is a function with compact support $\operatorname{supp} f$, then any finite system $\Delta$ of pairs $(x^j, \theta_j), j = 1, 2, \ldots, k$ is called a PU-partition provided $\theta_j$ are functions of class $C^1$ with compact supports, $0 \leqslant \theta_j(x) \leqslant 1$, $\operatorname{Int}\{x \in \mathbb{R}^n;\ \sum_{j=1}^k \theta_j(x) = 1\} \supset \operatorname{supp} f$. We define the integral sum corresponding to $f, \Delta$ by

$$S(f, \Delta) = \sum_{j=1}^{k} f(x^j) \int \theta_j(x)\, \mathrm{d}x$$

and introduce the PU-integral of $f$ as the number $q$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ such that

$$|q - S(f, \Delta)| < \varepsilon$$

for every $\delta$-fine PU-partition $\Delta$. (Here a gauge is any positive function on $\operatorname{supp} f$ and a PU-partition is $\delta$-fine if $\operatorname{supp} \theta_j \subset B(x^j, \delta(x^j))$ for $j = 1, 2, \ldots, k$.) It is easy to show that the number $q$ is uniquely determined provided it exists. In order to obtain a suitable concept of integral, the family of admissible partitions is reduced by imposing a certain analogue of the regularity condition for intervals (the ratio of the shortest and the longest edge has to be separated from zero). Roughly speaking, this condition ensures that the Stokes Theorem is valid for differentiable vector fields without further restrictions. This makes it possible to use the integral for integration on manifolds. Technically, it is required that the tag $x^j$ be located so that the corresponding function $\theta_j$ is not too small in its neighbourhood. The properties of each individual concept of the integral depend on the character of the regularity condition introduced. In particular, the condition used in [79] guarantees the validity of the Stokes Theorem for vector fields with discontinuities or even singularities. The condition introduced in [91] makes it possible to prove that $C^1$ functions $\psi$ with

$$\|\psi\|_1 = \sup\{|\psi(x)| + \|\mathrm{D}\psi(x)\|;\, x \in G\} < \infty$$

($G$ open bounded, $\operatorname{supp} f \subset G$) are multipliers, that is, if $\int f$ exists and $\psi$ is a function as above then $\int f\psi$ also exists. Moreover there exists $C = C(f) > 0$ such that

$$\left|(PU) \int f\psi \,\mathrm{d}x\right| \leqslant C\|\psi\|_1.$$

(This makes it possible to use this integral as a starting point for developing a theory of distributions.)

Two papers, [81] and [90] deal with one-dimensional generalized Perron integrals introduced via Riemann-type sums in which the partitions are subjected to a certain symmetry condition. Namely, the tag is required to be “not too far” from the centre of the interval in question. These integrals have some interesting properties (similar to the “valeur principale”) and in some cases allow to establish a standard transformation theorem, which is not possible for the classical Perron integral.

The largest number of papers from the decade 1986–96 is devoted to a thorough study of summation integrals in $\mathbb{R}^n$ over a compact interval. To obtain generalized Riemann integrals, the partitions of the integration interval are required to consist of regular intervals, i.e. intervals whose regularity (ratio of the shortest and the longest edge) is greater than a certain value $\varrho$ (a constant or, more generally, a function of the tag and/or the diameter of the interval of partition). The main problems considered are

(i) convergence theorems,

(ii) properties of the primitive function.

Besides general results they include a number of examples or rather counterexamples which clarify the relations between individual concepts of regular integrals. The beginnings of this line of research go back to the paper [73] from 1983, which again was inspired by J. Mawhin,[4] and its results have appeared in [95]–[102]. The main idea of [73] is presented in the anniversary paper in Czechoslovak Mathematical Journal, p. 160, see footnote 1. In the paper [95] Kurzweil goes back to the original Mawhin's definition, showing that the notion of the $\alpha$-regular integral actually depends on the bound for the regularity: if $\alpha < \beta$, then there exists a function $f$ that is $\beta$-integrable but not $\alpha$-integrable. On the other hand, the notion of $\alpha$-regular differentiability is independent of the value of $\alpha$, i.e. a function $f$ $\alpha$-regularly differentiable at a point is $\gamma$-regularly differentiable at the point for any $\gamma > 0$ (which of course does not mean that the regularity condition can be omitted). Paper [96] offers a comparison of various "regular integrals", among other those of W. Pfeffer[5], and shows that the "dangerous" points are those on the boundary of the integration interval. This led to the introduction of the so called extensive integral:

Let $I \subset \mathrm{Int}L \subset \mathbb{R}^n$, $L$ a compact interval. For $f\colon I \to \mathbb{R}$ define $f_{ex}\colon L \to \mathbb{R}$ by extending $f$ from $I$ to $L$ by $f_{ex}(x) = 0$ for $x \in L \setminus I$. The function $f\colon I \to \mathbb{R}$ is called *extensively integrable* if there is $L, I \subset \mathrm{Int}L$ such that $f_{ex}$ is Mawhin integrable (on $L$), cf. footnote 4.

In [97] the regularity condition is generalized in the sense that instead of measuring the regularity of the intervals of a partitions "uniformly", i.e. by a constant, it is measured by a function which may depend on the position of the tag $t$ of a pair $(t, J)$ and/or on the diameter of $J$. The paper [99] summarizes the properties of the regular integrals; in particular, it contains the descriptive definition and convergence theorems. Finally, the paper [101] deals with the problem whether the regular integral can be introduced via the Bochner approach, i.e. by extending the elementary integration of stepfunctions (piecewise constant functions) using a suitable limiting process. This required further modification of the concept of integral leading to the concept of strong integration. In [102] the strong integral is further modified by using the L-partitions which differ from the partitions used before by omitting the condition $t \in J$ for any pair $(t, J)$. (The partitions with $t \in J$ are more specifically called P-partitions. The letters L, P stand for Lebesgue, Perron, respectively, since the respective concepts of integral are connected with the classical Lebesgue and Perron integrals.)

In the common paper with J. Mawhin and W. Pfeffer [93] Kurzweil's idea of PU-partitions is combined with Pfeffer's one of the BV integration (BV for bounded

[4] J. Mawhin: *Generalized multiple Perron integrals and the Green-Goursat theorem for differentiable vector fields.* Czechoslovak Mathematical Journal *31* (1981), 614–632.

[5] W. F. Pfeffer: *A Riemann type integration and the fundamental theorem of calculus.* Rendiconti Circ. Mat. Palermo, Ser. II, *36* (1987), 482–506.

variation in the sense of DeGiorgi). This makes it possible to avoid the shortcomings and accentuate the advantages of both approaches.

The paper [94] is devoted mainly to the study of convergence theorems for generalized Perron integrals. The simplest assumption which (in addition to the pointwise convergence of the sequence of functions $f_k$ to a limit function $f$) is that of equiintegrability of the sequence, which of course means that the gauge in the definition of the integral can be chosen independently of $k$. Lee Peng Yee[6] introduced the notion of controlled convergence which involves a certain kind of generalized absolute continuity of the primitives $F_k$. In [94] for the more dimensional case analogous results are obtained by relaxing the notion of absolute continuity required. In order to obtain results in a more general setting, an axiomatic approach is chosen which allows to treat simultaneously various kinds of integrals.

It would be superfluous to dilate upon Kurzweil's merits in cultivating, fostering and developing Mathematics. The greater are his deserts that his work was done under a totalitarian regime which certainly did not create and ensure adequate conditions for scientific work in spite of its frequent big-mouthed declarations. For more than three decades Professor Kurzweil was head of Department of Ordinary Differential Equation of the Mathematical Institute of the Academy, members of which were also both the authors. So obvious was his integrity and natural authority that he was respected by practically everybody, except perhaps the most hardline party bosses. It was his merit that the microclimate in the Department as regarded both the scientific work and the human relations remained so exceptional even in the relatively favourable atmosphere of the Institute generally, during all the peripetias of four decades of communist reign. It was also a sense of humour of his own that helped him to get over the absurdities of the period. Let us just recall the opening ceremony of the EQUADIFF 7 Conference at which Kurzweil delivered an opening address. He started quite innocently: "Today we celebrate an extraordinary anniversary." Nevertheless, the audience (at least the Czechs and Slovaks, but many foreigners as well) held their breath: it was August 21, the day of Soviet invasion to Czechoslovakia in 1968. After a well-timed pause, Kurzweil went on: "Exactly two hundred years passed since the birth of one of the greatest mathematicians of all times, Augustin Cauchy..."

Soon after the revolution in 1989 J. Kurzweil was elected Director of the Mathematical Institute, and has held this office ever since. Since 1990 he has been chairman of the Board for Accreditation attached to the government of the Czech Republic which is an advisory body of the government for the scientific and teaching level of all institutions of higher education in the Czech Republic, approving among other their right to grant the academic degrees of Master and Doctor. In January 1996 Jaroslav Kurzweil was elected foreign member of the Belgian Royal Academy of Sciences.

[6] Lee Peng Yee: *Lanzhou Lectures on Henstock Integration*, Series in Real Analysis, Vol. 2, World Scientific Singapore, 1989.

The deep trace of Kurzweil's work and personality in Czech and world Mathematics is evident and incontestible to everybody who met him either as a mathematician or simply as a man. We hope that the Czech mathematical community will have an opportunity to profit from his knowledge and wisdom for many years to come. Let us extend to Jaroslav our wish of firm health, personal satisfaction and many new beautiful mathematical results.

## List of publications of Jaroslav Kurzweil

*A. Original papers* (Continued from Czechoslovak Mathematical Journal *36 (111)* (1986), p. 163)


[78] A New and More Powerful Concept of the PU-Integral. Czechoslovak Mathematical Journal *38* (1988), 8–48 (with J. Jarník).

[79] A non Absolutely Convergent Integral which Admits Transformation and can be used for Integration on Manifolds. Czechoslovak Mathematical Journal *35* (1985), 116–139 (with J. Jarník).

[80] Perron Integral, Perron Product Integral and Ordinary Linear Differential Equations. Equadiff 6., Lecture Notes in Mathematics 1192 (J. Vosmanský, M. Zlámal, eds.). Berlin, Springer, 1986, pp. 149–154 (with J. Jarník).

[81] On Some Extensions of the Perron Integral on one Dimensional Intervals. An Approach by Integral Sums Fulfilling a Symmetry Condition. Functiones et Approximatio Commentarii Mathematici *17* (1987), 49–55 (with J. Jarník).

[82] Limit Processes in Ordinary Differential Equations. Zeitschrift für Angewandte Mathematik und Physik *38* (1987), 241–256 (with J. Jarník).

[83] A General Form of the Product Integral and Linear Ordinary Differential Equations. Czechoslovak Mathematical Journal *37* (1987), 642–659 (with J. Jarník).

[84] Linear Differential Equations with Quasiperiodic Coefficients. Czechoslovak Mathematical Journal *37* (1987), 424–470 (with A. Vencovská).

[85] Iterated Lie Brackets in Limit Processes in Ordinary Differential Equations. Resultate der Mathematik *14* (1988), 125–137 (with J. Jarník).

[86] A Convergence Effect in Ordinary Differential Equations. Asymptotic Methods of Mathematical Physics (V. S. Korolyuk, ed.). Naukova Dumka, Kiev, 1988, pp. 134–144 (with J. Jarník).

[87] Structural Stability of Linear Discrete Systems via the Exponential Dichotomy. Czechoslovak Mathematical Journal *38* (1988), 280–284 (with G. Papaschinopoulos).

[88] The PU-Integral and its Properties Real Analysis Exchange *14* (1988–89), 34–43 (with J. Jarník).

[89] Ordinary Differential Equations the Solutions of which are $ACG_*$-functions. Archivum Mathematicum *26* (1990), 129–136 (with Š. Schwabik).

[90] On a Generalization of the Perron Integral on One-Dimensional Intervals. Annales Polonici Mathematici (Zdzislaw Opial in memoriam) *LI* (1990), 205–218 (with J. Jarník).

[91] The PU-integral: its Definitions and some Basic Properties. New Integrals, Proceedings of the Henstock Conference held in Coleraine, Northern Ireland, August 9–12, 1988 (P.S. Bullen, P.Y. Lee, J. Mawhin, P. Muldowney, W.F. Pfeffer, eds.). Lecture Notes im Math. 1419, Springer Verlag, Berlin, 1990, pp. 66–81 (with J. Jarník).

[92] Topological Equivalence and Structural Stability for Linear Difference Equations. Journal of Differential Equations *89* (1991), no. 1, 89–94 (with G. Papaschinopoulos).
[93] An Integral Defined by Approximating $BV$ Partitions of Unity. Czechoslovak Mathematical Journal *41* (1991), no. 4, 695–712 (with J. Mawhin and W. Pfeffer).
[94] Equiintegrability and Controlled Convergence of Perron-type Integrable Functions. Real Analysis Exchange *17* (1991–92), 110–139 (with J. Jarník).
[95] Differentiability and Integrability in $n$ Dimensions with Respect to $\alpha$-Regular Intervals. Results in Mathematics *21* (1992), no. 1, 138–151 (with J. Jarník).
[96] Equivalent Definitions of Regular Generalized Perron Integral. Czechoslovak Mathematical Journal *42* (1992), no. 2, 365–378 (with J. Jarník).
[97] Generalized Multidimensional Perron Integral Involving a New Regularity Condition. Results in Mathematics *23* (1993), no. 3, 363–373 (with J. Jarník).
[98] Pfeffer Integrability does not imply $M_1$-integrability. Czechoslovak Mathematical Journal *44* (1994), 47–56 (with J. Jarník).
[99] Perron-type Integration on n-dimensional Intervals and its Properties. Czechoslovak Mathematical Journal *45* (1995), 79–106 (with J. Jarník).
[100] Nonabsolutely Convergent Integrals via Integral Sums in $\mathbb{R}^n$. Tatra Mountains Math. Publications. To appear.
[101] Perron Type Integration on $n$-dimensional Intervals as an Extension of Integration of Stepfunctions by Strong Equiconvergence. Czechoslovak Mathematical Journal (with J. Jarník). To appear.
[102] Another Perron Type Integration in $n$-dimensions as an Extension of Integration of Stepfunctions. Czechoslovak Mathematical Journal (with J. Jarník). To appear.

*C. Popular and occasional articles* (Continued from Časopis pro pěstování matematiky *111* (1986), p. 110)

[17] O pojmu funkce ve školské matematice z pohledu matematika (On the concept of a function in highschool mathematics from the viewpoint of a mathematician). Terminologické otázky školské matematiky a fyziky. Praha, Jednota československých matematiků a fyziků, 1985, pp. 21–24. (In Czech.)
[18] Diferenciální rovnice v ČSR v letech 1945–1985 (Differential Equations in the ČSR in the years 1945–1985). Pokroky matematiky, fyziky a astronomie *32* (1987), 138–145. (In Czech.)
[19] Za profesorem Janem Maříkem (In memoriam professor Jan Mařík (1920–1994)). Mathematica Bohemica *119* (1994), no. 2, 213–215 (with J. Král, I. Netuka and Š. Schwabik). (In Czech.)
[20] In memoriam professor Jan Mařík (1920–1994). Czechoslovak Mathematical Journal *44* (1994), no. 1, 190–192 (with J. Král, I. Netuka and Š. Schwabik).