CYCLICALLY ORDERED SETS AS PARTIAL ALGEBRAS

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Abstract. A representation of cyclically ordered sets by means of partial semigroups with an additional unary operation is constructed.

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In [4], a representation of transitive ternary structures by partial semigroups with units is constructed. The construction proceeds as follows: If (G, t) is a transitive ternary structure, we put $c(t) = t \cup \{(x, y, x)$; there exists z such that $(x, y, z) \in t$ or $(z, y, x) \in t\}$ and define on $c(t)$ a partial binary operation $\cdot$ by

$$(x, y, z) \cdot (z, y, u) = (x, y, u).$$

Then $(c(t), \cdot)$ is a partial semigroup and (x, y, x) is a left unit of the element $(x, y, z) \in c(t)$, (z, y, z) is its right unit. This construction is not suitable for cyclically ordered sets, for if t is a cyclic order then $(x, y, x) \in t$ never holds. Here we present another construction which assigns to any cyclically ordered set a partial semigroup with an additional unary operation.

Another representation of transitive ternary structures is given in [5].

1. Partial semigroups

1.1. Let $S \neq \emptyset$ be a set, $\cdot$ a partial binary operation on S with the property: if one of the products $(x \cdot y) \cdot z$, $x \cdot (y \cdot z)$ is defined then the other is also defined and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. Then the structure $\mathbb{S} = (S, \cdot)$ is called a *partial semigroup*.

1.2. Homomorphisms and isomorphisms of partial semigroups are defined in a standard way. Thus, if $\mathbb{S} = (S, \cdot)$, $\mathbb{T} = (T, \cdot)$ are partial semigroups and $f\colon S \to T$, then f is a *homomorphism* of $\mathbb{S}$ into $\mathbb{T}$ if $x, y \in S$, $x \cdot y$ is defined in $\mathbb{S}$ then $f(x) \cdot f(y)$ is defined in $\mathbb{T}$ and $f(x \cdot y) = f(x) \cdot f(y)$. A bijective homomorphism of $\mathbb{S}$ onto $\mathbb{T}$ such that f^{-1} is a homomorphism of $\mathbb{T}$ onto $\mathbb{S}$ is an *isomorphism*; $\mathbb{S}$ and $\mathbb{T}$ are *isomorphic* if there exists an isomorphism of $\mathbb{S}$ onto $\mathbb{T}$.

1.3. Let $\mathbb{S} = (S, \cdot)$ be a partial semigroup, $e \in S$. The element e is a *unit* in $\mathbb{S}$ if $x \in S$, $e \cdot x$ is defined implies $e \cdot x = x$, and $y \in S$, $y \cdot e$ is defined implies $y \cdot e = y$. We denote by $E(\mathbb{S})$ the set of all units in $\mathbb{S}$.

The following four lemmas are trivial and known; we present them for the purpose of the subsequent text.

1.4. Lemma. *Let $\mathbb{S} = (S, \cdot)$ be a partial semigroup, $x \in S$, $e_1, e_2 \in E(\mathbb{S})$. If both $e_1 \cdot x$ and $e_2 \cdot x$ (both $x \cdot e_1$ and $x \cdot e_2$) are defined, then $e_1 = e_2$.*

Proof. Let both $e_1 \cdot x$ and $e_2 \cdot x$ be defined. As $e_2 \cdot x = x$, the product $e_1 \cdot (e_2 \cdot x)$ is defined. Hence $(e_1 \cdot e_2) \cdot x$, thus also $e_1 \cdot e_2$ is defined and then $e_1 \cdot e_2 = e_1 = e_2$. Dually in the case when $x \cdot e_1$, $x \cdot e_2$ are defined. $\square$

1.5. Let $\mathbb{S} = (S, \cdot)$ be a partial semigroup, $x \in S$. If there exists $e \in E(\mathbb{S})$ such that $e \cdot x$ is defined, we denote it $e = e_L(x)$; if there exists $e' \in E(\mathbb{S})$ such that $x \cdot e'$ is defined, we denote it $e' = e_R(x)$. $e_L(x)$ and $e_R(x)$ are called *left* and *right units* of x, respectively.

1.6. Lemma. *Let $\mathbb{S} = (S, \cdot)$ be a partial semigroup, $x, y \in S$ and let $x \cdot y$ be defined. Then $x \cdot y$ has the left (right) unit iff x (y) has the left (right) unit; in that case $e_L(x \cdot y) = e_L(x)$ ($e_R(x \cdot y) = e_R(y)$).*

Proof. If $e_L(x) = e$ exists then $e \cdot x = x$; thus $(e \cdot x) \cdot y$, hence $e \cdot (x \cdot y)$ is defined, i.e. $e = e_L(x \cdot y)$. Conversely, if $e_L(x \cdot y) = e$ exists then $e \cdot (x \cdot y)$, hence $(e \cdot x) \cdot y$ and $e \cdot x$ is defined and $e = e_L(x)$. Dually for the right units. $\square$

In the sequel we shall deal with partial semigroups $\mathbb{S} = (S, \cdot)$ with the property

$$x \in S \Rightarrow \text{ both } e_L(x) \text{ and } e_R(x) \text{ exist;}$$

partial semigroups with this property will be called *e-semigroups*.

1.7. Lemma. *Let $\mathbb{S}$ be an e-semigroup and $e \in E(\mathbb{S})$. Then $e_L(e) = e_R(e) = e$.*

Proof. By definition $e_L(e) \cdot e = e = e_L(e)$; similarly for $e_R(e)$. $\square$

1.8. Lemma. *Let $\mathbb{S} = (S, \cdot)$ be an e-semigroup, $x, y \in S$. If $x \cdot y$ is defined then $e_R(x) = e_L(y)$.*

Proof. As $x \cdot e_R(x) = x$, the product $(x \cdot e_R(x)) \cdot y$, thus $x \cdot (e_R(x) \cdot y)$ and $e_R(x) \cdot y$ are defined. Hence $e_R(x) = e_L(y)$. $\square$

2. Transitive ternary structures

2.1. Let $G \neq \emptyset$ be a set, T a ternary relation on G. The pair $\mathbb{G} = (G, T)$ is called a *ternary structure*. The ternary relation T (and the structure $\mathbb{G}$) is

reflexive if $(x, x, x) \in T$ for any $x \in G$

transitive if $(x, y, z) \in T$, $(z, y, u) \in T$ then $(x, y, u) \in T$.

2.2. Homomorphisms and isomorphisms of ternary structures are defined obviously. Thus, if $\mathbb{G} = (G, T)$, $\mathbb{H} = (H, S)$ are ternary structures and $f\colon G \to H$ then f is a *homomorphism* of $\mathbb{G}$ into $\mathbb{H}$ if

$$(x, y, z) \in T \Rightarrow (f(x), f(y), f(z)) \in S.$$

If f is a bijective homomorphism of $\mathbb{G}$ onto $\mathbb{H}$ such that f^{-1} is a homomorphism of $\mathbb{H}$ onto $\mathbb{G}$ then f is an *isomorphism* of $\mathbb{G}$ onto $\mathbb{H}$; $\mathbb{G}$ and $\mathbb{H}$ are *isomorphic* if there is an isomorphism of $\mathbb{G}$ onto $\mathbb{H}$.

2.3. Let $\mathbb{G} = (G, T)$ be a reflexive and transitive ternary structure. We define a partial binary operation $\cdot$ on the set T in the following way: if $m_1 = (x_1, y_1, z_1) \in T$, $m_2 = (x_2, y_2, z_2) \in T$ then $m_1 \cdot m_2$ is defined iff

$x_1 = y_1 = z_1 = x_2$ and then $m_1 \cdot m_2 = m_2$ or

$x_2 = y_2 = z_2 = z_1$ and then $m_1 \cdot m_2 = m_1$ or

$x_2 = z_1$, $y_2 = y_1$ and then $m_1 \cdot m_2 = (x_1, y_1, z_2)$.

In other words, we put

$$\begin{aligned}
(x, x, x) \cdot (x, y, z) &= (x, y, z) \\
(x, y, z) \cdot (z, z, z) &= (x, y, z) \\
(x, y, z) \cdot (z, y, u) &= (x, y, u).
\end{aligned}$$

2.4. Theorem. *Let $\mathbb{G} = (G, T)$ be a reflexive and transitive ternary structure, let $\cdot$ be a partial operation on T defined in 2.3. Then $\mathbb{T} = (T, \cdot)$ is an e-semigroup in which $E(\mathbb{T}) = \{(x, x, x); x \in G\}$ and $e_L(m) = (x, x, x)$, $e_R(m) = (z, z, z)$ for any $m = (x, y, z) \in T$.*

Proof. Let $m_1, m_2, m_3 \in T$ and let $(m_1 \cdot m_2) \cdot m_3$ be defined. If some of the elements m_1, m_2, m_3 has a form (x, x, x) then it is easy to show that $m_1 \cdot (m_2 \cdot m_3)$

is defined and $(m_1 \cdot m_2) \cdot m_3 = m_1 \cdot (m_2 \cdot m_3)$. If, for instance, $m_2 = (z,z,z)$ then necessarily $m_1 = (x,y,z)$ for some $x,y \in G$ and $m_1 \cdot m_2 = m_1$. Thus $m_1 \cdot m_3$ is defined so that $m_3 = (z,z,z)$ or $m_3 = (z,y,u)$ for some $u \in G$. In both cases $m_2 \cdot m_3$ is defined and $m_2 \cdot m_3 = m_3$ so that $m_1 \cdot (m_2 \cdot m_3)$ is defined and $m_1 \cdot (m_2 \cdot m_3) = m_1 \cdot m_3 = (m_1 \cdot m_2) \cdot m_3$. In the other cases $m_1 = (x,y,z)$, $m_2 = (z,y,u)$ so that $m_1 \cdot m_2 = (x,y,u)$; then necessarily $m_3 = (u,y,v)$ so that $(m_1 \cdot m_2) \cdot m_3 = (x,y,v)$. We see that $m_2 \cdot m_3$ is defined and $m_2 \cdot m_3 = (z,y,v)$ so that $m_1 \cdot (m_2 \cdot m_3)$ is defined and $m_1 \cdot (m_2 \cdot m_3) = (x,y,v) = (m_1 \cdot m_2) \cdot m_3$. Similarly in the case when $m_1 \cdot (m_2 \cdot m_3)$ is defined; thus $(T, \cdot)$ is a partial semigroup.

As T is reflexive, $(x,x,x) \in T$ for any $x \in G$. From the definition of the operation $\cdot$ it follows that $(x,x,x) \cdot m = m$ whenever $(x,x,x) \cdot m$ is defined for some $m \in T$; similarly $m \cdot (x,x,x) = m$ if $m \cdot (x,x,x)$ is defined. Hence $\{(x,x,x); x \in G\} \subseteq E(\mathbb{T})$. Let $m = (x,y,z) \in T$ and assume that $x = y = z$ does not hold. Then $(z,z,z) \in T$ and $m \cdot (z,z,z) = m \neq (z,z,z)$ so that $m \notin E(\mathbb{T})$. Thus $E(\mathbb{T}) = \{(x,x,x); x \in G\}$.

If $m = (x,y,z) \in T$ then $(x,x,x) = e_L(m)$, $(z,z,z) = e_R(m)$ by the definition of the operation $\cdot$. Thus $\mathbb{T}$ is an e-semigroup. □

3. Cyclically ordered sets

3.1. Let $\mathbb{G} = (G,T)$ be a ternary structure. The relation T (and the structure $\mathbb{G}$) is called

asymmetric if $(x,y,z) \in T \Rightarrow (z,y,x) \notin T$

antisymmetric if $(x,y,z) \in T$, $(z,y,x) \in T \Rightarrow x = y = z$

cyclic if $(x,y,z) \in T \Rightarrow (y,z,x) \in T$.

A ternary structure $\mathbb{G}$ is called a *cyclically ordered set* ([1], [3], [6]) if it is asymmetric, cyclic and transitive.

3.2. Remark. *Let $\mathbb{G} = (G,T)$ be a cyclic ternary structure. Then:*
(1) If T is asymmetric and $(x,y,z) \in T$, then $x \neq y \neq z \neq x$.
(2) If T is antisymmetric and $(x,y,z) \in T$, then either $x = y = z$ or $x \neq y \neq z \neq x$.

Proof. (1) If T is asymmetric and $(x,y,z) \in T$, then $x = z$ is impossible since $(x,y,x) \in T$ contradicts the asymmetry of T. If $x = y$, i.e. $(x,x,z) \in T$ then the cyclicity of T implies $(x,z,x) \in T$, a contradiction. Similarly for $y = z$.

(2) Let T be antisymmetric and $(x,y,z) \in T$. If $x = z$ then $(x,y,x) \in T$ and the antisymmetry implies $x = y = z$. If $x = y$, i.e. $(x,x,z) \in T$ then $(x,z,x) \in T$ and we have also $x = y = z$. Similarly for $y = z$. Thus either $x = y = z$ or $x \neq y \neq z \neq x$.

□

The following two lemmas are trivial; their proofs are therefore omitted.

3.3. Lemma. *Let* $\mathbb{G} = (G, T)$ *be a cyclically ordered set. Put* $C = T \cup \{(x, x, x); x \in G\}$. *Then the ternary relation* C *is reflexive, antisymmetric, cyclic and transitive.*

3.4. Lemma. *Let* $\mathbb{G} = (G, C)$ *be a ternary structure which is reflexive, antisymmetric, cyclic and transitive. Put* $T = C - \{(x, x, x); x \in G\}$. *Then* (G, T) *is a cyclically ordered set.*

By 3.3 and 3.4, cyclically ordered sets can be defined as ternary structures which are reflexive, antisymmetic, cyclic and transitive. In the sequel, this notion will be used just in this sense (see e.g. [2], where such a relation is called "extended cyclic order").

3.5. Let $\mathbb{G} = (G, C)$ be a cyclically ordered set, let $\cdot$ be a partial binary operation on C defined in 2.3. By 2.4, $\mathbb{C} = (C, \cdot)$ is an e-semigroup. Further, put $\varphi(m) = (y, z, x)$ for any $m = (x, y, z) \in C$; thus φ is a unary operation on the set C.

3.6. Lemma. *Let* $\mathbb{G} = (G, C)$ *be a cyclically ordered set,* $\mathbb{C} = (C, \cdot)$ *the e-semigroup from* 2.3 *and* $\varphi\colon C \to C$ *the unary operation on* C *defined in* 3.5. *Then*

(1) $\varphi^3 = id_C$ *and* $\varphi|_{E(\mathbb{C})} = id_{E(\mathbb{C})}$.

(2) $e_R(m) = e_L(\varphi^2(m))$ *for any* $m \in C$.

(3) *If* $e_R(m_1) = e_L(m_2)$, $e_L(m_1) = e_R(m_2)$, $e_L(\varphi(m_1)) = e_L(\varphi(m_2))$ *for some* $m_1, m_2 \in C$ *then* $m_1 = m_2 \in E(\mathbb{C})$.

(4) *For* $m_1, m_2 \in C - E(\mathbb{C})$ *the product* $m_1 \cdot m_2$ *is defined iff* $e_R(m_1) = e_L(m_2)$, $e_L(\varphi(m_1)) = e_L(\varphi(m_2))$.

(5) *If* $m_1, m_2 \in C - E(\mathbb{C})$ *and* $m_1 \cdot m_2$ *is defined then* $e_L(\varphi(m_1 \cdot m_2)) = e_L(\varphi(m_1))$.

Proof. (1) and (2) follow directly from the definiton of the operation φ.

(3): If $m_1 = (x, y, z)$, $m_2 = (u, v, w)$ then by 2.4 $e_L(m_1) = (x, x, x)$, $e_R(m_1) = (z, z, z)$, $e_L(\varphi(m_1)) = (y, y, y)$, and $e_L(m_2) = (u, u, u)$, $e_R(m_2) = (w, w, w)$, $e_L(\varphi(m_2)) = (v, v, v)$. Thus $(z, z, z) = (u, u, u)$, $(x, x, x) = (w, w, w)$, $(y, y, y) = (v, v, v)$, i.e. $u = z$, $v = y$, $w = x$ so that $m_2 = (z, y, x)$ and antisymmetry of C implies $x = y = z$. Thus $m_1 = m_2 = (x, x, x) \in E(\mathbb{C})$.

(4): For $m_1, m_2 \in C - E(\mathbb{C})$ the product $m_1 \cdot m_2$ is defined iff $m_1 = (x, y, z)$, $m_2 = (z, y, u)$ for suitable $x, y, z, u \in G$, i.e. iff $e_R(m_1) = e_L(m_2)$, $e_L(\varphi(m_1)) = e_L(\varphi(m_2))$.

(5): If $m_1, m_2 \in C - E(\mathbb{C})$ and $m_1 \cdot m_2$ is defined then $m_1 = (x, y, z)$, $m_2 = (z, y, u)$, and $m_1 \cdot m_2 = (x, y, u)$. Thus $e_L(\varphi(m_1 \cdot m_2)) = e_L(\varphi(m_1))$. □

3.7. Let $\mathbb{S} = (S, \cdot, \varphi)$ be a partial algebra such that $(S, \cdot)$ is an e-semigroup and φ is a unary operation on S having properties (1)–(5) from 3.6. Then $\mathbb{S}$ will be called a *c-algebra*. If $\mathbb{S} = (S, \cdot, \varphi)$, $\mathbb{T} = (T, \cdot, \psi)$ are c-algebras and $f\colon S \to T$, then f is a *homomorphism* of $\mathbb{S}$ into $\mathbb{T}$ iff it is a homomorphism of the e-semigroup $(S,\cdot)$ into the e-semigroup $(T,\cdot)$ and simultaneously a homomorphism of the unar (S,φ) into the unar (T,ψ), i.e. iff

$x,y \in S$, $x\cdot y$ is defined in $\mathbb{S} \Rightarrow f(x)\cdot f(y)$ is defined in $\mathbb{T}$ and $f(x\cdot y) = f(x)\cdot f(y)$

$x \in S \Rightarrow (f\circ\varphi)(x) = (\psi\circ f)(x)$.

Isomorphisms of c-algebras are defined in the obvious way.

3.8. Theorem. *Let $\mathbb{S} = (S,\cdot,\varphi)$ be a c-algebra. Put $G = E(\mathbb{S})$ and define a ternary relation C on G by*

$(x,y,z) \in C \Leftrightarrow$ there is $m \in S$ with $x = e_L(m)$, $y = e_L(\varphi(m))$, $z = e_R(m)$.

Then (G,C) is a cyclically ordered set.

Proof. If $x \in G$ then $x \in E(\mathbb{S})$. By 1.7, $e_L(x) = x$, $e_R(x) = x$. As $\varphi|_{E(\mathbb{S})} = \mathrm{id}_{E(\mathbb{S})}$, we have $\varphi(x) = x$ and $e_L(\varphi(x)) = x$. Thus $(x,x,x) \in C$ and C is reflexive.

Let $(x,y,z) \in C$, $(z,y,x) \in C$. Then there are $m_1, m_2 \in S$ such that $x = e_L(m_1)$, $y = e_L(\varphi(m_1))$, $z = e_R(m_1)$, $z = e_L(m_2)$, $y = e_L(\varphi(m_2))$, $x = e_R(m_2)$. By (3) we obtain $m_1 = m_2 \in E(\mathbb{S})$ and by 1.7, $e_L(m_1) = e_R(m_1) = m_1$. Further, $\varphi(m_1) = m_1$ and $e_L(\varphi(m_1)) = m_1$. Thus $x = y = z = m_1$ and C is antisymmetric.

Let $(x,y,z) \in C$ so that $x = e_L(m)$, $y = e_L(\varphi(m))$, $z = e_R(m)$ for some $m \in S$. By (2), $e_R(m) = e_L(\varphi^2(m))$. Further, from $\varphi^3 = \mathrm{id}_S$ and from (2) we conclude $e_R(\varphi(m)) = e_L(\varphi^3(m)) = e_L(m)$. Thus we have $y = e_L(\varphi(m))$, $z = e_L(\varphi(\varphi(m)))$, $x = e_R(\varphi(m))$ so that $(y,z,x) \in C$ and C is cyclic.

Let $(x,y,z) \in C$, $(z,y,u) \in C$. By 3.2 either $x = y = z$ or $x \neq y \neq z \neq x$ and either $z = y = u$ or $z \neq y \neq u \neq z$. If $x = y = z$ or $z = y = u$ then $x = y = z = u$ and $(x,y,u) \in C$. Thus assume $x \neq y \neq z \neq x$, $z \neq y \neq u \neq z$. There are $m_1, m_2 \in S$ such that $x = e_L(m_1)$, $y = e_L(\varphi(m_1))$, $z = e_R(m_1)$, $z = e_L(m_2)$, $y = e_L(\varphi(m_2))$, $u = e_R(m_2)$. Among other this implies $m_1, m_2 \in S - E(\mathbb{S})$ for if, e.g., $m_1 \in E(\mathbb{S})$ then $e_L(m_1) = e_R(m_1) = m_1$ by 1.7, i.e. $x = z$, a contradiction. By (4) the product $m_1\cdot m_2$ is defined. Further, by 1.6, $e_L(m_1\cdot m_2) = e_L(m_1) = x$, $e_R(m_1\cdot m_2) = e_R(m_2) = u$ and by (5), $e_L(\varphi(m_1\cdot m_2)) = e_L(\varphi(m_1)) = y$. Thus we have $x = e_L(m_1\cdot m_2)$, $y = e_L(\varphi(m_1\cdot m_2))$, $u = e_R(m_1\cdot m_2)$ so that $(x,y,u) \in C$ and C is transitive. □

4. Mappings $\mathcal{S}$ and $\mathcal{C}$

4.1. Let $\mathbb{G} = (G,C)$ be a cyclically ordered set, let $\cdot$ be a partial binary operation on C defined in 2.3 and let φ be a unary operation on C defined in 3.5. By 3.6, $\mathbb{C} = (C,\cdot,\varphi)$ is a c-algebra; we denote it by $\mathcal{S}(\mathbb{G})$. Thus, if $\mathfrak{C}$ is the class of cyclically ordered sets and $\mathfrak{A}$ is the class of c-algebras, $\mathcal{S}$ is a mapping of $\mathfrak{C}$ into $\mathfrak{A}$:

$$\mathcal{S}\colon \mathfrak{C} \to \mathfrak{A}.$$

4.2. Let $\mathbb{S} = (S, \cdot, \varphi)$ be a c-algebra. Let $G = E(\mathbb{S})$ and let C be a ternary relation on G defined in 3.8. By 3.8, $\mathbb{G} = (G, C)$ is a cyclically ordered set; we denote it by $\mathcal{C}(\mathbb{S})$. Thus $\mathcal{C}$ is a mapping of $\mathfrak{A}$ into $\mathfrak{C}$:

$$\mathcal{C} \colon \mathfrak{A} \to \mathfrak{C}.$$

4.3. Theorem. *Let $\mathbb{G} = (G, C)$ be a cyclically ordered set. Then $\mathbb{G}$ is isomorphic with $(\mathcal{C} \quad)(\mathbb{G})$.*

P r o o f. By 2.4, $E(\mathcal{S}(\mathbb{G})) = \{(x, x, x); x \in G\}$. Define a mapping $f \colon G \to E(\mathcal{S}(\mathbb{G}))$ by $f(x) = (x, x, x)$. Trivially, f is a bijection of G onto $E(\mathcal{S}(\mathbb{G}))$. By definition, $\mathcal{C}(\mathcal{S}(\mathbb{G})) = (E(\mathcal{S}(\mathbb{G})), C')$ where $(m_1, m_2, m_3) \in C' \Leftrightarrow$ there is $m \in C$ with $m_1 = e_L(m)$, $m_2 = e_L(\varphi(m))$, $m_3 = e_R(m)$. Let $x, y, z \in G$, $(x, y, z) \in C$. If we denote $(x, y, z) = m$ then $(x, x, x) = e_L(m)$, $(y, y, y) = e_L(\varphi(m))$, $(z, z, z) = e_R(m)$ in the c-algebra $\mathcal{S}(\mathbb{G})$. Thus $((x, x, x), (y, y, y), (z, z, z)) \in C'$, i.e. $(f(x), f(y), f(z)) \in C'$ and f is a homomorphism of $\mathbb{G}$ onto $(\mathcal{C} \quad)(\mathbb{G})$.

Let $x, y, z \in G$, $(f(x), f(y), f(z)) \in C'$, i.e. $((x, x, x), (y, y, y), (z, z, z)) \in C'$. By definition there is $m \in C$ with $(x, x, x) = e_L(m)$, $(y, y, y) = e_L(\varphi(m))$, $(z, z, z) = e_R(m)$. Then $m = (x, y, z)$ and $(x, y, z) \in C$. Hence f is an isomorphism of $\mathbb{G}$ onto $(\mathcal{C} \quad)(\mathbb{G})$. $\square$

4.4. Theorem. *Let $\mathbb{S} = (S, \cdot, \varphi)$ be a c-algebra. Then there exists a surjective homomorphism of $\mathbb{S}$ onto $(\mathcal{S} \circ \mathcal{C})(\mathbb{S})$.*

P r o o f. By 3.8, $\mathcal{C}(\mathbb{S}) = (E(\mathbb{S}), C)$ where $(e_1, e_2, e_3) \in C \Leftrightarrow$ there is $m \in S$ with $e_1 = e_L(m)$, $e_2 = e_L(\varphi(m))$, $e_3 = e_R(m)$. Further, by 4.1, $\mathcal{S}(\mathcal{C}(\mathbb{S})) = (C, \cdot, \psi)$ where $\cdot$ is the partial binary operation on C defined in 2.3 and $\psi \colon C \to C$ is defined by $\psi(e_1, e_2, e_3) = (e_2, e_3, e_1)$. Define a mapping $f \colon S \to C$ by

$$f(m) = (e_L(m), e_L(\varphi(m)), e_R(m)).$$

Clearly, $f \colon S \to C$ is surjective. Let $m_1, m_2 \in S$ and $m_1 \cdot m_2$ be defined. If $m_1 \in E(\mathbb{S})$ then $m_1 = e_L(m_2)$ so that $m_1 \cdot m_2 = m_2$ and $\varphi(m_1) = m_1$ by the property (1). Further, by 1.7, $e_L(m_1) = e_R(m_1) = m_1 = e_L(m_2)$. Thus $f(m_1) = \big(e_L(m_2), e_L(m_2), e_L(m_2)\big)$, $f(m_2) = \big(e_L(m_2), e_L(\varphi(m_2)), e_R(m_2)\big)$ so that $f(m_1) \cdot f(m_2)$ is defined in $(C, \cdot)$ and $f(m_1) \cdot f(m_2) = f(m_2) = f(m_1 \cdot m_2)$. Analogously in the case $m_2 \in E(\mathbb{S})$. Thus suppose $m_1, m_2 \in S - E(\mathbb{S})$. By 1.8, $e_R(m_1) = e_L(m_2)$, by the property (4), $e_L(\varphi(m_1)) = e_L(\varphi(m_2))$. Thus $f(m_1) = \big(e_L(m_1), e_L(\varphi(m_1)), e_R(m_1)\big)$, $f(m_2) = \big(e_R(m_1), e_L(\varphi(m_1)), e_R(m_2)\big)$. Hence $f(m_1) \cdot f(m_2)$ is defined in $(C, \cdot)$ and $f(m_1) \cdot f(m_2) = \big(e_L(m_1), e_L(\varphi(m_1)), e_R(m_2)\big)$. By 1.6, $e_L(m_1) = e_L(m_1 \cdot m_2)$, $e_R(m_2) = e_R(m_1 \cdot m_2)$ and by the property (5), $e_L(\varphi(m_1)) = e_L(\varphi(m_1 \cdot m_2))$. Thus

$f(m_1)\cdot f(m_2) = \big(e_L(m_1\cdot m_2), e_L(\varphi(m_1\cdot m_2)\big), e_R(m_1\cdot m_2)) = f(m_1\cdot m_2)$ and f is a homomorphism of $(S,\cdot)$ onto $(C,\cdot)$.

Let $m\in S$ be any element. Then $f(m) = \big(e_L(m), e_L(\varphi(m)), e_R(m)\big)$ and $f(\varphi(m)) = \big(e_L(\varphi(m)), e_L(\varphi^2(m)), e_R(\varphi(m))\big)$. Further, $\psi(f(m)) = \big(e_L(\varphi(m)), e_R(m), e_L(m)\big)$. By the property (2) and (1), $e_R(m) = e_L(\varphi^2(m))$ and $e_R(\varphi(m)) = e_L(\varphi^3(m)) = e_L(m)$. Thus $\psi(f(m)) = \big(e_L(\varphi(m)), e_L(\varphi^2(m)), e_R(\varphi(m))\big)$ so that $(\psi\circ f)(m) = (f\circ\varphi)(m)$ and f is a homomorphism of (S,φ) onto (C,ψ). Hence f is a homomorphism of $\mathbb{S}$ onto $(\mathcal{S}\circ\mathcal{C})(\mathbb{S})$. $\square$

5. Examples

5.1. Let $S = \{e_1, e_2, e_3, m_1, m_2, m_3\}$ where $e_1\cdot e_1 = e_1, e_2\cdot e_2 = e_2, e_3\cdot e_3 = e_3$ and

$$\begin{aligned} &e_1\cdot m_1 = m_1 \quad && e_2\cdot m_2 = m_2 \quad && e_3\cdot m_3 = m_3\\ &m_2\cdot e_1 = m_2 && m_3\cdot e_2 = m_3 && m_1\cdot e_3 = m_1. \end{aligned}$$

Trivially, $(S,\cdot)$ is an e-semigroup in which $E(S,\cdot) = \{e_1, e_2, e_3\}$ and $e_1 = e_L(m_1) = e_R(m_2)$, $e_2 = e_L(m_2) = e_R(m_3)$, $e_3 = e_L(m_3) = e_R(m_1)$. Define further $\varphi\colon S\to S$ by $\varphi(e_i) = e_i$ for $i = 1,2,3$ and $\varphi(m_1) = m_2$, $\varphi(m_2) = m_3$, $\varphi(m_3) = m_1$.

We show that $\mathbb{S} = (S,\cdot,\varphi)$ is a c-semigroup.

(1) $\varphi^3 = \mathrm{id}_S$ and $\varphi|_{E(\mathbb{S})} = \mathrm{id}_{E(\mathbb{S})}$ is trivial.

(2) Clearly $e_R(e_i) = e_L(\varphi^2(e_i))$ for $i = 1,2,3$ and further

$$\begin{aligned} e_R(m_1) &= e_3 = e_L(m_3) = e_L(\varphi^2(m_1))\\ e_R(m_2) &= e_1 = e_L(m_1) = e_L(\varphi^2(m_2))\\ e_R(m_3) &= e_2 = e_L(m_2) = e_L(\varphi^2(m_3)). \end{aligned}$$

(3) For $i\neq j$ the relation $e_R(e_i) = e_L(e_j)$ does not hold. Further

$$\begin{aligned} &e_L(m_1) = e_R(e_1) \;\text{ but }\; && e_R(m_1)\neq e_L(e_1)\\ &e_L(m_2) = e_R(e_2), && e_R(m_2)\neq e_L(e_2)\\ &e_L(m_3) = e_R(e_3), && e_R(m_3)\neq e_L(e_3) \end{aligned}$$

and analogously

$$\begin{aligned} &e_L(m_1) = e_R(m_2), \quad && e_R(m_1)\neq e_L(m_2)\\ &e_L(m_2) = e_R(m_3), && e_R(m_2)\neq e_L(m_3)\\ &e_L(m_3) = e_R(m_1), && e_R(m_3)\neq e_L(m_1) \end{aligned}$$

so that if $m,n\in S$, $e_R(m) = e_L(n)$, $e_L(m) = e_R(n)$, $e_L(\varphi(m)) = e_L(\varphi(n))$ then $m = n\in E(\mathbb{S})$.

(4), (5) is useless, as the product in $\mathbb{S}$ is defined only with units.

We shall construct $(\mathcal{S}\circ\mathcal{C})(\mathbb{S})$. Put $G = E(\mathbb{S}) = \{e_1, e_2, e_3\}$. By 3.8, $\mathcal{C}(\mathbb{S}) = (G, C)$ where $C = \{(e_1, e_1, e_1), (e_2, e_2, e_2), (e_3, e_3, e_3), (e_1, e_2, e_3), (e_2, e_3, e_1), (e_3, e_1, e_2)\}$. Further, $(\mathcal{S}\circ\mathcal{C})(\mathbb{S}) = (C, \cdot, \psi)$ where by 2.3, $(e_i, e_i, e_i)\cdot(e_i, e_i, e_i) = (e_i, e_i, e_i)$ for $i = 1, 2, 3$, and

$$\begin{aligned}
(e_1, e_1, e_1)\cdot(e_1, e_2, e_3) &= (e_1, e_2, e_3)\\
(e_1, e_2, e_3)\cdot(e_3, e_3, e_3) &= (e_1, e_2, e_3)\\
(e_2, e_2, e_2)\cdot(e_2, e_3, e_1) &= (e_2, e_3, e_1)\\
(e_2, e_3, e_1)\cdot(e_1, e_1, e_1) &= (e_2, e_3, e_1)\\
(e_3, e_3, e_3)\cdot(e_3, e_1, e_2) &= (e_3, e_1, e_2)\\
(e_3, e_1.e_2)\cdot(e_2, e_2, e_2) &= (e_3, e_1, e_2)
\end{aligned}$$

and $\psi(e_i, e_i, e_i) = (e_i, e_i, e_i)$ for $i = 1, 2, 3$, $\psi(e_1, e_2, e_3) = (e_2, e_3, e_1)$, $\psi(e_2, e_3, e_1) = (e_3, e_1, e_2)$, $\psi(e_3, e_1, e_2) = (e_1, e_2, e_3)$.

The mapping $f\colon S \to C$ constructed in the proof of Theorem 4.4 is

$$\begin{gathered}
f(e_i) = (e_i, e_i, e_i) \quad \text{for } i = 1, 2, 3,\\
f(m_1) = (e_1, e_2, e_3),\ f(m_2) = (e_2, e_3, e_1),\ f(m_3) = (e_3, e_1, e_2),
\end{gathered}$$

and it is an isomorphism of $\mathbb{S}$ onto $(\mathcal{S}\circ\mathcal{C})(\mathbb{S})$.

5.2. Let $S = \{e_1, e_2, e_3, e_4, e_5, m_1, m_2, \ldots, m_{30}\}$ where

$$\begin{array}{llll}
e_1\cdot m_1 = m_1 & m_{11}\cdot e_1 = m_{11} & e_2\cdot m_{11} = m_{11} & m_{21}\cdot e_2 = m_{21}\\
e_1\cdot m_4 = m_4 & m_{14}\cdot e_1 = m_{14} & e_2\cdot m_{12} = m_{12} & m_{22}\cdot e_2 = m_{22}\\
e_1\cdot m_6 = m_6 & m_{16}\cdot e_1 = m_{16} & e_2\cdot m_{13} = m_{13} & m_{23}\cdot e_2 = m_{23}\\
e_1\cdot m_7 = m_7 & m_{17}\cdot e_1 = m_{17} & e_2\cdot m_{14} = m_{14} & m_{24}\cdot e_2 = m_{24}\\
e_1\cdot m_9 = m_9 & m_{19}\cdot e_1 = m_{19} & e_2\cdot m_{15} = m_{15} & m_{25}\cdot e_2 = m_{25}\\
e_1\cdot m_{10} = m_{10} & m_{20}\cdot e_1 = m_{20} & e_2\cdot m_{16} = m_{16} & m_{26}\cdot e_2 = m_{26}\\
e_3\cdot m_2 = m_2 & m_1\cdot e_3 = m_1 & e_4\cdot m_3 = m_3 & m_2\cdot e_4 = m_2\\
e_3\cdot m_5 = m_5 & m_7\cdot e_3 = m_7 & e_4\cdot m_{20} = m_{20} & m_4\cdot e_4 = m_4\\
e_3\cdot m_8 = m_8 & m_{10}\cdot e_3 = m_{10} & e_4\cdot m_{22} = m_{22} & m_8\cdot e_4 = m_8\\
e_3\cdot m_{21} = m_{21} & m_{12}\cdot e_3 = m_{12} & e_4\cdot m_{24} = m_{24} & m_9\cdot e_4 = m_9\\
e_3\cdot m_{27} = m_{27} & m_{15}\cdot e_3 = m_{15} & e_4\cdot m_{28} = m_{28} & m_{13}\cdot e_4 = m_{13}\\
e_3\cdot m_{30} = m_{30} & m_{18}\cdot e_3 = m_{18} & e_4\cdot m_{29} = m_{29} & m_{30}\cdot e_4 = m_{30}
\end{array}$$

$$\begin{array}{ll}
e_5\cdot m_{17} = m_{17} & m_3\cdot e_5 = m_3\\
e_5\cdot m_{18} = m_{18} & m_5\cdot e_5 = m_5\\
e_5\cdot m_{19} = m_{19} & m_6\cdot e_5 = m_6\\
e_5\cdot m_{23} = m_{23} & m_{27}\cdot e_5 = m_{27}\\
e_5\cdot m_{25} = m_{25} & m_{28}\cdot e_5 = m_{28}\\
e_5\cdot m_{26} = m_{26} & m_{29}\cdot e_5 = m_{29},
\end{array}$$

further $e_i \cdot e_i = e_i$ for $i = 1, 2, 3, 4, 5$, and
$m_1 \cdot m_2 = m_4$, $m_2 \cdot m_3 = m_5$, $m_4 \cdot m_3 = m_6$, $m_7 \cdot m_8 = m_9$, $m_{14} \cdot m_{10} = m_{12}$, $m_1 \cdot m_5 = m_6$, $m_{15} \cdot m_8 = m_{13}$, $m_{16} \cdot m_7 = m_{15}$, $m_{19} \cdot m_{10} = m_{18}$, $m_{27} \cdot m_{26} = m_{21}$, $m_{16} \cdot m_9 = m_{13}$, $m_{29} \cdot m_{26} = m_{24}$, $m_{30} \cdot m_{24} = m_{21}$, $m_{30} \cdot m_{29} = m_{27}$.
As

$$(m_1 \cdot m_2) \cdot m_3 = m_4 \cdot m_3 = m_6 = m_1 \cdot m_5 = m_1 \cdot (m_2 \cdot m_3)$$
$$(m_{16} \cdot m_7) \cdot m_8 = m_{15} \cdot m_8 = m_{13} = m_{16} \cdot m_9 = m_{16} \cdot (m_7 \cdot m_8)$$
$$(m_{30} \cdot m_{29}) \cdot m_{26} = m_{27} \cdot m_{26} = m_{21} = m_{30} \cdot m_{24} = m_{30} \cdot (m_{29} \cdot m_{26})$$

and the other products $(m_i \cdot m_j) \cdot m_k$, $m_i \cdot (m_j \cdot m_k)$ are not defined, $(S, \cdot)$ is an e-semigroup in which $E(S, \cdot) = \{e_1, e_2, e_3, e_4, e_5\}$ and

$e_L(m_1) = e_1, e_R(m_1) = e_3, e_L(m_2) = e_3, e_R(m_2) = e_4, e_L(m_3) = e_4, e_R(m_3) = e_5,$
$e_L(m_4) = e_1, e_R(m_4) = e_4, e_L(m_5) = e_3, e_R(m_5) = e_5, e_L(m_6) = e_1, e_R(m_6) = e_5,$
$e_L(m_7) = e_1, e_R(m_7) = e_3, e_L(m_8) = e_3, e_R(m_8) = e_4, e_L(m_9) = e_1, e_R(m_9) = e_4,$
$e_L(m_{10}) = e_1, e_R(m_{10}) = e_3, e_L(m_{11}) = e_2, e_R(m_{11}) = e_1, e_L(m_{12}) = e_2, e_R(m_{12}) = e_3,$
$e_L(m_{13}) = e_2, e_R(m_{13}) = e_4, e_L(m_{14}) = e_2, e_R(m_{14}) = e_1, e_L(m_{15}) = e_2, e_R(m_{15}) = e_3,$
$e_L(m_{16}) = e_2, e_R(m_{16}) = e_1, e_L(m_{17}) = e_5, e_R(m_{17}) = e_1, e_L(m_{18}) = e_5, e_R(m_{18}) = e_3,$
$e_L(m_{19}) = e_5, e_R(m_{19}) = e_1, e_L(m_{20}) = e_4, e_R(m_{20}) = e_1, e_L(m_{21}) = e_3, e_R(m_{21}) = e_2,$
$e_L(m_{22}) = e_4, e_R(m_{22}) = e_2, e_L(m_{23}) = e_5, e_R(m_{23}) = e_2, e_L(m_{24}) = e_4, e_R(m_{24}) = e_2,$
$e_L(m_{25}) = e_5, e_R(m_{25}) = e_2, e_L(m_{26}) = e_5, e_R(m_{26}) = e_2, e_L(m_{27}) = e_3, e_R(m_{27}) = e_5,$
$e_L(m_{28}) = e_4, e_R(m_{28}) = e_5, e_L(m_{29}) = e_4, e_R(m_{29}) = e_5, e_L(m_{30}) = e_3, e_R(m_{30}) = e_4.$

Put further $\varphi(e_i) = e_i$ for $i = 1, 2, 3, 4, 5$ and $\varphi(m_i) = m_{i+10}$ where summation is mod 30. We show that $\mathbb{S} = (S, \cdot, \varphi)$ is a c-semigroup.

(1) is trivial and follows directly from the definition of φ.

(2): its verification is a routine; for instance, $e_R(m_1) = e_3 = e_L(m_{21}) = e_L(\varphi^2(m_1))$, $e_R(m_{14}) = e_1 = e_L(m_4) = e_L(\varphi^2(m_{14}))$, $e_R(m_{29}) = e_5 = e_L(m_{19}) = e_L(\varphi^2(m_{29}))$.

(3): As $e_L(m_i) \neq e_R(m_i)$ for any $i \leqslant 30$, the relation $e_L(m_i) = e_R(e_j)$, $e_R(m_i) = e_L(e_j)$ holds for no $i \leqslant 30$, $j \leqslant 5$. Further, by simple computation we see that $e_L(m_i) = e_R(m_j)$, $e_R(m_i) = e_L(m_j)$ hold only for the following pairs (m_i, m_j): (m_4, m_{20}), (m_5, m_{18}), (m_6, m_{17}), (m_6, m_{19}), (m_9, m_{20}), (m_{12}, m_{21}), (m_{13}, m_{22}), (m_{13}, m_{24}), (m_{15}, m_{21}), (m_{18}, m_{27}). In all these cases, however, $e_L(\varphi(m_i)) \neq e_L(\varphi(m_j))$. Thus really $m, n \in S$, $e_L(m) = e_R(n)$, $e_R(m) = e_L(n)$, $e_L(\varphi(m)) = e_L(\varphi(n)) \Rightarrow m = n \in E(\mathbb{S})$ and (3) holds.

(4): If $m, n \in S - E(\mathbb{S})$ and $m \cdot n$ is defined then it is easy to verify $e_R(m) = e_L(n)$, $e_L(\varphi(m)) = e_L(\varphi(n))$. E.g. for m_1, m_5: $e_R(m_1) = e_3 = e_L(m_5)$, $e_L(\varphi(m_1)) = e_L(m_{11}) = e_2 = e_L(m_{15}) = e_L(\varphi(m_5))$; for m_{30}, m_{24}: $e_R(m_{30}) = e_4 = e_L(m_{24})$, $e_L(\varphi(m_{30})) = e_L(m_{10}) = e_1 = e_L(m_4) = e_L(\varphi(m_{24}))$. On the other hand, by a simple computation we find that $e_R(m) = e_L(n)$, $e_L(\varphi(m)) = e_L(\varphi(n))$ hold for no other pairs $m, n \in S - E(\mathbb{S})$.

(5) can be verified also by a simple computation. E.g. $e_L(\varphi(m_1 \cdot m_2)) = e_L(\varphi(m_4)) = e_L(m_{14}) = e_2 = e_L(m_{11}) = e_L(\varphi(m_1))$, $e_L(\varphi(m_{19} \cdot m_{10})) = e_L(\varphi(m_{18})) = e_L(m_{28}) = e_4 = e_L(m_{29}) = e_L(\varphi(m_{19}))$. Thus $\mathbb{S} = (S, \cdot, \varphi)$ is a c-semigroup. We construct $(\mathcal{S} \circ \mathcal{C})(\mathbb{S})$. We have $G = E(\mathbb{S}) = \{e_1, e_2, e_3, e_4, e_5\}$ and $\mathcal{C}(\mathbb{S}) = (G, C)$ where $C = \{(e_1, e_1, e_1), (e_2, e_2, e_2), (e_3, e_3, e_3), (e_4, e_4, e_4), (e_5, e_5, e_5),$ $(e_1, e_2, e_3), (e_3, e_2, e_4), (e_4, e_2, e_5), (e_1, e_2, e_4), (e_3, e_2, e_5), (e_1, e_2, e_5), (e_1, e_5, e_3),$ $(e_3, e_5, e_4), (e_1, e_5, e_4), (e_1, e_4, e_3), (e_2, e_3, e_1), (e_2, e_4, e_3), (e_2, e_5, e_4), (e_2, e_4, e_1),$ $(e_2, e_5, e_3), (e_2, e_5, e_1), (e_5, e_3, e_1), (e_5, e_4, e_3), (e_5, e_4, e_1), (e_4, e_3, e_1), (e_3, e_1, e_2),$ $(e_4, e_3, e_2), (e_5, e_4, e_2), (e_4, e_1, e_2), (e_5, e_3, e_2), (e_5, e_1, e_2), (e_3, e_1, e_5), (e_4, e_3, e_5),$ $(e_4, e_1, e_5), (e_3, e_1, e_4)\}$;
here $(e_1, e_2, e_3) = \big(e_L(m_1), e_L(\varphi(m_1)), e_R(m_1)\big), (e_3, e_2, e_4) = \big(e_L(m_2), e_L(\varphi(m_2)), e_R(m_2)\big)$ etc. Further $(\mathcal{S} \circ \mathcal{C})(\mathbb{S}) = (C, \cdot, \psi)$ where $(e_i, e_i, e_i) \cdot (e_i, e_i, e_i) = (e_i, e_i, e_i)$ for $i = 1, 2, 3, 4, 5$, $(e_i, e_i, e_i) \cdot (e_i, e_j, e_k) = (e_i, e_j, e_k) \cdot (e_k, e_k, e_k) = (e_i, e_j, e_k)$ whenever $(e_i, e_j, e_k) \in C$ and

$$\begin{aligned}
(e_1, e_2, e_3) \cdot (e_3, e_2, e_4) &= (e_1, e_2, e_4)\\
(e_1, e_2, e_3) \cdot (e_3, e_2, e_5) &= (e_1, e_2, e_5)\\
(e_3, e_2, e_4) \cdot (e_4, e_2, e_5) &= (e_3, e_2, e_5)\\
(e_1, e_2, e_4) \cdot (e_4, e_2, e_5) &= (e_1, e_2, e_5)\\
(e_1, e_5, e_3) \cdot (e_3, e_5, e_4) &= (e_1, e_5, e_4)\\
(e_2, e_4, e_1) \cdot (e_1, e_4, e_3) &= (e_2, e_4, e_3)\\
(e_2, e_5, e_3) \cdot (e_3, e_5, e_4) &= (e_2, e_5, e_4)\\
(e_2, e_5, e_1) \cdot (e_1, e_5, e_3) &= (e_2, e_5, e_3)\\
(e_2, e_5, e_1) \cdot (e_1, e_5, e_4) &= (e_2, e_5, e_4)\\
(e_5, e_4, e_1) \cdot (e_1, e_4, e_3) &= (e_5, e_4, e_3)\\
(e_3, e_1, e_5) \cdot (e_5, e_1, e_2) &= (e_3, e_1, e_2)\\
(e_4, e_1, e_5) \cdot (e_5, e_1, e_2) &= (e_4, e_1, e_2)\\
(e_3, e_1, e_4) \cdot (e_4, e_1, e_5) &= (e_3, e_1, e_5)\\
(e_3, e_1, e_4) \cdot (e_4, e_1, e_2) &= (e_3, e_1, e_2)
\end{aligned}$$

and $\psi(e_i, e_j, e_k) = (e_j, e_k, e_i)$ if $(e_i, e_j, e_k) \in C$. The mapping $f \colon S \to C$ from the proof of Theorem 4.4 is $f(e_i) = (e_i, e_i, e_i)$ for $i = 1, 2, 3, 4, 5$ and $f(m_1) = (e_1, e_2, e_3)$, $f(m_2) = (e_3, e_2, e_4)$, $f(m_3) = (e_4, e_2, e_5)$, $f(m_4) = (e_1, e_2, e_4)$, $f(m_5) = (e_3, e_2, e_5)$, $f(m_6) = (e_1, e_2, e_5)$, $f(m_7) = (e_1, e_5, e_3)$, $f(m_8) = (e_3, e_5, e_4)$, $f(m_9) = (e_1, e_5, e_4)$, $f(m_{10}) = (e_1, e_4, e_3)$, $f(m_{11}) = (e_2, e_3, e_1)$, $f(m_{12}) = (e_2, e_4, e_3)$, $f(m_{13}) = (e_2, e_5, e_4)$, $f(m_{14}) = (e_2, e_4, e_1)$, $f(m_{15}) = (e_2, e_5, e_3)$, $f(m_{16}) = (e_2, e_5, e_1)$, $f(m_{17}) = (e_5, e_3, e_1)$, $f(m_{18}) = (e_5, e_4, e_3)$, $f(m_{19}) = (e_5, e_4, e_1)$, $f(m_{20}) = (e_4, e_3, e_1)$, $f(m_{21}) = (e_3, e_1, e_2)$, $f(m_{22}) = (e_4, e_3, e_2)$, $f(m_{23}) = (e_5, e_4, e_2)$, $f(m_{24}) = (e_4, e_1, e_2)$, $f(m_{25}) = (e_5, e_3, e_2)$, $f(m_{26}) = (e_5, e_1, e_2)$, $f(m_{27}) =$

(e_3, e_1, e_5), $f(m_{28}) = (e_4, e_3, e_5)$, $f(m_{29}) = (e_4, e_1, e_5)$, $f(m_{30}) = (e_3, e_1, e_4)$; this mapping is also an isomorphism of $\mathbb{S}$ onto $(\mathcal{S} \circ \mathcal{C})(\mathbb{S})$.

In the light of Examples 5.1 and 5.2 we can formulate

P r o b l e m. Let $\mathbb{S}$ be a c-algebra. What are the necessary and sufficient conditions for $\mathbb{S}$ to be isomorphic with $(\mathcal{S} \circ \mathcal{C})(\mathbb{S})$?

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