GENERATING FUNCTIONS AND BÉZOUTIANS

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(Received August 16, 1995)

Abstract. Using the idea of the generating function of a matrix in an extended sense we establish a Bézoutian type formula for a matrix M satisfying an intertwining relation of the form $MA^T = AM$. In the particular case of classical generating functions this formula gives a simple proof of Lander's theorem on the inverse of a Hankel matrix.

Keywords: generating function, Bézoutian, Hankel matrix

MSC 1991: 15A09, 15A57

In a paper devoted to the study of similarities between a given matrix and its transpose O. Taussky and H. Zassenhaus proved the following theorem.

Let A be an n by n matrix. Denote by $S(A)$ the set of all nonsingular matrices S satisfying the intertwining relation

$$A^T S = SA.$$

Then:

1° the set $S(A)$ contains symmetric matrices

2° all matrices in $S(A)$ are symmetric if and only if A is nonderogatory.

In the present note we intend to show that, in the case of a nonderogatory A, the second assertion may be considerably strengthened, in particular, we establish a close relation of the elements of $S(A)$ to the characteristic polynomial of A. This relation appears in the form of a Bézoutian type formula for a certain generating function of S^{-1}.

The generating function of a (not necessarily square) matrix M is defined by the formula $p(y)^T M p(x)$ where $p(z)$ stands for the column vector $(1, z, \ldots, z^{m-1})^T$, the lengths being chosen so as to make the above expression meaningful. It is the purpose of the present note to point out that useful formulae may be obtained if the notion of generating function is taken in a suitably extended sense. Indeed, replacing the polynomial vector $p(z)$ by $g(z) = (g_0(z), \ldots, g_{n-1}(z))^T$, the g_i being polynomials in z, the corresponding expression $g(y)^T M g(z)$ may assume a form more suitable for the problem considered.

In the present note we do not impose further conditions on the polynomial g_j; of course M cannot be recovered from the function $g(y)^T M g(x)$ unless the g_j are linearly independent.

We intend to show that, for matrices H satisfying an intertwining relation of the form $A^T H = HA$ a polynomial vector $g(z)$ may be chosen in such a manner that the corresponding (generalized) generating function of H^{-1} is a Bézoutian. In this manner we obtain more insight into the well known relationship between Hankel and Bézoutian matrices. Also, the formula obtained throws more light onto the manner in which the whole of H^{-1} is obtained from the solution of one particular equation $Hx = u$.

The intertwining relation $A^T S = SA$ has been investigated extensively in [1]. As opposed to the present note which concentrates on the construction of a suitable generating function of S^{-1} the main emphasis in [1] was on the characterization of matrices A that possess the following (testing) property: If S satisfies the intertwining relation $A^T S = SA$ then S is a Hankel matrix.

1. PRELIMINARIES AND NOTATION

Given a polynomial of degree n

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0,$$

we assign to it a sequence of polynomials

$$\begin{aligned} f^{(0)}(z) &= a_n z^{n-1} + \ldots + a_1 \\ f^{(1)}(z) &= a_n z^{n-2} + \ldots + a_2 \\ &\vdots \\ f^{(n-1)}(z) &= a_n. \end{aligned}$$

These polynomials satisfy the relations

$$\begin{aligned} z f^{(0)}(z) &= f(z) - a_0 \\ z f^{(j)}(z) &= f^{(j-1)}(z) - a_j \quad \text{for} \quad 1 \leqslant j \leqslant n-1, \\ \sum_0^{n-1} y^j f^{(j)}(z) &= \frac{f(y) - f(z)}{y - z}. \end{aligned}$$

It is easy to verify these identities directly; for a conceptual proof, see [4].

If $a_n = 1$ we denote by $C(f)$ the companion matrix of the polynomial f

$$C(f) = \begin{pmatrix} 0 & \dots & 0 & -a_0 \\ 1 & \dots & 0 & -a_1 \\ & & & \\ 0 & \dots & 1 & -a_{n-1} \end{pmatrix}.$$

The relations for the $f^{(j)}$ yield the following fact. If f is the characteristic polynomial of a nonderogatory matrix A and if e is a vector for which the vectors

$$e,\ Ae,\ A^2e,\ \dots,\ A^{n-1}e$$

are linearly independent then the matrix of the operator A in the basis $f^{(0)}(A)e, \dots, f^{(n-1)}(A)e$ is $C(f)$. Indeed,

$$A(e, Ae, \dots, A^{n-1}e) = (e, Ae, \dots, A^{n-1}e)C(f).$$

The space C^n is identified with column vectors of length n, the indices running from 0 to $n-1$; the standard basis of C^n consists of the vectors

$$\begin{aligned} e_0 &= (1, 0, \dots \quad 0)^T \\ e_1 &= (0, 1, \dots \quad 0)^T \\ \dots \end{aligned}$$

All matrices will be complex of type (n, n).

Suppose $f(A) = 0$. Define $B(z) = \sum\ z^j f^{(j)}(A)$; then $B(z)\ A = z\ B(z) - f(z)$.

Choose two vectors v and w and set

$$g(z) = B(z)^T v, \qquad h(y) = B(y)^T w.$$

It follows that

$$\begin{aligned} g(z)^T A &= zg(z)^T - f(z)v^T \\ A^T h(y) &= yh(y) - f(y)w. \end{aligned}$$

These two formulae will be used to compute the function

$$F(z, y) = g(z)^T H^{-1} h(y)$$

for a matrix H satisfying the intertwining relation $A^T H = HA$, the matrix A being nonderogatory; f will be the monic minimal polynomial of A; since A is nonderogatory f is of degree n.

In this section we formulate the main result. In its formulation, we use two possibly different polynomial vectors; this further step in generality does not require a more complicated proof. Surprisingly enough, the resulting polynomial of two variables remains symmetric.

Theorem. *Suppose A is nonderogatory and let f be its characteristic polynomial. Suppose H is invertible and*

$$A^T H = HA.$$

Choose two vectors v and w and set

$$g(z) = B(z)^T v, \quad h(y) = B(y)^T w.$$

Then

$$g(z)^T H^{-1} h(y) = \frac{f(z)m(y) - m(z)f(y)}{z - y}$$

the coefficients of m being

$$m_j = v^T f^{(j)}(A) H^{-1} w, \quad j = 0, 1, \ldots, n-1.$$

Proof.

$$\begin{aligned} 0 &= g(z)^T (AH^{-1} - H^{-1}A^T) h(y) \\ &= (zg(z)^T - f(z)v^T) H^{-1} h(y) - g(z)^T H^{-1} (yh(y) - f(y)w) \\ &= (z - y) F(z, y) - f(z) v^T H^{-1} h(y) + f(y) g(z)^T H^{-1} w. \end{aligned}$$

We now prove that, for every x

$$v^T H^{-1} h(x) = g(x)^T H^{-1} w.$$

Indeed,

$$\begin{aligned} v^T H^{-1} h(x) &= \sum x^j v^T H^{-1} f^{(j)}(A^T) w \\ &= \sum x^j v^T f^{(j)}(A)\, H^{-1} w = g(x)^T H^{-1} w. \end{aligned}$$

Denoting this polynomial by m, the identity above assumes the form

$$0 = (z - y) F(z, y) - f(z) m(y) + f(y) m(z).$$

This completes the proof. □

The theorem holds for arbitrary vectors v, w; of course, in order to obtain useful applications a suitable choice of these vectors has to be made. As an illustration, we consider, in the following section, the case of Hankel matrices.

The relation between Hankel matrices and Bézoutians established first by Lander [3] appears as a particular case of the preceding considerations with a simple and transparent proof.

Suppose H is a Hankel matrix of type (n, n)

$$H = \begin{pmatrix} s_0 & \dots & s_{n-1} \\ \\ s_{n-1} & \dots & s_{2n-2} \end{pmatrix}$$

Let f be a polynomial of degree n

$$f(x) = x^n - (a_0 + a_1 x + \ldots + a_{n-1} x^{n-1})$$

and denote by C its companion matrix

$$C(f) = \begin{pmatrix} 0 & \dots & 0 & -a_0 \\ 1 & \dots & 0 & -a_1 \\ \\ 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

The following conditions are equivalent

1° $H(a_0 \ldots a_{n-1})^T = (s_n, \ldots, s_{2n-2}, t)^T$ for some t

2° $HC(f) = C(f)^T H$

A (monic) polynomial of degree n satisfying these conditions is said to be *compatible with H.* Given two such polynomials, their difference is a polynomial $m_0 + m_1 x + \ldots + m_{n-1} x^{n-1}$ such that the vector $m = (m_0, \ldots, m_{n-1})^T$ is a solution of $Hx = e_{n-1}$

The following criterion of invertibility is well known, see e.g. [2]. Let us show how it may be established using the intertwining relation 2° and the polynomials $f^{(j)}$.

Suppose H is a Hankel matrix of type (n, n) such that the following two conditions are satisfied

1° *the equation $Hx = e_{n-1}$ has a solution*

2° *the equation $Hx = (s_n, \ldots, s_{2n-2}, t)^T$ has a solution for some t.*

Then H is invertible.

P r o o f. It follows from condition 2^0 that H possesses a polynomial f of degree n compatible with H; if C stands for its companion matrix the matrix H satisfies the intertwining relation $HC = C^T H$. To prove the invertibility of H it suffices to prove the existence of solutions for each of the equations $Hx = e_j$ for $j = 0, 1, \ldots, n-1$. Observing that $f^{(j)}(C^T)e_{u-1} = e_j$, these solutions may be obtained from the solution m of $Hx = e_{n-1}$ by taking $f^{(j)}(C)m$. Indeed,

$$H\ f^{(j)}(C)m = f^{(j)}(C^T)Hm = f^{(j)}(C^T)e_{n-1} = e_j.$$

□

In the particular case of Hankel matrices the main result of the preceding section reduces to the following

Theorem. *Suppose H is an invertible Hankel matrix. Then the generating function of H^{-1} is the B zoutian*

$$p(x)^T H^{-1} p(y) = \frac{f(x)m(y) - m(x)f(y)}{x - y}$$

where m is the solution of $Hm = e_{n-1}$ and f is a monic polynomial compatible with H.

Proof. Since $f^{(j)}(C^T)e_{n-1} = e_j$ we have $\sum z^j f^{(j)}(C^T)e_{n-1} = \sum z^j e_j = p(z)$. Furthermore

$$e_{n-1}^T H^{-1} f^{(j)}(C^T) e_{n-1} = e_{n-1}^T H^{-1} e_j = e_j^T H^{-1} e_{n-1} = m_j.$$

□

To conclude, let us remark that, in the case of a Hankel matrix, the choice $v = w = e_{n-1}$ simplifies the argument considerably. Indeed, having established the two relations

$$p(x)^T C = x\, p(x)^T - f(x) e_{n-1}^T$$
$$C^T p(y) = y\, p(y) - f(y) e_{n-1}$$

we obtain

$$\begin{aligned} 0 &= p(x)^T (CH^{-1} - H^{-1}C^T) p(y) \\ &= (xp(x)^T - f(x)e_{n-1}^T) H^{-1} p(y) - p(x)^T H^{-1} (yp(y) - f(x)e_{n-1}) \\ &= (x-y) p(x)^T H^{-1} p(y) - f(x) e_{n-1}^T H^{-1} p(y) + f(y) p(x)^T H^{-1} e_{n-1} \\ &= (x-y) p(x)^T H^{-1} p(y) - f(x) m^T p(y) + f(y) p(x)^T m. \end{aligned}$$

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