TWO TYPES OF RETRACT IRREDUCIBILITY OF CONNECTED MONOUNARY ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a connected monounary algebra. The paper deals with retract irreducibility of $\mathcal{A}$ in the class of all connected monounary algebras and in the class of all monounary algebras.

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This paper can be regarded as a continuation of author's articles [2] and [3]. For further references, cf. [2].

Some questions concerning irreducibility of posets were dealt with by Duffus and Rival [1]; the notion of irreducibility is defined by means of retracts of a poset.

Let (A, f) be a monounary algebra. A nonempty subset M of A is said to be a retract of (A, f) if there is a mapping h of A onto M such that h is an endomorphism of (A, f) and $h(x) = x$ for each $x \in M$. The mapping h is then called a retraction endomorphism corresponding to the retract M. Further, let $R(A, f)$ be the system of all monounary algebras (B, g) such that (B, g) is isomorphic to (M, f) for some retract M of (A, f).

Let $\mathcal{K}$ be a class of monounary algebras. A monounary algebra $\mathcal{A}$ will be said to be retract irreducible in $\mathcal{K}$ if, whenever $\mathcal{A} \in R\big(\prod\limits_{i\in I} \mathcal{A}_i\big)$ for some monounary algebras $\mathcal{A}_i \in \mathcal{K}$, $i \in I$, then there exists $j \in I$ such that $\mathcal{A} \in R\mathcal{A}_j$. If the condition is not satisfied, then $\mathcal{A}$ will be called retract reducible in $\mathcal{K}$.

Let (A, f) be a connected monounary algebra. Retract reducibility of (A, f) in the class of all connected monounary algebras was investigated in [2] and [3]. The aim of this paper is to deal with retract reducibility of (A,f) in the class of all (not only connected) monounary algebras. The following result will be proved.

Theorem. *Let (A,f) be a connected monounary algebra.*

(a) *If (A,f) is a cycle with p^n elements (p being a prime, $n \in \mathbb{N}$), then (A,f) is retract irreducible in the class of all connected monounary algebras and, at the same time, it is retract reducible in the class of all monounary algebras.*

(b) *If (A,f) does not satisfy the assumption of* (a)*, then (A,f) is retract reducible in the class of all connected monounary algebras if and only if (A,f) is retract reducible in the class of all monounary algebras.*

1. Assertion (a)

In what follows let (A,f) be a connected monounary algebra. We have the following obvious

1.1. Lemma. *If (A,f) is retract reducible in the class of all connected monounary algebras, then (A,f) is retract reducible in the class of all monounary algebras.*

1.2. Lemma. *Suppose that (A,f) is a cycle with p^n elements, where p is a prime and $n \in \mathbb{N}$. Then (A,f) is retract irreducible in the class of all connected monounary algebras.*

Proof. The assertion was proved in [3], Thm. (R1). $\square$

1.3. Lemma. *Let the assumption of* 1.2 *hold. Then (A,f) is retract reducible in the class of all monounary algebras.*

Proof. Take a prime $q \neq p$ and for each $i \in \mathbb{N}$ consider a monounary algebra (B_i,f) such that B_i consists of two cycles, one of them with p^n elements and the other with q^i elements. Put

$$(B,f) = \prod_{i\in\mathbb{N}} (B_i,f).$$

If (K,f) is a connected component of (B,f), then either (K,f) contains no cycle or it contains a cycle with $q^j p^n$ elements, where $j \in \mathbb{N} \cup \{0\}$. Further, there exists a cycle in (B,f) possessing p^n elements, denote it by M. Then we have

$$(A,f) \cong (M,f)$$

and $(A,f) \in R(B,f)$ in view of [2], Thm. 1.3. $\square$

1.4. Corollary. *The assertion* (a) *of Theorem is valid.*

2. Assertion (b)

Now suppose that (A, f) is retract reducible in the class of all monounary algebras. Then there are monounary algebras $(B_i, f), i \in \mathbb{N}$, such that

$$(A, f) \in R\Big(\prod_{i \in I}(B_i, f)\Big), \tag{1}$$

$$(A, f) \notin R(B_i, f) \text{ for each } i \in I. \tag{2}$$

Denote

$$\prod_{i \in I}(B_i, f) = (B, f). \tag{3}$$

According to (1), there is a subalgebra (M, f) of (B, f) and a mapping h of B onto M such that

(4) $(A, f) \cong (M, f)$,

(5) h is a homomorphism,

(6) $h(x) = x$ for each $x \in M$.

If $i \in I$, let ν_i be the natural i-th projection of (B, f) onto (B_i, f) and put

$$M_i = \nu_i(M) \text{ for each } i \in \mathbb{N}. \tag{7}$$

The following assertion is obvious:

2.1. Lemma. *If $i \in \mathbb{N}$, then (M_i, f) is a connected subalgebra of (B_i, f).*

Now let us denote

$$\prod_{i \in I}(M_i, f) = (D, f). \tag{8}$$

2.2. Lemma. *(M, f) is a subalgebra of (D, f).*

Proof. Let $z \in M, f(z) = u$. We have $\nu_i(z) \in M_i$ and 2.1 implies that $f(\nu_i(z)) \in M_i$. Then

$$\nu_i(u) = \nu_i(f(z)) = f(\nu_i(z)) \in M_i,$$

i.e., $u \in \prod_{i \in I} M_i = D$. □

2.3. Lemma. $(A, f) \in R(D, f)$.

Proof. By (4) and 2.2, (A, f) is isomorphic to a subalgebra (M, f) of (D, f). Further, 2.1 and the relations (3) and (8) yield that (D, f) is a subalgebra of (B, f). Consider the mapping $h_1 = h \upharpoonright D$. Then h_1 is a homomorphism of (D, f) onto (M, f) in view of (5), $h_1(x) = x$ for each $x \in M$ by (6). Therefore we obtain that $(A, f) \in R(D, f)$. □

2.4. Lemma. *(A, f) is retract irreducible in the class of all connected monounary algebras if and only if one of the following conditions is satisfied:*

(i) *$(A, f) \cong (\mathbb{N}, f)$, where $f(n+1) = n$ for each $n \in \mathbb{N}$, $f(1) = 1$;*
(ii) *there is $k \in \mathbb{N}$ such that $(A, f) \cong (\{1, 2, \ldots, k\}, f)$, where $f(k) = k-1, \ldots, f(2) = f(1) = 1$;*
(iii) *$(A, f) \cong (\mathbb{N}, f)$, where $f(n) = n+1$ for each $n \in \mathbb{N}$;*
(iv) *the assumption of* (a) *of Theorem is valid.*

Proof. It is a corollary of [2], (R) and of [3], (R1). □

2.5. Lemma. *If (A, f) is retract irreducible in the class of all connected monounary algebras, then there is $j \in I$ with*

(9)
$$(A, f) \in R(M_j, f).$$

Proof. Let the assumption hold. By 2.3 and (8),

$$(A, f) \in R(D, f) = R\Big(\prod_{i \in \mathbb{N}} (M_i, f)\Big).$$

Further, (M_i, f) is a connected monounary algebra for each $i \in I$ in view of 2.1. According to the definition, $(A, f) \in R(M_j, f)$ for some $j \in I$. □

2.6. Lemma. *If (A, f) is retract irreducible in the class of all connected monounary algebras, then the condition* (i) *of* 2.4 *fails to hold.*

Proof. Suppose that (i) of 2.4 is valid and that (9) holds, $j \in I$. Then there is a subalgebra (M', f) of (M_j, f) such that $(M', f) \cong (A, f)$. Since (M', f) fulfils the condition (i), in view of [2], Thm. 1.3 we obtain that $(A, f) \in R(B_j, f)$, which is a contradiction to (2). □

2.7. Lemma. *Let (A, f) satisfy the condition* (ii) *of* 2.4 *and $k > 1$. Further, let (T, f) be a monounary algebra. Then $(A, f) \in R(T, f)$ if and only if there is $a \in T$ such that*

$$f^k(a) = f^{k-1}(a) \neq f^{k-2}(a),$$
$$f^{-(k-1)}(f^{k-2}(a)) = \emptyset.$$

P r o o f. Let the assumption hold. Suppose that $(A, f) \in R(T, f)$. By (ii) there are distinct elements $a_1, \ldots, a_k \in T$ with $f(a_k) = a_{k-1}, \ldots, f(a_2) = f(a_1) = a_1$. Denote $S = \{a_1, \ldots, a_k\}, a = a_k$. We have $(S, f) \cong (A, f)$ and $f^k(a) = f^k(a_k) = a_1 = f^{k-1}(a_k) \neq a_2 = f^{k-2}(a_k) = f^{k-2}(a)$. Assume that $x \in f^{-(k-1)}(f^{k-2}(a))$. There is a retraction endomorphism g of (T, f) onto (S, f), which implies

$$f^{k-1}(g(x)) = g(f^{k-1}(x)) = g(f^{k-2}(a)) = g(a_2) = a_2,$$
$$g(x) \in f^{-(k-1)}(a_2) \cap S = \emptyset,$$

a contradiction.

Conversely, let there be $a \in S$ with the above properties. Put $a_k = a, a_{k-1} = f(a), \ldots, a_1 = f^{k-1}(a), S = \{a_1, \ldots, a_k\}$. According to the assumption, the elements $a_1, \ldots, a_k$ are mutually distinct and

$$(S, f) \cong (A, f).$$

We are going to define a retraction endomorphism g of (T, f) onto (S, f). Consider $x \in T$. We distinguish two cases.

a) First assume that

$$f^n(x) \notin S - \{a_1\} \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

Then we put $g(x) = a_1$.

b) Next suppose that the assumption from a) fails to hold. Let n be the smallest non-negative integer with $f^n(x) \in S - \{a_1\}$. There is $m \in \{2, \ldots, k\}$ such that $f^n(x) = a_m$. Then

$$f^n(x) = f^{k-m}(a_k) = f^{k-m}(a),$$
$$x \in f^{-n}(f^{k-m}(a)) \subseteq f^{-(n+m-2)}(f^{k-2}(a))$$

and the assumption implies that $n+m-2 < k-1$, i.e., $m+n \leqslant k$. Put $g(x) = a_{m+n}$.

It is easy to see that g is a retraction endomorphism of (T, f) onto (S, f). Hence $(A, f) \in R(T, f)$. $\square$

2.8. Lemma. *If (A,f) is retract irreducible in the class of all connected monounary algebras, then the condition* (ii) *of* 2.4 *fails to hold.*

P r o o f. Suppose that (ii) of 2.4 is valid. If $k=1$, then card $M=$ card $M_i=1$ for each $i\in I$ and then it is obvious that $(A,f)\cong(M_i,f)\in R(B_i,f)$, which contradicts (2). Let $k>1$. By (ii) and (4), there are distinct elements $\{a_1,\dots,a_k\}=M$ such that $f(a_k)=a_{k-1},\dots,f(a_2)=f(a_1)=a_1$. Further, 2.7 implies

$$f^{-(k-1)}(f^{k-2}(a_k))=\emptyset.$$

If $i\in I$, then the elements $a_1(i),\dots,a_k(i)$ are mutually distinct if and only if $a_2(i)\neq a_1(i)$, i.e., if and only if $f^k(a_k(i))=f^{k-1}(a_k(i))\neq f^{k-2}(a_k(i))$. Let J be the set of all $i\in I$ such that $a_1(i),\dots,a_k(i)$ are mutually distinct. According to (2) and 2.7 we get that if $i\in J$, then there exists $x_i\in f^{-(k-1)}(f^{k-2}(a_k(i)))$. The set J is nonempty in view of 2.5. Let $y\in T$ be such that

$$y(i)=\begin{cases} a_k(i) & \text{if } i\in I-J,\\ x_i & \text{if } i\in J.\end{cases}$$

Then

$$(f^{k-1}(y))(i)=\begin{cases} f^{k-1}(a_k(i))=a_1(i)=a_2(i) & \text{if } i\in I-J,\\ f^{k-2}(a_k(i))=a_2(i) & \text{if } i\in J.\end{cases}$$

Hence

$$f^{k-1}(y)=a_2=f^{k-2}(a_k),$$
$$y\in f^{-(k-1)}(f^{k-2}(a_k)),$$

which is a contradiction. □

2.9. Lemma. *Let* (iii) *of* 2.4 *hold. Suppose that (T,f) is a monounary algebra. Then $(A,f)\in R(T,f)$ if and only if each connected component (K,f) of (T,f) fulfils the following condition:*

(cond) *there is $x\in K$ such that $f^{-(n+1)}(f^n(x))=\emptyset$ for each $n\in\mathbb{N}\cup\{0\}$.*

P r o o f. Let the assumption of the lemma be valid. First suppose that $(A,f)\in R(T,f)$, and let (K,f) be a connected component of (T,f). Then there is a subalgebra of (T,f) isomorphic to (A,f), i.e., by (iii) of 2.4 there exist distinct elements $\{a_n\colon n\in\mathbb{N}\}$ in T with $f(a_n)=a_{n+1}$ for each $n\in\mathbb{N}$. Further, there is an endomorphism g of (T,f) onto $\{a_n\colon n\in\mathbb{N}\}$ with $g(a_n)=a_n$ for each $n\in\mathbb{N}$. Take the smallest $m\in\mathbb{N}$ such that

$$g^{-1}(a_m)\cap K\neq\emptyset;$$

such m obviously exists. Let x be an arbitrary element of the set $g^{-1}(a_m)\cap K$ and $n\in\mathbb{N}\cup\{0\}$. Suppose that there is $z\in f^{-(n+1)}(f^n(x))$. Then

$$f^{n+1}(g(z))=g(f^{n+1}(z))=g(f^n(x))=f^n(g(x))=f^n(a_m)=a_{m+n},$$

i.e.

$$g(z)\in f^{-(n+1)}(a_{m+n}).$$

Since $f^{-(m+n)}(a_{m+n})=\emptyset$, this implies that $m+n>n+1, m>1$. Further, $g(z)\in\{a_i\colon i\in\mathbb{N}\}$, thus

$$g(z)=a_{m-1}.$$

We have

$$z\in g^{-1}(a_{m-1})\cap K\neq\emptyset,$$

which is a contradiction. Therefore the condition (cond) is satisfied.

Conversely, let (cond) be fulfilled for each connected component (K,f) of (T,f). Then no connected component of (T,f) contains a cycle. In each connected component (K,f) of (T,f) let $x=x_K$ be a fixed element of K satisfying the condition (cond). Take one of these elements and denote it by a_1. Then there are distinct elements $\{a_n\colon n\in\mathbb{N}\}$ in T such that $f(a_n)=a_{n+1}$ for each $n\in\mathbb{N}$. Let us prove that $\{a_n\colon n\in\mathbb{N}\}$ forms a retract of (T,f) (it is obviously isomorphic to (A,f)). We are going to define a retraction endomorphism g as follows. Let $y\in K$, where (K,f) is a connected component of (T,f). Then there are $i,j\in\mathbb{N}\cup\{0\}$ with $f^i(y)=f^j(x_K)$. Further, if i_1,j_1 are such that $f^{i_1}(y)=f^{j_1}(x_K)$, then the fact that K contains no cycle yields that $j-i=j_1-i_1$. According to (cond), $i<j+1$. Put

$$g(y)=a_{j+1-i};$$

the mapping g is correctly defined. For $n\in\mathbb{N}$ we have $f^0(a_n)=f^{n-1}(a_1)$, thus

$$g(a_n)=a_n.$$

The mapping g is a homomorphism and therefore $(A,f)\in R(T,f)$. $\square$

Let $(K,f),(L,g)$ be connected monounary algebras and let ν be a surjective homomorphism of (K,f) onto (L,g). If (K,f) satisfies (cond), then there exists $x\in K$ such that $f^n(x)=f^m(y)$ implies $m\leqslant n$ for any $n,m\in N\cup\{0\}$ and any $y\in K$. This yields that $g^n(\nu(x))=g^m(t)$ implies $m\leqslant n$ for any $n,m\in N\cup\{0\}$ and any $t\in L$, because $t=\nu(y)$ for some $y\in K$. Hence, (L,g) satisfies (cond).

2.10. Lemma. *If (A,f) is retract irreducible in the class of all connected monounary algebras, then the condition* (iii) *of* 2.4 *fails to hold.*

P r o o f. Suppose that (iii) of 2.4 is valid. We have $(A, f) \in R(B, f)$, thus in view of 2.9 each connected component of (B, f) fulfils the condition (cond). Let $i \in I$. The projection ν_i is a homomorphism of (B, f) onto (B_i, f) and therefore each connected component of (B_i, f) fulfils the condition (cond), too. Hence, by 2.9, $(A, f) \in R(B_i, f)$, which is a contradiction with (2). $\square$

2.11. Corollary. *The assertion* (b) *of Theorem is valid.*

P r o o f. Let the assumption of (b) hold. Consider the following conditions:

(c1) (A, f) is retract reducible in the class of all connected monounary algebras;

(c2) (A, f) is retract reducible in the class of all monounary algebas.

In view of 1.1,

$$(c1) \Longrightarrow (c2).$$

Now suppose that (c2) holds and (c1) is not satisfied. According to 2.4 we obtain that either (i) or (ii) or (iii) of 2.4 is valid. By 2.6, 2.8 and 2.10, respectively, the condition (i), (ii) and (iii) is not satisfied. Therefore we have got a contradiction, hence

$$(c2) \Longrightarrow (c1).$$

$\square$

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