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ON KURZWEIL-HENSTOCK EQUIINTEGRABLE SEQUENCES

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his seventieth birthday

Abstract. For the Kurzweil-Henstock integral the equiintegrability of a pointwise convergent sequence of integrable functions implies the integrability of the limit function and the relation

$$\lim_{m\to\infty}\int_a^b f_m(s)\,\mathrm{d}s = \int_a^b \lim_{m\to\infty} f_m(s)\,\mathrm{d}s.$$

Conditions for the equiintegrability of a sequence of functions pointwise convergent to an integrable function are presented. These conditions are given in terms of convergence of some sequences of integrals.

Keywords: equiintegrable sequences, Kurzweil-Henstock integral

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A finite system of points

$$a \leqslant \beta_1 \leqslant \xi_1 \leqslant \gamma_1 \leqslant \beta_2 \leqslant \xi_2 \leqslant \gamma_2 \leqslant \ldots \leqslant \beta_m \leqslant \xi_m \leqslant \gamma_m \leqslant b$$

is called a *P-system* in the interval $[a,b]$.

This P-system is called a *P-partition* of the interval $[a,b]$ if

$$\bigcup_{i=1}^{m} [\beta_i,\gamma_i] = [a,b].$$

Any positive function $\delta\colon [a,b]\to(0,\infty)$ is called a *gauge on* $[a,b]$.

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If
$$[\beta_j,\gamma_j]\subset[\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j)],\quad j=1,\dots,m,$$
then the P-system is called *δ-fine.*

In other words, a finite system of points
$$\{\alpha_0,\tau_1,\alpha_1,\tau_2,\dots,\alpha_{k-1},\tau_k,\alpha_k\}$$
such that
$$a=\alpha_0<\alpha_1<\ldots<\alpha_{k-1}<\alpha_k=b$$
and
$$\tau_j\in[\alpha_{j-1},\alpha_j]\text{ for }j=1,\dots,k$$
is a *P-partition* of the interval $[a,b]$.

For a given gauge δ on $[a,b]$ a P-partition $\{\alpha_0,\tau_1,\alpha_1,\tau_2,\dots,\alpha_{k-1},\tau_k,\alpha_k\}$ of $[a,b]$ is δ-fine if
$$[\alpha_{j-1},\alpha_j]\subset(\tau_j-\delta(\tau_j),\tau_j+\delta(\tau_j))\text{ for }j=1,\dots,k.$$

Cousin's Lemma. *Given an arbitrary gauge δ on $[a,b]$ there is a δ-fine P-partition of $[a,b]$.*

(See e.g. [3], Lemma 9.2 or [4], [5], [7], [9].) Cousin's lemma is crucial for the following definition.

1. Definition. Assume that a function $f\colon[a,b]\to\mathbb{R}$ is given. The Kurzweil-Henstock integral $\int_a^b f(s)\,\mathrm{d}s$ exists if there is an element $I\in\mathbb{R}$ such that for every $\varepsilon>0$ there is a gauge δ on $[a,b]$ such that for
$$S(f,D)=\sum_{j=1}^{k}f(\tau_j)(\alpha_j-\alpha_{j-1})$$
we have
$$|S(f,D)-I|<\varepsilon$$
provided D is a δ-fine P-partition of $[a,b]$. We denote $I=\int_a^b f(s)\,\mathrm{d}s$. For the case $a=b$ it is convenient to set $\int_a^b f(s)\,\mathrm{d}s=0$.

The following proposition is easy to show by the definition of the integral.

2. Proposition. *Assume that $f,g\colon[a,b]\to\mathbb{R}$ are given such that the integrals $\int_a^b f(s)\,\mathrm{d}s$, $\int_a^b g(s)\,\mathrm{d}s$ exist.*

Then for every $c_1, c_2 \in \mathbb{R}$ the function $c_1 f + c_2 g\colon [a,b] \to \mathbb{R}$ is integrable and

$$\int_a^b [c_1 f(s) + c_2 g(s)]\,\mathrm{d}s = c_1 \int_a^b f(s)\,\mathrm{d}s + c_2 \int_a^b g(s)\,\mathrm{d}s.$$

If $g(\tau) \leqslant f(\tau)$ for $\tau \in [a,b]$ then

$$\int_a^b g(s)\,\mathrm{d}s \leqslant \int_a^b f(s)\,\mathrm{d}s.$$

If $[c,d] \subset [a,b]$ then the integral $\int_c^d f(s)\,\mathrm{d}s$ exists.

The next statement provides an operative tool in the theory of Kurzweil-Henstock integral. Its original version belongs to S. Saks and it was formulated for generalized integrals using Riemann-like sums by R. Henstock.

3. Lemma (Saks-Henstock). *Assume that a function $f\colon [a,b] \to \mathbb{R}$ is given such that the integral $\int_a^b f(s)\,\mathrm{d}s$ exists. Given $\varepsilon > 0$ assume that the gauge δ on $[a,b]$ is such that*

$$\left|\sum_{j=1}^k f(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b f(s)\,\mathrm{d}s\right| < \varepsilon$$

for every δ-fine P-partition $D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ of $[a,b]$. If

$$a \leqslant \beta_1 \leqslant \xi_1 \leqslant \gamma_1 \leqslant \beta_2 \leqslant \xi_2 \leqslant \gamma_2 \leqslant \ldots \leqslant \beta_m \leqslant \xi_m \leqslant \gamma_m \leqslant b$$

represents a δ-fine system $\{(\xi_j, [\beta_j, \gamma_j]),\ j = 1, \ldots, m\}$, i.e.

$$\xi_j \in [\beta_j, \gamma_j] \subset [\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)], \quad j = 1, \ldots, m,$$

then

$$\left|\sum_{j=1}^m \left[f(\xi_j)(\gamma_j - \beta_j) - \int_{\beta_j}^{\gamma_j} f(s)\,\mathrm{d}s\right]\right| \leqslant \varepsilon.$$

P r o o f. Without any loss of generality it can be assumed that $\beta_j < \gamma_j$ for every $j = 1, \ldots, m$.

Denote $\gamma_0 = a$ and $\beta_{m+1} = b$. If $\gamma_j < \beta_{j+1}$ for some $j = 0, 1, \ldots, m$ then the integral $\int_{\gamma_j}^{\beta_{j+1}} f(s)\,\mathrm{d}s$ exists and therefore for every $\eta > 0$ there exists a gauge δ_j on $[\gamma_j, \beta_{j+1}]$ such that $\delta_j(\tau) < \delta(\tau)$ for $\tau \in [\gamma_j, \beta_{j+1}]$ and for every δ_j-fine P-partition D^j of $[\gamma_j, \beta_{j+1}]$ we have

$$\left|S(f, D^j) - \int_{\gamma_j}^{\beta_{j+1}} f(s)\,\mathrm{d}s\right| < \tfrac{\eta}{m+1}.$$

If $\gamma_j = \beta_{j+1}$ then we take $S(f, D^j) = 0$.

The expression

$$\sum_{j=1}^{m} f(\xi_j)(\gamma_j - \beta_j) + \sum_{j=1}^{m} S(f, D^j)$$

represents an integral sum which corresponds to a certain δ-fine P-partition of $[a, b]$ and consequently

$$\left|\sum_{j=1}^{m} f(\xi_j)(\gamma_j - \beta_j) + \sum_{j=1}^{m} S(f, D^j) - \int_a^b f(s)\,\mathrm{d}s\right| < \varepsilon.$$

Hence

$$\begin{aligned}
&\left|\sum_{j=1}^{m}\left[f(\xi_j)(\gamma_j - \beta_j) - \int_{\beta_j}^{\gamma_j} f(s)\,\mathrm{d}s\right]\right| \\
&\leqslant \left|\sum_{j=1}^{m} f(\xi_j)(\gamma_j - \beta_j) + \sum_{j=1}^{m} S(f, D^j) - \int_a^b f(s)\,\mathrm{d}s\right| + \sum_{j=1}^{m}\left|S(f, D^j) - \int_{\gamma_j}^{\beta_{j+1}} f(s)\,\mathrm{d}s\right| \\
&< \varepsilon + (m+1)\tfrac{\eta}{m+1} = \varepsilon + \eta.
\end{aligned}$$

Since this inequality holds for every $\eta > 0$, we immediately obtain the inequality from the statement. $\square$

Convergence results for a given integration theory are important for estimating the power of the theory. For the Kurzweil-Henstock integral we have a result given by Theorem 4 bellow. It should be mentioned that this theorem is based on the classical principle that "two limiting processes are interchangeable if one of them is uniform with respect to the variable of the second". Classical theorems known for the Lebesgue integral (the dominated convergence theorem and the monotone convergence result of B. Levi) can be derived from Theorem 4 (see e.g. [1], [3], [5], [6], [8], [9]).

4. Theorem. *Let functions $f, f_m\colon [a, b] \to \mathbb{R}$, $m = 1, 2, \ldots$ be given where the integral $\int_a^b f_m(s)\,\mathrm{d}s$ exists for every $m = 1, 2, \ldots$. Assume that*

$$\lim_{m\to\infty} f_m(\tau) = f(\tau) \tag{1}$$

for $\tau \in [a, b]$. Assume further that

for every $\eta > 0$ there is a gauge δ on $[a, b]$ such that

$$\left|S(f_m, D) - \int_a^b f_m(s)\,\mathrm{d}s\right| < \eta \tag{2}$$

for every δ-fine P-partition D of $[a,b]$ and every $m = 1,2,\ldots$.
Then the function $f\colon [a,b] \to \mathbb{R}$ is integrable and

$$\lim_{m\to\infty} \int_a^b f_m(s)\,\mathrm{d}s = \int_a^b f(s)\,\mathrm{d}s. \tag{3}$$

P r o o f. Let $\varepsilon > 0$ be given. By (2) there is a gauge δ on $[a,b]$ such that for every δ-fine P-partition

$$D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

of $[a,b]$ we have

$$\left| S(f_m, D) - \int_a^b f_m(s)\,\mathrm{d}s \right| < \tfrac{\varepsilon}{2}$$

for $m = 1,2,\ldots$. By (1) for every fixed P-partition D of $[a,b]$ there exists a positive integer m_0 such that for $m > m_0$ the inequality

$$|S(f_m, D) - S(f, D)| = \left| \sum_{j=1}^{k} [f_m(\tau_j) - f(\tau_j)](\alpha_j - \alpha_{j-1}) \right| < \tfrac{\varepsilon}{2}$$

holds and this means that

$$\lim_{m\to\infty} S(f_m, D) = S(f, D).$$

Therefore for any δ-fine P-partition D of $[a,b]$ there is a positive integer m_0 such that for $m > m_0$ we have

$$\left| S(f, D) - \int_a^b f_m(s)\,\mathrm{d}s \right| < \varepsilon. \tag{4}$$

First we get from (4) that for all positive integers $m, l > m_0$ the inequality

$$\left| \int_a^b f_m(s)\,\mathrm{d}s - \int_a^b f_l(s)\,\mathrm{d}s \right| < 2\varepsilon$$

holds. This means that $(\int_a^b f_m(s)\,\mathrm{d}s)_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and therefore it has a limit

$$\lim_{m\to\infty} \int_a^b f_m(s)\,\mathrm{d}s = I \in \mathbb{R}. \tag{5}$$

Another consequence of (4) is the inequality

$$\begin{aligned}|S(f,D)-I| &\leqslant \left|S(f,D)-\int_a^b f_m(s)\,\mathrm{d}s\right| + \left|\int_a^b f_m(s)\,\mathrm{d}s - I\right| \\ &< \varepsilon + \left|\int_a^b f_m(s) - I\right| \quad \text{for} \quad m > m_0.\end{aligned}$$

By (5) we obtain immediately from this inequality that for every δ-fine P-partition D of $[a,b]$ we have

$$|S(f,D)-I| < \varepsilon$$

and this means that the integral $\int_a^b f(s)\,\mathrm{d}s$ exists and (3) is satisfied. □

5. Definition. A sequence of integrable functions $f_m\colon [a,b]\to\mathbb{R}$, $m=1,2,\dots$ is called *equiintegrable* if the condition (2) of Theorem 4 is satisfied.

6. Remark. In some texts (see e.g. [4]) an equiintegrable sequence is called *uniformly integrable.*

Theorem 4 gives a sufficient condition for a sequence of integrable functions to tend to an integrable limit and for the integrals of the members of the sequence to tend to the integral of the limit function. The convergence of the functions f_m to f given by (1) is the pointwise convergence and the sufficient condition is the equiintegrability (2) of the sequence (f_m).

Using the concept of an equiintegrable sequence of functions $f_m\colon [a,b]\to\mathbb{R}$ Theorem 4 can be reformulated as follows:

Let functions f, $f_m\colon [a,b]\to\mathbb{R}$, $m=1,2,\dots$ be such that the functions f_m are integrable for every $m=1,2,\dots$. Assume that

$$\lim_{m\to\infty} f_m(\tau) = f(\tau)$$

for $\tau\in[a,b]$. Assume further that the sequence $f_m\colon [a,b]\to\mathbb{R}$, $m=1,2,\dots$ is equiintegrable, then the function $f\colon [a,b]\to\mathbb{R}$ is integrable and

$$\lim_{m\to\infty}\int_a^b f_m(s)\,\mathrm{d}s = \int_a^b f(s)\,\mathrm{d}s.$$

7. Proposition. *Let functions $f_n\colon [a,b]\to\mathbb{R}, n=1,2,\dots$ be given where the integrals $\int_a^b f_n(s)\,\mathrm{d}s$ exist for $n=1,2,\dots$. Assume that*

$$\lim_{n\to\infty} f_n(\tau) = 0 \tag{6}$$

for $\tau \in [a,b]$ and that the sequence of functions $f_n\colon [a,b] \to \mathbb{R}$ is equiintegrable.

Then

(C) *for every $\varepsilon > 0$ and for every finite system of nonoverlapping*[2] *intervals $[\beta_j, \gamma_j] \subset [a,b]$, $j = 1,2,\dots,l$ there is an $N \in \mathbb{N}$ such that for every $n > N$ the inequality*

$$\left| \sum_{j=1}^{l} \int_{\beta_j}^{\gamma_j} f_n(s)\,\mathrm{d}s \right| < \varepsilon$$

holds.

P r o o f. Let $\varepsilon > 0$ be given. By the equiintegrability given by the condition (2) there is a gauge δ on $[a,b]$ such that for every δ-fine P-partition

$$D = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

of $[a,b]$ we have

$$\left| S(f_n, D) - \int_a^b f_n(s)\,\mathrm{d}s \right| < \tfrac{\varepsilon}{3} \tag{7}$$

for $n = 1,2,\dots$. Assume that an arbitrary finite system of nonoverlapping intervals $[\beta_j, \gamma_j] \subset [a,b]$, $j = 1,2,\dots,l$ is given. Let

$$D^j = \{\alpha_0^j, \tau_1^j, \alpha_1^j, \dots, \alpha_{k_j-1}^j, \tau_{k_j}^j, \alpha_{k_j}^j\}$$

be a δ-fine P-partition of $[\beta_j, \gamma_j]$ for $j = 1,2,\dots,l$.

Using (7) and the Saks-Henstock lemma we obtain

$$\begin{aligned} &\left| \sum_{j=1}^{l} \left[S(f_n, D^j) - \int_{\beta_j}^{\gamma_j} f_n(s)\,\mathrm{d}s \right] \right| \\ &\quad = \left| \sum_{j=1}^{l} \sum_{i=1}^{k_j} f_n(\tau_i^j)(\alpha_i^j - \alpha_{i-1}^j) - \sum_{j=1}^{l} \int_{\beta_j}^{\gamma_j} f_n(s)\,\mathrm{d}s \right| \leqslant \tfrac{\varepsilon}{3} < \tfrac{\varepsilon}{2}. \end{aligned} \tag{8}$$

By (6) for every fixed system of P-partitions D^j, $j = 1,2,\dots,l$ with the properties given above there exists a positive integer $N \in \mathbb{N}$ such that for $n > N$ the inequality

$$\left| \sum_{j=1}^{l} S(f_n, D^j) \right| = \left| \sum_{j=1}^{l} \sum_{i=1}^{k_j} f_n(\tau_i^j)(\alpha_i^j - \alpha_{i-1}^j) \right| < \tfrac{\varepsilon}{2}$$

[2] This means in our case that $(\beta_j, \gamma_j) \cap (\beta_i, \gamma_i) = \emptyset$ provided $i \neq j$.

holds. Hence using (8) we get

$$\left|\sum_{j=1}^{l}\int_{\beta_j}^{\gamma_j} f_n(s)\,\mathrm{d}s\right| \leqslant \left|\sum_{j=1}^{l}\left[\int_{\beta_j}^{\gamma_j} f_n(s)\,\mathrm{d}s - S(f_n, D^j)\right]\right| + \left|\sum_{j=1}^{l} S(f_n, D^j)\right| < \varepsilon$$

for $n > N$. □

Remark. The proof of Proposition 7 is a slight modification of the proof of the Convergence theorem 4.

8. Theorem. *Let functions $f_n\colon [a,b] \to \mathbb{R}$, $n = 1, 2, \dots$ be given where the integrals $\int_a^b f_n(s)\,\mathrm{d}s$ exist for $n = 1, 2, \dots$. Assume that*

(9) $$\lim_{n\to\infty} f_n(\tau) = 0$$

for $\tau \in [a,b]$ and that

(UC) *for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for every $n > N$ and every finite system of nonoverlapping intervals $[\beta_j, \gamma_j] \subset [a,b]$, $j = 1, 2, \dots, l$ the inequality*

(10) $$\left|\sum_{j=1}^{l}\int_{\beta_j}^{\gamma_j} f_n(s)\,\mathrm{d}s\right| < \varepsilon$$

Then the sequence of functions $f_n\colon [a,b] \to \mathbb{R}$, $n = 1, 2, \dots$ is equiintegrable.

Proof. Let $\varepsilon > 0$ be given. Let us set

$$A_n = \{t \in I\colon |f_n(t)| \geqslant \varepsilon\}.$$

From (9) we get

(11) $$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

By the definition of the integral, for the given ε and $n = 1, 2, \dots$ there exist gauges δ_n such that

(12) $$\left|S(f_n, D) - \int_a^b f_n(t)\,\mathrm{d}t\right| < \varepsilon,$$

whenever the P-partition D is δ_n-fine. Setting $l = 1$ and $[\beta_1, \gamma_1] = [a,b]$ in (10) we obtain

(13) $$\left|\int_a^b f_n(s)\,\mathrm{d}s\right| < \varepsilon \quad \text{for} \quad n > N.$$

Let us define new gauges $\hat{\delta}_k$ such that $\hat{\delta}_k(t) = \delta_k(t)$ if $t \in A_k$ and $\hat{\delta}_k(t) = 1$ if $t \notin A_k$.

Define further $\Delta\colon [a,b] \to \mathbb{R}$ by

$$\Delta(t) = \inf_k \hat{\delta}_k(t). \tag{14}$$

If $t \in [a,b]$ then by (9) there is an index M such that $|f_n(t)| < \varepsilon$ for $n > M$; hence $t \notin A_n$ for $n > M$ and this implies that t belongs to a finite number of sets A_n only, e.g. $t \in \bigcap\limits_{i=1}^{L} A_{k_i}$ and therefore we have $\Delta(t) = \min\{\hat{\delta}_{k_1}(t), \dots \hat{\delta}_{k_L}(t), 1\}$. Hence

$$\Delta(t) > 0 \quad \text{for all} \quad t \in [a,b]$$

and Δ is a gauge on $[a,b]$. We will prove that Δ is the required gauge from (2) corresponding to ε.

Assume that $D = \{\alpha_0, \tau_1, \alpha_1, \tau_2, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$ is a Δ-fine P-partition of the interval $[a,b]$. Let us consider the integral sum $S(f_n, D) = \sum_{j=1}^{k} f_n(\tau_j)(\alpha_j - \alpha_{j-1})$. We have

$$\sum_{j=1}^{k} f_n(\tau_j)(\alpha_j - \alpha_{j-1}) = \sum_{\substack{j=1\\ \tau_j \notin A_n}}^{k} f_n(\tau_j)(\alpha_j - \alpha_{j-1}) + \sum_{\substack{j=1\\ \tau_j \in A_n}}^{k} f_n(\tau_j)(\alpha_j - \alpha_{j-1}) \tag{15}$$

For the first sum on the right hand side of (15) we have $|f_n(\tau_j)| < \varepsilon$, hence

$$\Bigg| \sum_{\substack{j=1\\ \tau_j \notin A_n}}^{k} f_n(\tau_j)(\alpha_j - \alpha_{j-1}) \Bigg| < \varepsilon \sum_{j=1}^{k} (\alpha_j - \alpha_{j-1}) = \varepsilon(b-a)$$

while by (10) we get

$$\Bigg| \sum_{\substack{j=1\\ \tau_j \notin A_n}}^{k} \int_{\alpha_{j-1}}^{\alpha_j} f_n(s)\,\mathrm{d}s \Bigg| < \varepsilon \text{ for } n > N.$$

Hence

$$\Bigg| \sum_{\substack{j=1\\ \tau_j \notin A_n}}^{k} \left[f_n(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_{\alpha_{j-1}}^{\alpha_j} f_n(s)\,\mathrm{d}s \right] \Bigg| < \varepsilon(b-a) + \varepsilon = \varepsilon(b-a+1) \tag{16}$$

If $\tau_j \in A_n$ then $\hat{\delta}_n(\tau_j) = \delta_n(\tau_j)$ and $\Delta(\tau_j) < \delta_n(\tau_j)$. Hence the system

$$\alpha_{j-1} \leqslant \tau_j \leqslant \alpha_j \text{ with } \tau_j \in A_n$$

is such that $[\alpha_{j-1},\alpha_j] \subset (\tau_j-\Delta(\tau_j),\tau_j+\Delta(\tau_j)) \subset (\tau_j-\delta_n(\tau_j),\tau_j+\delta_n(\tau_j))$. Since (12) holds for any δ_n-fine P-partition, Lemma 3 yields

$$\Bigl|\sum_{\substack{j=1\\ \tau_j\in A_n}}^{k}\Bigl[f_n(\tau_j)(\alpha_j-\alpha_{j-1})-\int_{\alpha_{j-1}}^{\alpha_j} f_n(s)\,\mathrm{d}s\Bigr]\Bigr| \leqslant \varepsilon. \tag{17}$$

Taking into account (15), by (16) and (17) we have

$$\begin{aligned}&\Bigl|\sum_{j=1}^{k} f_n(\tau_j)(\alpha_j-\alpha_{j-1})-\int_a^b f_n(s)\,\mathrm{d}s\Bigr|\\ &\quad=\Bigl|\sum_{j=1}^{k}\Bigl[f_n(\tau_j)(\alpha_j-\alpha_{j-1})-\int_{\alpha_{j-1}}^{\alpha_j} f_n(s)\,\mathrm{d}s\Bigr]\Bigr| < \varepsilon+\varepsilon(b-a+1)\\ &\quad=\varepsilon(b-a+2)\end{aligned} \tag{18}$$

for $n > N$.

Let us now set

$$\delta(t)=\min\{\delta_1(t),\delta_2(t),\dots,\delta_N(t),\Delta(t)\}$$

and assume that D is a δ-fine P-partition of $[a,b]$. Then using (12) for $n=1,\dots,N$ and (18) we obtain

$$\Bigl|\sum_{j=1}^{k} f_n(\tau_j)(\alpha_j-\alpha_{j-1})-\int_a^b f_n(s)\,\mathrm{d}s\Bigr| < \varepsilon(3+b-a)$$

for every $n\in\mathbb{N}$ and this means that the sequence of functions $f_n\colon [a,b]\to\mathbb{R}$ is equiintegrable. $\square$

9. Remark. It is easy to see that the conclusion (C) of Proposition 7 can be stated in the form

(C) *for every finite system of nonoverlapping intervals* $[\beta_j,\gamma_j]\subset[a,b]$, $j=1,2,\dots,l$

$$\lim_{n\to\infty}\sum_{j=1}^{l}\int_{\beta_j}^{\gamma_j} f_n(s)\,\mathrm{d}s=0$$

holds

while the condition (UC) from Theorem 8 can read

(UC)

$$\lim_{n\to\infty}\sum_{j=1}^{l}\int_{\beta_j}^{\gamma_j} f_n(s)\,\mathrm{d}s=0$$

uniformly with respect to finite systems of nonoverlapping intervals

$$[\beta_j, \gamma_j] \subset [a, b],\ j = 1, 2, \ldots, l.$$

Looking at Proposition 7 we can ask whether the conclusion (C) can be replaced by its uniform version (UC) given in Theorem 8. In other words: we are asking whether for a sequence of functions $f_n\colon [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ with $\lim\limits_{n\to\infty} f_n(\tau) = 0$ for $\tau \in [a, b]$ the equiintegrability of this sequence is equivalent to the condition (UC).

The following example shows that the answer to this question is negative.

10. Example. Let us define the function $f\colon [0,1] \to \mathbb{R}$ by

$$f(\tau) = \frac{(-1)^k}{\tau} \text{ for } \frac{1}{k+1} < \tau \leqslant \frac{1}{k},\ k = 1, 2, \ldots$$
$$f(0) = 0.$$

The function $F(\tau) = \ln \tau$ is for $\tau > 0$ the primitive to the function $\frac{1}{\tau}$. Hence

$$\int_{\frac{1}{k+1}}^{\frac{1}{k}} f(s)\,\mathrm{d}s = (-1)^k \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{1}{s}\,\mathrm{d}s = (-1)^k \Big(\ln \frac{1}{k} - \ln \frac{1}{k+1} \Big)$$

and

$$\int_{\frac{1}{n}}^{1} f(s)\,\mathrm{d}s = \sum_{k=1}^{n-1} \int_{\frac{1}{k+1}}^{\frac{1}{k}} f(s)\,\mathrm{d}s = \sum_{k=1}^{n-1} (-1)^k \ln \frac{k+1}{k}.$$

The series $\sum\limits_{k=1}^{\infty} (-1)^k \ln \frac{k+1}{k}$ converges (to the value $-\ln \frac{\pi}{2}$) and therefore the limit

$$\lim_{c\to 0+} \int_c^1 f(s)\,\mathrm{d}s = -\ln \tfrac{\pi}{2}$$

exists. Using Hake's theorem[3] we obtain the existence of the integral $\int_0^1 f(s)\,\mathrm{d}s = -\ln \frac{\pi}{2}$.

[3] **Hake's theorem.** If $f\colon [a,b] \to \mathbb{R}$ is such that the integral $\int_c^b f(s)\,\mathrm{d}s$ exists for every $c \in (a, b]$ and that there exist a finite limit $\lim\limits_{c\to a+} \int_c^b f(s)\,\mathrm{d}s = I$, then the function f is integrable over $[a, b]$ and

$$\int_a^b f(s)\,\mathrm{d}s = I.$$

(See e.g. [9], Theorem 1.14)

On the other hand we have $|f(\tau)| = \frac{1}{\tau}$ for $\tau \in (0,1]$, $|f(0)| = 0$ and the integral

$$\int_0^1 |f(s)|\,\mathrm{d}s = \int_0^1 \frac{1}{s}\,\mathrm{d}s$$

does not exist because

$$\int_0^1 \frac{1}{s}\,\mathrm{d}s = \ln 1 - \lim_{c\to 0+} \ln c = +\infty.$$

The function $f\colon [0,1] \to \mathbb{R}$ given above is a typical representative of functions which are not absolutely integrable.

Let us now define a sequence of functions $f_n\colon [0,1] \to \mathbb{R}$, $n = 1,2,\ldots$ in such a way that

$$f_n(\tau) = f(\tau) \text{ for } \tau \in [0, \tfrac{1}{2n}],$$
$$f(0) = 0 \text{ for } \tau > \tfrac{1}{2n}.$$

Then $\lim\limits_{n\to\infty} f_n(\tau) = 0$ and the sequence (f_n) is equiintegrable (see e.g. Theorem 8 in [1]).

If $n_0 \in \mathbb{N}$ and $n > n_0$, then

$$\int_{\frac{1}{2n+1}}^{\frac{1}{2n}} f_{n_0}(s)\,\mathrm{d}s = \int_{\frac{1}{2n+1}}^{\frac{1}{2n}} \frac{1}{s}\,\mathrm{d}s = \ln\frac{1}{2n} - \ln\frac{1}{2n+1} = \ln\frac{2n+1}{2n}.$$

Since the series $\sum\limits_{l=1}^{\infty} \ln\frac{2l+1}{2l}$ diverges, for every $n_0 \in \mathbb{N}$ there is a finite $k(n_0) \in \mathbb{N}$, $k(n_0) > n_0$ such that

$$\sum_{n=n_0}^{k(n_0)} \ln\frac{2n+1}{2n} > 1$$

and therefore also

$$\left|\sum_{n=n_0}^{k(n_0)} \int_{\frac{1}{2n+1}}^{\frac{1}{2n}} f_{n_0}(s)\,\mathrm{d}s\right| = \sum_{n=n_0}^{k(n_0)} \int_{\frac{1}{2n+1}}^{\frac{1}{2n}} f_{n_0}(s)\,\mathrm{d}s > 1.$$

Hence for every $n_0 \in \mathbb{N}$ there is a finite system of nonoverlapping intervals

$$[\beta_j, \gamma_j] = \Big[\frac{1}{2j+1}, \frac{1}{2j}\Big], \quad j = n_0, \ldots, k(n_0)$$

such that

$$\left|\sum_{n=n_0}^{k(n_0)} \int_{\beta_j}^{\beta_j} f_{n_0}(s)\,\mathrm{d}s\right| > 1$$

and this shows that even if $\lim\limits_{n\to\infty} f_n(\tau) = 0$ and the sequence (f_n) is equiintegrable, the condition (UC) from Theorem 8 does not hold.

11. E x a m p l e. We construct a sequence of functions $(f_n(t))_{n=1}^{\infty}$ defined and integrable on the interval $[0,1]$ fulfilling

$$\lim_{n\to\infty} f_n(t) = 0 \tag{6}$$

and

$$\lim_{n\to\infty} \int_a^b f_n(t)\,\mathrm{d}t = 0 \tag{19}$$

for every interval $[a,b]\subset[0,1]$ such that the sequence $(f_n)_{n=1}^{\infty}$ is not equiintegrable.

Denote

$$\begin{aligned} I_n^+ &= (0, 4^{-n}),\\ I_n^- &= (4^{-n}, 2.4^{-n}). \end{aligned}$$

Define

$$\begin{aligned} f_n(t) &= 4^n \quad \text{for} \quad x\in I_n^+,\\ f_n(t) &= -4^n \quad \text{for} \quad x\in I_n^-,\\ f_n(t) &= 0 \quad\ \ \text{otherwise.} \end{aligned}$$

Since $I_n^+\cup I_n^-$ is monotone and

$$\bigcap_{n=1}^{\infty}(I_n^+\cup I_n^-) = \emptyset,$$

the condition (6) is fulfilled.

Let $0<a<b\leqslant 1$ be fulfilled. There exists n_1 so that $(I_n^+\cup I_n^-)\cap[a,b]=\emptyset$ for $n>n_1$. The property (19) immediately follows for the interval $[a,b]$.

Let $0=a<b\leqslant 1$ be fulfilled, then there exists n_2 so that $(I_n^+\cup I_n^-)\subset[a,b]$ for $n>n_2$ and the property (19) for the interval $[a,b]$ follows again. It is also easy to see that our sequence (f_n) satisfies the condition (C) given in Proposition 7.

Assume now that the sequence $\{f_n(t)\}_{n=1}^{\infty}$ is equiintegrable, i.e. for every $\varepsilon>0$ there is a gauge δ on $[0,1]$ such that

$$\left|S(f_m,D) - \int_0^1 f_m(s)\,\mathrm{d}s\right| < \varepsilon$$

for every $m=1,2,\ldots$ and every δ-fine P-partition D of $[0,1]$.

Denote $d = \delta(0)$. Choose an integer m so that $2 \cdot 4^{-m} < d$ and let us set

$$\beta = 0, \quad \gamma = 4^{-m}.$$

Then $\{\beta, 0, \gamma\}$ is δ-fine, i.e. $[\beta, \gamma] \subset [-\delta(0), \delta(0)]$. The Saks-Henstock lemma yields

$$\left| f_m(0)(\gamma - \beta) - \int_\beta^\gamma f_m(x)\,\mathrm{d}x \right| = \left| f_m(0)4^{-m} - \int_0^{4^{-m}} f_m(x)\,\mathrm{d}x \right| \leqslant \varepsilon.$$

On the contrary we have $f_m(0) = 0$ and $\int_0^{4^{-m}} f_m(x)\,\mathrm{d}x = 1$.

Taking into account the simple statement given in Remark 9, this example shows that even if $\lim\limits_{n\to\infty} f_n(\tau) = 0$ and the sequence (f_n) satisfies the condition (C), the sequence (f_n) is not equiintegrable.

12. Remark. For a given sequence of functions $f_n\colon [a,b] \to \mathbb{R}$ which are integrable over $[a,b]$ and such that $\lim\limits_{n\to\infty} f_n(\tau) = 0$, we denote by (EI) the property that this sequence is equiintegrable in the sense of definition 5 and we denote by (L) the property that

$$\lim_{n\to\infty} \int_a^b f_n(s)\,\mathrm{d}s = 0.$$

Using the results given above, we have

$$(\mathrm{UC}) \Rightarrow (\mathrm{EI}) \Rightarrow (\mathrm{C}) \Rightarrow (\mathrm{L}).$$

The first implication is given by Theorem 8, the second by Proposition 7 and the third is trivial ((L) is in fact (C) for the case of the single interval $[\beta, \gamma] = [a, b]$).

On the other hand by Example 10 we have

$$(\mathrm{UC}) \not\Leftarrow (\mathrm{EI})$$

and Example 11 shows that

$$(\mathrm{EI}) \not\Leftarrow (\mathrm{C}).$$

Hence we have also (L) $\not\Rightarrow$ (UC). If it were (L) $\Rightarrow$ (UC) then all (UC), (EI), (C) and (L) would be equivalent but this cannot hold because of (EI) $\not\Leftarrow$ (C), or (UC) $\not\Leftarrow$ (EI).

13. Proposition. *Let nonnegative functions $f_n\colon [a,b] \to [0, +\infty)$, $n = 1, 2, \ldots$ be given where the integrals $\int_a^b f_n(s)\,\mathrm{d}s$ exist for $n = 1, 2, \ldots$. Assume that*

$$\lim_{n\to\infty} f_n(\tau) = 0 \tag{6}$$

for $\tau \in [a,b]$ *and*

$$\lim_{n\to\infty} \int_a^b f_n(s)\,\mathrm{d}s = 0. \tag{20}$$

Then the sequence of functions $f_n\colon [a,b] \to \mathbb{R}$, $n = 1,2,\ldots$ *is equiintegrable.*

P r o o f. By Theorem 8 it is sufficient to show that the condition (UC) is satisfied in this case.

Since $f_n(\tau) \geqslant 0$ for every $\tau \in [a,b]$ we get by Proposition 2 the inequality $\int_a^b f_n(s)\,\mathrm{d}s \geqslant 0$ for $n = 1,2,\ldots$. By (20) for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > N$ we have $\int_a^b f_n(s)\,\mathrm{d}s < \varepsilon$.

Assume that $[\beta_j,\gamma_j] \subset [a,b]$, $j = 1,2,\ldots,l$ is an arbitrary finite system of nonoverlapping intervals. Define

$$g_n(\tau) = f_n(\tau) \text{ for } \tau \in [\beta_j,\gamma_j]$$
$$g_n(\tau) = 0 \text{ for } \tau \in [a,b] \setminus \bigcup_{j=1}^{l} [\beta_j,\gamma_j].$$

Then

$$0 \leqslant g_n(\tau) \leqslant f_n(\tau) \text{ for } \tau \in [a,b]$$

and

$$0 \leqslant \int_a^b g_m(s)\,\mathrm{d}s = \sum_{j=1}^{l} \int_{\beta_j}^{\gamma_j} f_m(s)\,\mathrm{d}s \leqslant \int_a^b f_m(s)\,\mathrm{d}s < \varepsilon$$

for $n > N$ and therefore also

$$\left| \int_a^b g_m(s)\,\mathrm{d}s \right| = \left| \sum_{j=1}^{l} \int_{\beta_j}^{\gamma_j} f_m(s)\,\mathrm{d}s \right| < \varepsilon,$$

i.e. the condition (UC) from Theorem 8 is satisfied and by this proposition we obtain the equiintegrability of the sequence of functions $f_n\colon [a,b] \to \mathbb{R}$, $n = 1,2,\ldots$. □

R e m a r k. For a given sequence of nonnegative functions $f_n\colon [a,b] \to \mathbb{R}$ which are integrable over $[a,b]$ and such that $\lim\limits_{n\to\infty} f_n(\tau) = 0$ we have proved in fact that (L) $\Rightarrow$ (UC). Using the results given in Remark 12, we have

$$\text{(UC)} \Leftrightarrow \text{(EI)} \Leftrightarrow \text{(C)} \Leftrightarrow \text{(L)}$$

for a sequence of nonnegative functions $f_n\colon [a,b] \to \mathbb{R}$.

For a given function $f\colon [a,b]\to\mathbb{R}$ and $\tau\in[a,b]$ define

$$\begin{aligned} f_+(\tau) &= \max(f(\tau),0)=\tfrac{1}{2}(|f(\tau)|+f(\tau)),\\ f_-(\tau) &= \max(-f(\tau),0)=\tfrac{1}{2}(|f(\tau)|-f(\tau)). \end{aligned}$$

Clearly

$$0\leqslant f_+(\tau)\leqslant |f(\tau)| \quad\text{and}\quad 0\leqslant f_-(\tau)\leqslant |f(\tau)|$$

for $\tau\in[a,b]$.

If the functions f and $|f|$ are integrable then, by Proposition 2 and the above definition the functions f_+ and f_- are also integrable.

Let a sequence of functions $f_m\colon [a,b]\to\mathbb{R}$, $m=1,2,\dots$ be given.

If for some $\tau\in[a,b]$ we have $f_n(\tau)\to 0$ for $n\to\infty$ then also $f_{n+}(\tau)\to 0$ and $f_{n-}(\tau)\to 0$ for $n\to\infty$.

14. Proposition. *Let functions $f_n\colon [a,b]\to\mathbb{R}, n=1,2,\dots$ be given where the integrals $\int_a^b f_n(s)\,\mathrm{d}s$, $\int_a^b |f_n(s)|\,\mathrm{d}s$ exist for $n=1,2,\dots$. Assume that*

$$\lim_{n\to\infty} f_n(\tau)=0 \tag{6}$$

for $\tau\in[a,b]$ and

$$\lim_{n\to\infty}\int_a^b |f_n(s)|\,\mathrm{d}s=0. \tag{21}$$

Then the sequence of functions $f_n\colon [a,b]\to\mathbb{R}$, $n=1,2,\dots$ is equiintegrable.

Proof. Let us consider the sequence of functions $f_{n+}\colon [a,b]\to\mathbb{R}$, $n=1,2,\dots$.

We have $f_{n+}(\tau)\geqslant 0$ for $\tau\in[a,b]$, the integrals $\int_a^b f_{n+}(s)\,\mathrm{d}s$, $n=1,2,\dots$ exist and $f_{n+}(\tau)\leqslant |f_n(\tau)|$ for $\tau\in[a,b]$. Hence (see Proposition 2)

$$0\leqslant \int_a^b f_{n+}(s)\,\mathrm{d}s\leqslant \int_a^b |f_n(s)|\,\mathrm{d}s$$

and therefore

$$\lim_{n\to\infty}\int_a^b f_{n+}(s)\,\mathrm{d}s=0.$$

Moreover, by definition

$$\lim_{n\to\infty} f_{n+}(\tau)=0$$

for $\tau\in[a,b]$.

By Proposition 13 we obtain that the sequence of functions $f_{n+}\colon [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ is equiintegrable.

Similarly it can be shown that also the sequence of functions $f_{n-}\colon [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ is equiintegrable. Hence for every $\varepsilon > 0$ there is a gauge $\delta\colon [a,b] \to (0, +\infty)$ such that

$$\left| \sum_{j=1}^{k} f_{n+}(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b f_{n+}(s)\,\mathrm{d}s \right| < \tfrac{\varepsilon}{2}$$

and

$$\left| \sum_{j=1}^{k} f_{n-}(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b f_{n-}(s)\,\mathrm{d}s \right| < \tfrac{\varepsilon}{2}$$

for every $n \in \mathbb{N}$ and any δ-fine P-partition $\{\alpha_0, \tau_1, \alpha_1, \tau_2, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$.

Hence

$$\begin{aligned} &\left| \sum_{j=1}^{k} f_n(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b f_n(s)\,\mathrm{d}s \right| \\ &\qquad \leqslant \left| \sum_{j=1}^{k} f_{n+}(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b f_{n+}(s)\,\mathrm{d}s \right| \\ &\qquad\quad + \left| \sum_{j=1}^{k} f_{n-}(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b f_{n-}(s)\,\mathrm{d}s \right| \\ &\qquad < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for every $n \in \mathbb{N}$, and by Definition 5 this yields the equiintegrability of the sequence (f_n). $\square$

15. Corollary. *Let functions $f, f_n\colon [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ be given where the integrals $\int_a^b f(s)\,\mathrm{d}s$, $\int_a^b |f(s)|\,\mathrm{d}s$ $\int_a^b f_n(s)\,\mathrm{d}s$, $\int_a^b |f_n(s)|\,\mathrm{d}s$ exist for $n = 1, 2, \ldots$. Assume that*

$$\lim_{n\to\infty} f_n(\tau) = f(\tau) \tag{22}$$

for $\tau \in [a,b]$ and

$$\lim_{n\to\infty} \int_a^b |f_n(s) - f(s)|\,\mathrm{d}s = 0. \tag{23}$$

Then the sequence of functions $f_n\colon [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ is equiintegrable.

P r o o f. Let us set $g_n = f_n - f\colon [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$. Then the functions $f_n - f$ are absolutely integrable and the sequence g_n, $n = 1, 2, \ldots$ satisfies the assumptions of Proposition 13 and by this proposition the sequence $g_n = f_n - f$, $n = 1, 2, \ldots$ is equiintegrable. Hence also the sequence f_n, $n = 1, 2, \ldots$ is equiintegrable. □

R e m a r k. It is well known that $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable (McShane integrable, see [3]) if and only if both f and $|f|$ are Kurzweil-Henstock integrable in the sense of Definition 1 (see e.g. [7], p. 22).

Therefore for example Corollary 15 can be reformulated as follows: *If the functions* $f, f_n\colon [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ *are Lebesgue integrable and*

$$\lim_{n\to\infty} f_n(\tau) = f(\tau)$$

for $\tau \in [a,b]$ *and*

$$\lim_{n\to\infty} \int_a^b |f_n(s) - f(s)| \, \mathrm{d}s = 0$$

is satisfied, then the sequence of functions $f_n\colon [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ *is equiintegrable.*

16. E x a m p l e. Let us define a sequence of functions $g_n\colon [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$ such that

$$g_n(\tau) = n \text{ for } \tau \in (0, \tfrac{1}{n})$$
$$g_n(\tau) = 0 \text{ for } \tau \in [0,1] \setminus (0, \tfrac{1}{n}).$$

It is easy to check that $\int_0^1 g_n(s)\,\mathrm{d}s = 1$ for every $n \in \mathbb{N}$.

Define further

$$f_n(\tau) = g_n(\tau) \text{ for } \tau \in [0,1]$$
$$f_n(\tau) = -g_n(-\tau) \text{ for } \tau \in [-1,0].$$

For the sequence of functions $f_n\colon [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$ we have

$$\lim_{n\to\infty} f_n(\tau) = 0 \quad \text{for every } \tau \in [-1,1],$$
$$\int_{-1}^1 f_n(s)\,\mathrm{d}s = 0 \quad \text{for every } n \in \mathbb{N}$$

and

$$\int_{-1}^1 |f_n(s)|\,\mathrm{d}s = 2 \quad \text{for every } n \in \mathbb{N}.$$

The sequence $f_n\colon [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$ is not equiintegrable. If it were equiintegrable, then also its restriction to the interval $[0,1]$, i.e. the functions $g_n\colon [0,1] \to \mathbb{R}$ would be equiintegrable and by Theorem 4 we would obtain $\lim\limits_{n\to\infty} \int_0^1 g_n(s)\,\mathrm{d}s = 0$ but this contradicts the fact that $\int_0^1 g_n(s)\,\mathrm{d}s = 1$ for every $n \in \mathbb{N}$.

This example shows that in Proposition 14 the condition (21) cannot be replaced by

$$\lim_{n\to\infty} \int_a^b f_n(s)\,\mathrm{d}s = 0$$

and also that (23) in Corollary 15 cannot be replaced by

$$\lim_{n\to\infty} \int_a^b f_n(s) = \int_a^b f(s)\,\mathrm{d}s.$$

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