ON SOLVABILITY OF NONLINEAR OPERATOR EQUATIONS AND EIGENVALUES OF HOMOGENEOUS OPERATORS

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(Received September 27, 1995)

Abstract. Notions as the numerical range $W(S,T)$ and the spectrum $\sigma(S,T)$ of couple (S,T) of homogeneous operators on a Banach space are used to derive theorems on solvability of the equation $Sx - \lambda Tx = y$. Conditions for the existence of eigenvalues of the couple (S,T) are given.

Keywords: Banach and Hilbert space, homogeneous, polynomial, symmetric, monotone operator, numerical range, spectrum, eigenvalue.

MSC 1991: 47H15

Introduction

The development of modern science and technology is associated with passing from linear to nonlinear models. In mathematics and particularly in functional analysis it reflects the tendency for studying nonlinear operator equations. In the present paper we deal with a special class of nonlinear operator equations involving homogeneous and polynomial operators. These operators arise as a natural generalization of linear operators and conserve a number of their properties. For example, any polynomial operator is continuous if and only if it is bounded. Also notions as the norm, symmetry, selfadjointness, spectrum and numerical range can be transfered from linear theory to homogeneous operators. We have introduced these notions in [3], [4], [9] and [11] (in rather different way than it was done in [1], [12], [13], [14], [15]). In this paper we use these notions and their properties to derive some existence theorems for equations of the type $Sx - \lambda T = y$ with a couple (S,T) of homogeneous operators from a Banach space into its dual. Some conditions for the existence of eigenvalues and eigenvectors of the couple (S,T) (i.e., the existence of nontrivial solutions of the equation $Sx - \lambda T = 0$) both in the symmetric and nonsymmetric case are given. Finally we show, similarly as in the linear case, that the set of eigenvalues of the couple of selfadjoint homogeneous operators creates at most a countable set. These results can be applied to various boundary value problems as, for example, in nonlinear elasticity, fluid mechanics and others.

1. Notation and definitions

Throughout this paper let X denote a Banach space and X^* its dual space. By the symbol $\langle \cdot , \cdot \rangle$ we mean the pairing between X and X^*. In the case of a Hilbert space we use the same symbol for the inner product. For the norm or weak convergence of a sequence $\{x_n\} \subset X$ to a point $x_0 \in X$ we use the symbols $x_n \to x_0$ or $x_n \stackrel{w}{\to} x_0$, respectively.

Let $\mathbb{R}$ and $\mathbb{C}$ be the spaces of real and complex numbers, respectively. Further, we denote $S_1(0) = \{x \in X \colon \|x\| = 1\}$, the unit sphere in X.

Definition 1.1. We say that an operator F from X into a Banach space Y is

(a) a *positive homogeneous operator of a degree* k if there is a number $k \in \mathbb{R}$ such that $F(tx) = t^k \cdot F(x)$ for any $x \in X$ and any $t \in \mathbb{R}$, $t > 0$.

(b) a *homogeneous operator of a degree* k on a real (complex) space X if k is an integer and the equality $F(tx) = t^k \cdot F(x)$ holds for any $t \in \mathbb{R}$ ($t \in \mathbb{C}$), $t \neq 0$ and any $x \in X$.

(c) For any continuous positive homogeneous operator $F \colon X \to Y$ of a degree $k > 0$ we define the norm by

$$\|F\| = \sup_{x \in S_1(0)} \|F(x)\|.$$

Remark 1.2. It is easy to show that for any $x \in X$, the estimation $\|F(x)\| \leqslant \|F\| \cdot \|x\|^k$ holds.

Definition 1.3. We say that a homogeneous operator P of a degree $k \geqslant 1$ from X into a Banach space Y is a *homogeneous polynomial* operator if there is a k-linear symmetric operator $\mathscr{P} \colon X \times X \times \ldots \times X \to X$, (i.e., $\mathscr{P}(x_1, x_2, \ldots, x_k)$ is linear in any variable x_j, $j = 1, 2, \ldots, k$ and does not change its values under arbitrary permutation of all variables) such that $\mathscr{P}(x, x, \ldots, x) = P(x)$ for any $x \in X$.

The operator $\mathscr{P}$ is called the *polar operator* to the operator P. For any continuous k-linear operator $\mathscr{P}$ we define the norm by

$$\|\mathscr{P}\| = \sup_{\substack{x_i \in S_1(0),\\ i=1,2,\ldots,k}} \|\mathscr{P}(x_1, x_2, \ldots, x_k)\|.$$

R e m a r k 1.4. Polynomial operators have some properties which are similar to the properties of linear operators. For example, it is easy to prove that for any polynomial operator P the following conditions are equivalent.

(a) P is continuous at the point zero,
(b) P is continuous at any point,
(c) P is bounded (i.e., P has a finite norm $\|P\|$),
(d) P is uniformly continuous on every bounded set,
(e) P is Frechet differentiable at any point.

Definition 1.5. We say that a homogeneous operator $F\colon X \to X^*$ is

a) *positive* if $\langle Fx, x\rangle > 0$ holds for any $x \in X$, $x \neq 0$,
b) *positively defined* if there is a number $c \in \mathbb{R}$, $c > 0$ such that $\inf\limits_{x\in S_1(0)} \operatorname{Re}\langle Fx, x\rangle = c > 0$.

Definition 1.6. An *operator* $F\colon X \to Y$ is called *hemi-continuous* at $x_0 \in X$, if for any sequence $\{t_n\} \subset \mathbb{R}$, $t_n \to 0$, and for any $h \in X$ we have $(F(x_0+t_nh)) \xrightarrow{w} F(x_0)$ in the weak topology of the space Y. We say that the *Gateaux-derivative* F' of a differentiable operator $F\colon X \to Y$ is *hemi-continuous* at a point $x \in X$, if for any sequence $\{t_n\} \subset [0, +\infty)$ such that $t_n \to 0$ and for arbitrary points $h \in X$, $y \in X$ the sequence $\big\{F'(x+t_nh)y\big\} \subset X^*$ converges in the weak*-topology of the space X^* to the point $F'(x)y \in X^*$.

A notion of the adjoint operator to a nonlinear operator was defined and some of its properties were studied in papers [3] and [4].

Definition 1.7. ([3], Definition 3.) Let $D \subset X$ be an open set which is star-shaped with respect to the origin (i.e., for any $x \in D$ and all $t \in [0,1]$ we have $tx \in D$). Let an operator $F\colon D \subset X \to X^*$ have the Gateaux-derivative $F'(x)$ at any point $x \in D$ and let F satisfy the following conditions:

(1) $F(0) = 0$
(2) The function $\langle F'(tx), x\rangle$ of the variable $t \in [0,1]$ is integrable for arbitrary (but fixed) points $x \in D$, $h \in X$.

Let us suppose, further, that for any $x \in D$ there exists a unique point $x^*(x) \in X^*$ such that for all $h \in X$ we have

$$\langle x^*(x), h\rangle = \int_0^1 \langle F'(tx)h, x\rangle \,\mathrm{d}t.$$

Then we call the operator $F^*\colon D \subset X \to X^*$ defined for $x \in D$ by $F^*(x) = x^*(x)$ the *adjoint operator* to operator F.

R e m a r k 1.8. In the case of a real Banach space X the adjoint operator F^* from Definition 1.7 can be written in the form

$$F^*(x) = \int_0^1 \bigl[F'(tx)\bigr]^*(x)\,\mathrm{d}t,$$

where $\bigl[F'(tx)\bigr]^*$ denotes the adjoint operator to the continuous linear operator $F'(tx)$.

R e m a r k 1.9. According to [4] (Theorem 2.6) the adjoint operator F^* to a nonlinear operator F exists if F satisfies the conditions (1) and (2) from Definition 1.7 and, moreover, F has at any point $x \in X$ a hemi-continuous Gateaux-derivative. Then both F and F^* are hemi-continuous and the following estimation holds:

$$\|F^*(x)\| \leqslant \|x\| \int_0^1 \|F'(tx)\|\,\mathrm{d}t, \quad x \in X.$$

The following proposition is a direct consequence of Definition 1.7 and results from [3] and [4] applied to the class of homogeneous operators.

Proposition 1.10. *Let $F\colon X \to X^*$ be a homogeneous operator of a degree $k \geqslant 1$ having a hemi-continuous Gateaux-derivative F'. Then for any $x \in X$ the following assertions hold:*

(1) $F^*(x) = \frac{1}{k}\bigl[F'(x)\bigr]^* x$, *where F^* is the adjoint operator to the operator F and $\bigl[F'(x)\bigr]^*$ is the adjoint operator to the continuous linear operator $F'(x)$.*

(2) $F(x) = H(x) + R(x)$. *Here H, $R\colon X \to X^*$ are hemi-continuous operators which can be written as*

$$H(x) = \frac{1}{k+1}\bigl[F(x) + kF^*(x)\bigr],$$
$$R(x) = \frac{k}{k+1}\bigl[F(x) - F^*(x)\bigr].$$

(3) *H is a potential operator, $H = \operatorname{grad}\varphi$, where*

$$\varphi(x) = \frac{1}{k+1}\langle F(x), x\rangle$$

and the operator R fulfils the equality

$$\langle R(x), x\rangle = \frac{2k\mathrm{i}}{k+1}\operatorname{Im}\bigl\{\langle F(x), x\rangle\bigr\}.$$

(4) $\|F^*(x)\| \leqslant \frac{1}{k}\|F'(x)\| \cdot \|x\|$.

(5) *If S, $T\colon X \to X^*$ are homogeneous operators with their adjoint operators S^*, T^* then*
$$(S - \lambda T)^* = S^* - \overline{\lambda} T^* \text{ for any } \lambda \in \mathbb{C}.$$

Definition 1.11. Let $M \subset X$ be an open set which is starshaped with respect to the origin. We shall say that an operator $F\colon M \to X^*$ satisfying $F(0) = 0$ is *symmetric* on M if its adjoint operator F^* exists and, for any $x \in M$, $F(x) = F^*(x)$. Let X be a Hilbert space. We say that $F\colon X \to X^*$ is *selfadjoint* if F is symmetric on the whole space X.

Definition 1.12. ([9] Definition 3.1.) Let S, $T\colon X \to X^*$ be homogeneous operators of a degree k. By the *numerical range of the couple* (S,T) we mean the set $W(S,T)$ of complex numbers defined by
$$W(S,T) = \left\{ \frac{\langle Sx, x\rangle}{\langle Tx, x\colon \rangle} : x \in S_1(0), \langle Tx, x\rangle \neq 0 \right\}.$$
It is evident that if X is a Hilbert space, S is a continuous linear operator and T is the identity operator then we obtain the well-known Hausdorff's and Toeplitz's definition of the numerical range.

Definition 1.13. ([11] Definition 3.1) Let S, $T\colon X \to Y$ be positive homogeneous operators. By the *approximative spectrum* (briefly spectrum) of the couple (S,T) we understand the set $\sigma(S,T)$ of complex numbers defined by
$$\sigma(S,T) = \left\{ \lambda \in \mathbb{C}\colon \inf_{x \in S_1(0)} \|Sx - \lambda Tx\| = 0 \right\}.$$

Definition 1.14. ([11] Definition 3.8.) We say that $\lambda_0 \in \mathbb{C}$ is an *eigenvalue of a couple* (S,T) of positive homogeneous operators S, $T\colon X \to Y$, if there is a point $x_0 \in S_1(0) \subset X$ such that $Sx_0 - \lambda_0 Tx_0 = 0$. the point x_0 is called the *eigenvector* of the couple (S,T) *related to* λ_0. The set of all eigenvalues of the couple (S,T) is denoted by $\Lambda(S,T)$.

Proposition 1.15. (Proposition 3.14 [11]) *Let S, $T\colon X \to X^*$ be positive homogeneous operators. Let S be continuous and T positively defined. Then the following assertions hold:*

(1) *$W(S,T)$ is a bounded set and for any $\lambda \in W(S,T)$ we have*
$$|\lambda| \leqslant \frac{\|S\|}{c}, \text{ where } c = \inf_{x \in S_1(0)} \operatorname{Re}\left\{\langle Tx, x\rangle\right\}.$$

(2) *If, in addition, S and T are polynomial operators then $W(S,T)$ is a convex set. (A generalization of Hausdorff and Toeplitz theorem on the convexity of numerical range.)*

Definition 1.16. Let $\omega\colon [0,+\infty) \to [0,+\infty)$ be a continuous increasing function such that $\omega(0) = 0$, $\lim_{t\to+\infty} \omega(t) = +\infty$. Then the mapping $J\colon X \to 2^{X^*}$ defined on a real Banach space X by

$$J(0) = 0,$$
$$J(x) = \bigl\{x^* \in X^*\colon \langle x, x^*\rangle = \omega\bigl(\|x\|\bigr) \cdot \|x\|, \|x^*\| = \omega\bigl(\|x\|\bigr), x \neq 0\bigr\}$$

is called the *duality mapping on X with the gauge function ω.*

2. Equations with homogeneous operators and eigenvalues

Proposition 2.1. *Let $S, T\colon X \to X^*$ be homogeneous operators of a degree $k > 0$ and let T be positively defined. Suppose that $\lambda \in \mathbb{C}$ is a number having a positive distance $d > 0$ from the range $W(S,T)$ and let the mapping $(S - \lambda T)$ be surjective. Suppose, further, that there exists a singlevalued branch $(S - \lambda T)^{-1}$ of the inverse operator to $(S - \lambda T)$. Then the following assertions are true.*

(1) *$(S - \lambda T)^{-1}$ is a bounded homogeneous operator of the degree $\frac{1}{k}$ and for any $y \in X^*$ we have*
$$\bigl\|(S - \lambda T)^{-1} y\bigr\| \leqslant \left(\frac{\|y\|}{dc}\right)^{\frac{1}{k}},$$
where
$$c = \inf_{x\in S_1(0)} \operatorname{Re}\bigl\{\langle Tx, x\rangle\bigr\}.$$

(2) *If, in addition, $(S - \lambda T)^{-1}$ is a homogeneous polynomial operator, then it is continuous on X^*.*

Proof. Using the definition of $W(S,T)$ we obtain, by virtue of the assumptions of the present theorem, that for any $x \in S_1(0)$ the inequality

$$\left|\frac{\langle Sx, x\rangle}{\langle Tx, x\rangle} - \lambda\right| \geqslant d$$

holds. This yields $\bigl|\langle Sx - \lambda Tx, x\rangle\bigr| \geqslant d \cdot \bigl|\langle Tx, x\rangle\bigr| \geqslant d \cdot c$ for any $x \in S_1(0)$.

Choosing a nonzero point $x \in X$ and substituting the point $\frac{x}{\|x\|} \in S_1(0)$ into the above inequality we obtain after simple calculations

$$\bigl|\langle Sx - \lambda Tx, x\rangle\bigr| \geqslant d \cdot c \cdot \|x\|^{k+1}.$$

This implies that

$$\|Sx - \lambda Tx\| \geqslant d \cdot c \cdot \|x\|^k \text{ for any } x \in X.$$

Now, let $y \in X^*$ be an arbitrary point. Due to surjectivity there is $x \in X$ such that $y = Sx - \lambda Tx$ and $x = (S - \lambda T)^{-1}y$. Thus $\|y\| \geqslant d \cdot c \cdot \|(S - \lambda T)^{-1}y\|^k$, which gives the inequality in the assertion (1).

To show that $(S-\lambda T)^{-1}$ is homogeneous of the degree k^{-1} let us put $F = S-\lambda T$. Then, due to homogeneity of the operator F, for any $t \in \mathbb{R}$, $t > 0$ we have

$$F\big(t^{\frac{1}{k}} \cdot x\big) = t \cdot F(x) = t \cdot y,$$

so that

$$F^{-1}(ty) = F^{-1}\big[F\big(t^{\frac{1}{k}} \cdot x\big)\big] = t^{\frac{1}{k}} \cdot x = t^{\frac{1}{k}} \cdot F^{-1}(y).$$

The inequality in (1) and the homogeneity provide the boundedness of the operator $(S - \lambda T)^{-1}$ and assertion (1) is proved.

The assertion (2) follows from Remark 1.4. □

Theorem 2.2. *Let X be a real reflexive Banach space and let $S, T\colon X \to X^*$ be a hemi-continuous homogeneous operators of a degree $k \geqslant 1$. Let us suppose that there is a $\lambda \in \mathbb{R} \setminus \sigma(S,T)$ such that the operator $S - \lambda T$ is monotone.*

Then the following assertions hold.

(1) *For any $y^* \in X^*$ there is a solution $x_0 \in X$ of the equation $Sx - \lambda Tx = y^*$.*

(2) *$x_0 = 0$ if and only if $y^* = 0$.*

Proof. Because $\lambda \notin \sigma(S,T)$, there is a positive real number $C > 0$ such that for any $x \in S_1(0)$,

$$\|Sx - \lambda Tx\| \geqslant c.$$

Using homogeneity we immediately obtain that

$$\|Sx - \lambda Tx\| \geqslant c \cdot \|x\|^k \text{ for any } x \in X,$$

so that

$$\lim_{\|x\|\to\infty} \|Sx - \lambda Tx\| = +\infty.$$

Hence, the operator $S-\lambda T$ is weakly coercive. Being hemi-continuous and defined on the whole space X the operator $S-\lambda T$ is maximal monotone, and applying a theorem on surjectivity of maximal monotone operators (see for example [2]) we obtain the assertion (1). Hence, for any $y^* \in X^*$ there is $x_0 \in X$ such that $(S - \lambda T)(x_0) = y^*$.

To prove (2) we first realize that if $x_0 = 0$, then $y^* = 0$ because S and T are homogeneous operators. Now, let $y^* = 0$ and suppose $x_0 \neq 0$. Then the point $\frac{x_0}{\|x_0\|} \in S_1(0)$ is an eigenvector and λ is an eigenvalue of the couple (S,T). Hence $\lambda \in \sigma(S,T)$ which contradicts the assumption of the theorem and the proof is complete. □

Corollary 2.3. *Let X be a real reflexive Banach space and let $S,T\colon X\to X^*$ be homogeneous polynomial operators. Suppose S is continuous and T is positively defined.*

Then for any $\lambda\in\mathbb{R}\setminus\overline{W(S,T)}$ for which the operator $S-\lambda T$ is strongly monotone there is a singlevalued inverse operator $(S-\lambda T)^{-1}$ which is defined and bounded on X^*.

Proof. According to Theorem 3.13 in [11] we have $\sigma(S,T)\subset\overline{W(S,T)}$ so that $\lambda\notin\sigma(S,T)$. Being polynomial, the operators S and T are hemi-continuous and applying Theorem 2.2, we obtain a solution $x_0\in X$ of the equation $Sx-\lambda Tx=y^*$ for any $y^*\in X^*$. This solution must be unique because $S-\lambda T$ is strongly monotone. Using Proposition 2.1 we complete the proof. ☐

Remark 2.4. If S, $T\colon X\to X$ are linear operators on a real Hilbert space X then the operator $S-\lambda T$ is strictly monotone if and only if for any $x\in S_1(0)$ the inequality $\langle Sx-\lambda Tx,x\rangle>0$ holds. Moreover, if S, T are selfadjoint and T is positively defined, then the last inequality is satisfied for any $\lambda<m$ where

$$m=\inf_{x\in S_1(0)}\frac{\langle Sx,x\rangle}{\langle Tx,x\rangle}$$

and the opposite inequality holds for any $\lambda>M$, where

$$M=\sup_{x\in S_1(0)}\frac{\langle Sx,x\rangle}{\langle Tx,x\rangle}.$$

In the latter case the operator $S-\lambda T$ is strongly monotone for any $\lambda<m$ and the operator $-(S-\lambda T)$ is strongly monotone for any $\lambda>M$. According to the definition of $W(S,T)$ for selfadjoint operators we have $\overline{W(S,T)}\subset[m,M]$ so that applying Corollary 2.3 we obtain the existence and continuity of the inverse operator $(S-\lambda T)^{-1}$ for any $\lambda\in\mathbb{R}\setminus[m,M]$.

In [10] (Theorem 5.25) conditions are shown under which the aproximative spectrum $\sigma(S,T)$ of a couple (S,T) of homogeneous symmetric operators is nonempty and contains nonzero points. Another simple condition for the spectrum $\sigma(S,T)$ of homogeneous operators(not necessarily symmetric) to be nonempty is given in the following proposition.

Proposition 2.5. *Let X be a Hilbert space and let $S,T\colon X\to X$ be continuous homogeneous operators which are not identically zero and satisfy the condition*

$$(*)\qquad \sup_{x\in S_1(0)}|\langle Sx,Tx\rangle|=\|S\|\cdot\|T\|.$$

Then the aproximative spectrum $\sigma(S,T)$ contains a nonzero point $\lambda \in \sigma(S,T)$, $\lambda \neq 0$ such that

$$|\lambda| = \frac{\|S\|}{\|T\|}.$$

P r o o f. If the condition $(*)$ is fulfilled then, according to the definition of supremum, for any positive real number $\varepsilon > 0$ there is a point $x_0 \in S_1(0)$ such that

$$|\langle Sx_0, Tx_0\rangle| > \|S\| \cdot \|T\| - \varepsilon.$$

Choosing $\varphi \in [0, 2\pi)$ such that $\langle Sx_0, Tx_0\rangle = |\langle Sx_0, Tx_0\rangle| \cdot \mathrm{e}^{\mathrm{i}\varphi}$ and denoting $\lambda = \frac{\|S\|}{\|T\|} \cdot \mathrm{e}^{\mathrm{i}\varphi}$, we obtain

$$\begin{aligned}\|Sx_0 - \lambda Tx_0\|^2 &= \|Sx_0\|^2 + \|Tx_0\|^2 \cdot |\lambda|^2 - \Big[\overline{\lambda}\langle Sx_0, Tx_0\rangle + \lambda\overline{\langle Sx_0, Tx_0\rangle}\Big] \\ &< 2\|S\|^2 + 2\frac{\|S\|}{\|T\|}\big(\varepsilon - \|S\| \cdot \|T\|\big) = 2\frac{\|S\|}{\|T\|} \cdot \varepsilon.\end{aligned}$$

Hence $\inf\limits_{x \in S_1(0)} \|Sx - \lambda Tx\| = 0$, so that according to Definition 1.13, $\lambda \in \sigma(S,T)$ and $|\lambda| = \frac{\|S\|}{\|T\|} > 0$. The proof is complete. □

Recall that the condition $(*)$ in Proposition 2.5 is satisfied if S is a symmetric homogeneous polynomial operator on a Hilbert space X and T is the duality mapping defined for any $x \in X$ by $Tx = c \cdot \|x\|^{k-1} \cdot x$ where $c \in \mathbb{R}$ is a positive constant. (See [11] Theorem 3.15.)

An example of nonsymmetric operators S, T satisfying the condition $(*)$ is the following.

E x a m p l e 2.6. Let S, $T\colon L^2[0,1] \to L^2[0,1]$ be homogeneous operators of degree 2 defined by

$$Sx = \int_0^1 a(s) \cdot x^2(t)\,\mathrm{d}t = a(s) \cdot \|x\|^2,$$

$$Tx = \int_0^1 b(s) \cdot x^2(t)\,\mathrm{d}t = b(s) \cdot \|x\|^2$$

where $a(s)$, $b(s) \in L^2[0,1]$.

Then the condition $(*)$ will be satisfied if $b(s)=k\cdot a(s)$, where $k\in\mathbb{R}$ is a constant. Indeed, in this case we obtain

$$\|Sx\|=\|\int_0^1 a(s)\cdot x^2(t)\,\mathrm{d}t\|=\|a\|\cdot\|x\|^2\Rightarrow\|S\|=\sup_{x\in S_1(0)}\|Sx\|=\|a\|,$$
$$\|Tx\|=\|\int_0^1 b(s)\cdot x^2(t)\,\mathrm{d}t\|=\|b\|\cdot\|x\|^2\Rightarrow\|T\|=\sup_{x\in S_1(0)}\|Tx\|=\|b\|,$$
$$\begin{aligned}\sup_{x\in S_1(0)}|\langle Sx,Tx\rangle| &=\sup_{x\in S_1(0)}|\langle a(s)\cdot\|x\|^2,b(s)\cdot\|x\|^2\rangle|\\ &=|\langle a(s),b(s)\rangle|=|\langle a(s),k\cdot a(s)\rangle|=|k|\cdot\|a\|^2\\ &=\|a\|\cdot\|b\|=\|S\|\cdot\|T\|.\end{aligned}$$

At the same time we have

$$S'(x)h=\int_0^1 2a(s)x(t)h(t)\,\mathrm{d}t,\qquad \big[S'(x)\big]^*h=\int_0^1 2a(t)x(s)h(t)\,\mathrm{d}t,$$

hence, by virtue of Proposition 1.10, we obtain

$$S^*x=\tfrac{1}{2}\big[S'(x)\big]^*x=\int_0^1 a(t)x(s)x(t)\,\mathrm{d}t=x(s)\int_0^1 a(t)x(t)\,\mathrm{d}t\neq Sx$$

and thus the operator S (and similarly the operator T) is not symmetric.

Other conditions for the existence of nonzero eigenvalues of the couple (S,T) are given in [11] Theorem 3.16.

Theorem 2.7. ([10] theorem 5.24) *Let X be a Hilbert space and let $S,T\colon X\to X$ be homogeneous operators of a degree $k\geqslant 1$ which are both symmetric on $S_1(0)\subset X$. Suppose, further, that S and T have hemi-continuous Gateaux-derivatives on $S_1(0)$ and let T be positive.*

Then the following assertions hold:

(1) *The couple (S,T) has only a real aproximative spectrum $\sigma(S,T)$ which lies in the interval $[m,M]$, where*

$$m=\inf_{x\in S_1(0)}\frac{\langle Sx,x\rangle}{\langle Tx,x\rangle},\qquad M=\sup_{x\in S_1(0)}\frac{\langle Sx,x\rangle}{\langle Tx,x\rangle}$$

and both the boundary points m and M belong to the spectrum $\sigma(S,T)$.

(2) *If, in addition, there is a point $y \in S_1(0)$ such that the number $\lambda = \frac{\langle Sy,y\rangle}{\langle Ty,y\rangle}$ is equal to m or M then λ is an eigenvalue of the couple (S,T) with y its eigenvector.*

Theorem 2.8. *Let X be a real Hilbert space and let $S, T\colon X \to X$ be selfadjoint homogeneous operators which are continuously Gateaux differentiable on X. Suppose that at least one of the following assumptions is satisfied:*

(1) *S is a monotone not identically vanishing operator and T is a compact positively defined operator*
(2) *S is a compact not identically vanishing operator and T is a strongly monotone operator.*

Then the couple (S,T) has a nonzero eigenvalue.

Proof. According to Theorem 2.7 it suffices to show that the functional $f(x) = \frac{\langle Sx,x\rangle}{\langle Tx,x\rangle}$ assumes its minimum (or maximum) on $S_1(0)$. Using homogeneity we can easily transform this problem to the problem of finding extremes of $g(x) = \langle Sx,x\rangle$ on the set $\mathcal{U}$ defined by $\mathcal{U} = \bigl\{x \in X\colon \langle Tx,x\rangle = 1\bigr\}$.

Suppose that the assumption (a) is satisfied. Then, according to [19] (Example 8.7, p. 110), the functional $g(x)$ is weakly lower semicontinuous and, according to [19] (Lemma 8.7, p. 111), the set $\mathcal{U}$ is weakly closed. Because T is positively defined, $\mathcal{U}$ is also bounded and thus the functional $g(x)$ assumes its minimum on $\mathcal{U}$. (See [19], theorem 9.2, p. 114.)

There is a point $y \in S_1(0)$ such that $f(y) = \inf\limits_{x\in S_1(0)} f(x)$, and applying Theorem 2.7 we obtain the assertion. Suppose that (b) is satisfied. Then the functional $g(x)$ is weakly continuous and the set $\mathcal{V} = \bigl\{x \in X\colon \langle Tx,x\rangle \leqslant 1\bigr\}$ is convex and bounded because T is strongly monotone. This implies that $\mathcal{V}$ is weakly compact. If the functional g attains positive (negative) values then it attains its maximum (minimum) at a nonzero point $x_0 \in \mathcal{V}$.

At the same time $x_0 \in \mathcal{U}$ because, if this is not the case, then

$$\sup_{x\in\mathcal{V}} g(x) = g(x_0) = \langle Sx_0,x_0\rangle, \text{ where } 0 < \langle Tx_0,x_0\rangle < 1.$$

Putting $x_1 = \frac{x_0}{\langle Tx_0,x_0\rangle^{\frac{1}{k+1}}}$ we obtain $\langle Tx_1,x_1\rangle = 1$ and simultaneously $g(x_1) = \frac{g(x_0)}{\langle Tx_0,x_0\rangle} > g(x_0)$, which is impossible. Hence the functional $f(x)$ must attain its extreme on $S_1(0)$ and the proof follows as in the previous case. $\square$

Proposition 2.9. *Let S, $T\colon X \to X$ be homogeneous operators on a Hilbert space X mapping every weakly convergent sequence in $S_1(0) \subset X$ into the norm* *convergent sequence and satisfying the condition* $(*)$ *from Proposition* 2.5. *Then the couple* (S,T) *has a nonzero eigenvalue* λ *such that* $|\lambda| = \frac{\|S\|}{\|T\|}$.

Proof. First we show that the functional $g(x) = \langle Sx, Tx\rangle$ is weakly continuous on $S_1(0)$. Indeed, for a point $x_0 \in S_1(0)$ and any sequence $\{x_n\} \subset S_1(0)$, $x_n \xrightarrow{w} x_0$ we have $\|Sx_n - Sx_0\| \to 0$ and $\|Tx_n - Tx_0\| \to 0$.

Further,

$$\begin{aligned}
|\langle Sx_n, Tx_n\rangle - \langle Sx_0, Tx_0\rangle| &= |\langle Sx_n - Sx_0, Tx_0\rangle + \langle Sx_n, Tx_n - Tx_0\rangle| \\
&\leqslant \|Sx_n - Sx_0\| \cdot \|Tx_0\| + \|Sx_n\| \cdot \|Tx_n - Tx_0\| \\
&\leqslant \|Sx_n - Sx_0\| \cdot \|T\| + \|S\| \cdot \|Tx_n - Tx_0\|,
\end{aligned}$$

so that $\langle Sx_n, Tx_n\rangle \to \langle Sx_0, Tx_0\rangle$ and thus $g(x)$ is weakly continuous at x_0.

Due to condition $(*)$ from Proposition 2.5 we can find a sequence $\{y_n\} \subset S_1(0)$ such that $|\langle Sy_n, Ty_n\rangle| \to \|S\| \cdot \|T\|$. Using the Eberlein-Smuljan theorem (see [21]) we choose a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \xrightarrow{w} y_0$.

Then $\langle Sy_{n_k}, Ty_{n_k}\rangle \to \langle Sy_0, Ty_0\rangle$ and thus $|\langle Sy_0, Ty_0\rangle| = \|S\| \cdot \|T\|$. Putting $\lambda = \frac{\|S\|}{\|T\|} \cdot \mathrm{e}^{\mathrm{i}\varphi}$, where $\varphi \in [0, 2\pi]$ is such that $\langle Sy_0, Ty_0\rangle = |\langle Sy_0, Ty_0\rangle| \cdot \mathrm{e}^{\mathrm{i}\varphi}$, we obtain an eigenvalue λ of the couple (S,T) with the eigenvector $y_0 \in S_1(0)$.

The structure of the spectrum $\sigma(S,T)$ of a couple of symmetric homogeneous operators is, in a sense, similar to the structure of the spectrum of linear selfadjoint operators as the next proposition shows. □

Proposition 2.10. *Let* $S,T\colon X \to X$ *be symmetric homogeneous operators of a degree* $k > 0$ *on a Hilbert space* X *and let* T *be positive. Then for any nonzero eigenvalue* $\lambda_0 \in \Lambda(S,T)$ *there exists a neighbourhood* $U(\lambda_0) \subset \mathbb{R}$ *such that*

$$U(\lambda_0) \cap \Lambda(S,T) = \{\lambda_0\}.$$

Proof. Let $x_0 \in S_1(0)$ be an eigenvector belonging to the eigenvalue $\lambda_0 \neq 0$.

Suppose that equation $Sx - \lambda Tx = 0$ has a parametric solution $\big[\lambda(t), x(t)\big]$ defined and differentiable in a neighbourhood $\mathcal{U}(0) \subset \mathbb{R}$ of the origin such that $\lambda(0) = \lambda_0, x(0) = x_0$. We shall show that $\lambda(t)$ must be a constant function.

Consider a mapping $F\colon \mathbb{R} \times \mathbb{R} \times S_1(0) \to \mathbb{R}$ defined by the equation $F(t,\lambda,x) = \langle Sx, x\rangle - \lambda\langle Tx, x\rangle$. Then for the function

$$f(s) = F(t, \lambda_0 + s, x_0 + sh), \quad s \in \mathbb{R}, h \in X$$

we obtain

$$\begin{aligned}
f'(0) &= \langle S'(x_0)h, x_0\rangle + \langle Sx_0, h\rangle - \lambda_0\big[\langle T'(x_0)h, x_0\rangle + \langle Tx_0, h\rangle\big] - \langle Tx_0, x_0\rangle \\
&= (k+1) \cdot \langle Sx_0 - \lambda_0 Tx_0, h\rangle - \langle Tx_0, x_0\rangle = -\langle Tx_0, x_0\rangle \neq 0.
\end{aligned}$$

This implies that by using the Implicit Function Theorem we can find a positive real number $\varepsilon > 0$ and functions $\lambda\colon (-\varepsilon, +\varepsilon) \to \mathbb{R}$, $x\colon (-\varepsilon, +\varepsilon) \to S_1(0)$ which are both differentiable on the interval $(-\varepsilon, +\varepsilon)$, satisfying the identity

$$(**) \qquad \langle Sx(t) - \lambda(t) \cdot Tx(t), x(t)\rangle = 0,$$

or, equivalently,

$$\lambda(t) = \frac{\langle Sx(t), x(t)\rangle}{\langle Tx(t), x(t)\rangle}$$

for any $t \in (-\varepsilon, +\varepsilon)$, $\lambda(0) = \lambda_0$, $x(0) = x_0$. Using symmetry and homogeneity of the operators S, T we obtain after some calculations

$$\begin{aligned}\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} &= \frac{(k+1)}{\langle Tx(t), x(t)\rangle}\Big\langle Sx(t) - \frac{\langle Sx(t), x(t)\rangle}{\langle Tx(t), x(t)\rangle}Tx(t), \frac{\mathrm{d}x(t)}{\mathrm{d}t}\Big\rangle \\ &= \frac{(k+1)}{\langle Tx(t), x(t)\rangle}\Big\langle Sx(t) - \lambda(t)Tx(t), \frac{\mathrm{d}x(t)}{\mathrm{d}t}\Big\rangle.\end{aligned}$$

Due to $(**)$ we have $\frac{\mathrm{d}\lambda}{\mathrm{d}t} = 0$ for any $t \in (-\varepsilon, +\varepsilon)$, so that $\lambda(t) = \lambda_0$ is a constant function on the interval $(-\varepsilon, +\varepsilon)$. Hence, there is a neighbourhood $\mathcal{U}(\lambda_0)$ of λ_0 such that λ_0 is the unique eigenvalue of the couple (S, T) on $\mathcal{U}(\lambda_0)$ and the proof is complete. $\square$

Theorem 2.11. *Let $S, T\colon X \to X$ be symmetric homogeneous operators of a degree $k > 0$ on Hilbert space X. Suppose that S is completely continuous and T is a duality mapping with the gauge function $\omega(t) = c \cdot t^k$, where $c > 0$ is a constant. Then the following assertion hold:*

(1) *The aproximative spectrum $\sigma(S, T)$ of the couple (S, T) is at most a countable set, possibly with a unique limit point, zero.*
(2) *Every nonzero point $\lambda \in \sigma(S, T)$ is an eigenvalue of the couple (S, T).*

Proof. The assertion (1) is a direct consequence of Proposition 2.10 and the assertion (2) follows from Theorem 3.16 in [11]. $\square$

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