DIRECTLY INDECOMPOSABLE DIRECT FACTORS OF A LATTICE

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Abstract. In this paper we generalize a result of Libkin concerning direct product decompositions of lattices.

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1. Introduction

An element x of a lattice L is called strictly join-irreducible if, whenever $\emptyset \neq X \subseteq L$ and $x = \bigvee X$, then $x \in X$. A lattice in which every element is the join of strictly join-irreducible elements is called a V_1-lattice. Such lattices were investigated in [6].

The following theorem is the main result of [5].

(A) (Libkin [5], Theorem 2.) Every algebraic V_1-lattice is a direct product of directly indecomposable lattices.

A lattice is defined to be algebraic if it is complete and compactly generated (cf. [1]).

When investigating direct product decompositions of a lattice L having the least element 0 we can suppose without loss of generality that all direct factors under consideration are convex sublattices of L containing the element 0 (cf. Section 1 below). The set of all such direct factors of L will be denoted by $D(L)$. The system $D(L)$ is partially ordered by the set-theoretical inclusion.

In the present paper we prove

(B) Let L be a lattice such that

(i) L is conditionally complete and has the least element 0;

(ii) L is compactly generated;

(iii) L is a V_1-lattice.

Then $D(L)$ is atomistic.

From (B) we deduce a generalization of Theorem (A) above; this generalization concerns lattices which are conditionally complete and orthogonally complete. (Cf. Theorem 5.2.)

The method is essentially different from that of [5].

For a lattice L with the least element 0 we denote by $S(L)$ the set of all strictly join-irreducible elements of L. Let us consider the following condition for L.

(α) If $\{x_i\}_{i\in I}$ is a nonempty subset of L, $y = \bigvee\limits_{i\in I} x_i$, $x \in S(L)$, $i(0) \in I$, $x \wedge x_i = 0$ for each $i \in I \setminus \{i(0)\}$ and if $x \leqslant y$, then $x \leqslant x_{i(0)}$.

We show that the assertion of (B) remains valid if the condition (ii) is replaced by the condition (α).

Directly indecomposable direct factors of some types of partially ordered sets were investigated in [3] and [4].

2. Preliminaries

We recall some notions and the notation that we will use in the sequel.

Let L_1 be a lattice and let a be an element of L_1. Then a is called compact if $a \leqslant \bigvee X$ implies that $a \leqslant \bigvee X_1$ for some finite $X_1 \subseteq X$. If each element of L_1 is a join of compact elements, then L_1 is said to be compactly generated.

A lattice L with the least element 0 will be called atomistic if each its nonzero element exceeds some atom. If L is a Boolean algebra, then it is atomistic if every nonzero element of L is a join of atoms.

The notion of the direct product of lattices has the usual meaning. Let L be a lattice with the least element 0 and let φ be an isomorphism of L onto the direct product $A \times B$. If $x \in L$ and $\varphi(x) = (a, b)$, then we denote $a = x(A)$, $b = x(B)$. Put

$$A_0 = \{x \in L\colon x(B) = 0\}, \quad B_0 = \{x \in L\colon x(A) = 0\}.$$

Then A_0 and B_0 are convex sublattices of L with $A_0 \cap B_0 = \{0\}$. Also,

$$B_0 = \{x \in L\colon x \wedge a = 0 \quad \text{for each } a \in A_0\}.$$

The lattice A_0 is isomorphic to A and B_0 is isomorphic to B. The mapping

$$\varphi_0\colon L \longrightarrow A_0 \times B_0$$

defined by $\varphi_0(x) = (a', b')$ where

$$x(A) = a'(A), \quad x(B) = b'(B)$$

is an isomorphism of L onto $A_0 \times B_0$.

Hence without loss of generality we can suppose that $A = A_0$ and $B = B_0$. In such a case we write

$$L = A_0 \times B_0. \tag{1}$$

The lattice L is called directly indecomposable if, whenever (1) is valid, then either $A = \{0\}$ or $B = \{0\}$.

Analogous notation will be applied in the case when we consider the direct product decompositions having more than two factors; we write

$$L = A_1 \times A_2 \times \ldots \times A_n$$

or

$$L = \prod_{i \in I} A_i, \tag{1a}$$

where the power of the set $I \neq \emptyset$ can be arbitrary.

The following lemma can, in fact, be considered a folklore.

2.1. Lemma. *Let $\{0\} \neq A \in D(L)$. Then the following conditions are equivalent:*

(i) *A is directly indecomposable.*
(ii) *A is an atom of $D(L)$.*

Proof. Let (i) be valid. By way of contradiction, suppose that A fails to be an atom of $D(L)$. Hence there exists $\{0\} \neq A_1 \in D(L)$ with $A_1 < A$. Then there is a direct product decomposition

$$L = A_1 \times B_1. \tag{1'}$$

The direct product decompositions (1) and (1′) have a common refinement (cf., e.g., [2]) and thus

$$A = (A \cap A_1) \times (A \cap B_1). \tag{1''}$$

We have $A \cap A_1 = A_1 \neq \{0\}$ and $A_0 \neq A$. The last relation implies that $A \cap B_1 \neq \{0\}$ and we have arrived at a contradiction.

Conversely, suppose that (ii) is valid. Assume that (i) does not hold. Hence there exists a direct product decomposition

$$A = P \times Q$$

such that $P \neq \{0\} \neq Q$. Then $P < A$ and

$$L = P \times (Q \times B),$$

hence $P \in D(L)$, contradicting (ii). $\square$

2.2. Corollary. *Assume that $L \neq \{0\}$ is a direct product of directly indecomposable lattices. Then $D(L)$ is atomistic.*

Proof. Suppose that (1a) is valid and that all A_i are directly indecomposable. Let $\{0\} \neq A \in D(L)$. Then

$$A = \prod_{i \in I} (A \cap A_i).$$

There exists $i(1) \in I$ such that $A \cap A_{i(1)} \neq \{0\}$. Then $A \cap A_{i(1)} \in D(A_{i(1)})$, whence $A \cap A_{i(1)} = A_{i(1)}$. We conclude that $A_{i(1)} \leqslant A$. Thus in view of 2.1, $D(A)$ is atomistic. $\square$

With regard to the conditions (i), (ii), (iii) used in (B) and to the condition (α) let us consider the following two examples.

Let L_1 be the lattice consisting of elements u, v, a_i ($i = 1, 2, 3, \ldots$) such that $u < a_i < v$ and

$$a_{i(1)} \wedge a_{i(2)} = u, \quad a_{i(1)} \vee a_{i(2)} = v$$

whenever $i(1)$ and $i(2)$ are distinct positive integers. Then L_1 is an algebraic V_1-lattice which does not satisfy the condition (α).

Further, let L_2 be the lattice consisting of elements u_1, u_2, v, a_i, b_i ($i = 1, 2, 3, \ldots$) such that $u_1 < u_2 < a_1 < a_2 < \ldots < v$, $u_2 < b_1 < b_2 < \ldots < v$ and

$$a_i \wedge b_j = u_2, \quad a_i \vee b_j = v$$

whenever i and j are positive integers. This is a complete V_1-lattice satisfying the condition (α), but it fails to be algebraic.

In this section we suppose that L is a conditionally complete lattice with the least element 0. Further we assume that L is a V_1-lattice and $L \neq \{0\}$.

3.1. Lemma. *Let $s \in S(L)$ and let (1) be valid. Then either $s \in A$ or $s \in B$.*

Proof. From (1) we obtain that $s = s(A) \vee s(B)$. Then, since $s \in S(L)$, we must have either $s = s(A)$ or $s = s(B)$. □

For $x \in L$ we denote

$$[x]^0 = \bigcap_{i \in I} A_i,$$

where $\{A_i\}_{i \in I}$ is the set of all direct factors A_i of L with $x \in A_i$.

3.2. Lemma. *Let $x \in L$. Then $[x]^0$ is a closed sublattice of L and $0 \in [x]^0$.*

Proof. Let $\{A_i\}_{i \in I}$ be as above. Each A_i is a closed sublattice of L containing the element 0, thus the same is valid for $[x]^0$. □

3.3. Lemma. *Let $x, y \in S(L)$, $0 \neq x \in [y]^0$. Then $[x]^0 = [y]^0$.*

Proof. From the relation $x \in [y]^0$ we infer that $[x]^0 \subseteq [y]^0$. Let (1) be valid and suppose that $x \in A$. If $y \notin A$, then in view of 3.1 we have $y \in B$ and hence x belongs to B as well. Therefore $x \in A \cap B = \{0\}$, which is a contradiction. Thus $y \in A$ yielding that $[y]^0 \subseteq [x]^0$. □

3.4. Lemma. *Let $x \in L$, $y \in S(L)$, $0 \neq x \in [y]^0$. Then $[x]^0 = [y]^0$.*

Proof. Clearly $[x]^0 \subseteq [y]^0$. Since L is a V_1-lattice there exists $x_1 \in S(L)$ such that $0 < x_1 \leqslant x$. Then $[x_1]^0 \subseteq [x]^0$, thus $x_1 \in [y]^0$. Now 3.3 yields that $[x_1]^0 = [y]^0$. Hence $[x]^0 = [y]^0$. □

3.5. Lemma. *Let $x, y \in S(L)$, $0 < z \in [x]^0 \cap [y]^0$. Then $[x]^0 = [y]^0$.*

Proof. This is an immediate consequence of 3.4. □

Let us denote by $\{C_j\}_{j \in J}$ the system of all sublattices $[x]^0$ of L, where x runs over the set $S(L) \setminus \{0\}$.

For $t \in L$ and $j \in J$ we denote

$$t_j = \sup\{x \in C_j : x \leqslant t\}.$$

Since L is conditionally complete and in view of 3.2, the element t_j does exist and belongs to C_j. Also, for $t_1, t_2 \in L$ we have

$$t_1 \leqslant t_2 \Longrightarrow (t_1)_j \leqslant (t_2)_j. \tag{2}$$

There exists a subset $\{x_k\}_{k\in K} \subseteq S(L) \setminus \{0\}$ such that

$$t = \bigvee_{k\in K} x_k. \tag{3}$$

For $k_1 \in K$ we put

$$K(k_1) = \{k_2 \in K\colon [x_{k_1}]^0 = [x_{k_2}]^0\},$$
$$x(k_1) = \bigvee_{k\in K(k_i)} x_k.$$

Then in view of 3.2 we obtain $x(k_1) \in [x_{k_1}]^0$. Moreover, $x(k_1) \leqslant t$ and hence $x(k_1) \leqslant t_j$ for $C_j = [x_{k_1}]^0$. Therefore according to (3) we get

$$t = \bigvee_{j\in J} t_j. \tag{4}$$

3.6. Lemma. *Let $j(0), j(1), \ldots, j(n)$ be distinct elements of J and let $x^k \in C_{j(k)}$ for $k = 0, 1, 2, \ldots, n$, $y = x^1 \vee x^2 \vee \ldots \vee x^n$, $x^{01} \in C_{j(0)}$, $x^{01} \leqslant x^0 \vee y$. Then $x^{01} \leqslant x^0$.*

Proof. Let $i \in \{1, 2, \ldots, n\}$. Then $x^i \notin C_{j(0)}$. Hence there exists a direct product decomposition

$$L = A_i \times B_i$$

such that $x^0, x^{01} \in A_i$ and $x^i \in B_i$. Put $A = A_{j(1)} \cap A_{j(2)} \cap \ldots \cap A_{j(n)}$. Then $A \in D(L)$. Hence there exists $B \in D(L)$ such that

$$L = A \times B.$$

Since $A_i \cap B_i = \{0\}$ we get $A \cap B_i = \{0\}$. Further,

$$B_i = (B_i \cap A) \times (B_i \cap B),$$

thus $B_i = B_i \cap B$ and hence $B_i \subseteq B$ for $i \in \{1, 2, \ldots, n\}$, implying that $y \in B$. We have

$$x^0 \in A, \quad x^{01} \in A,$$
$$x^0 = x^0(A), \quad x^{01} = x^{01}(A), \quad 0 = y(A),$$
$$x^{01}(A) \leqslant x^0(A) \vee y(A) = x^0(A).$$

$\square$

Since 0 is an element of $C_{j(0)}$, we obtain

3.7. Corollary. *Let $j(0), j(1), j(2), \ldots, j(n)$ be distinct elements of J, $x^k \in C_{j(k)}$ $(k = 0, 1, 2, \ldots, n)$. Suppose that*

$$x^0 \leqslant x^1 \vee x^2 \vee \ldots \vee x^n.$$

Then $x^0 = 0$.

Again, let $j(0)$ be a fixed element of J. We denote by B the set of all elements $t \in L$ such that $t_{j(0)} = 0$.

In the remaining part of this section we suppose that L is compactly generated.

3.8. Lemma. *Let $x \in S(L)$. Then s is compact.*

Proof. Since L is compactly generated, s is a join of compact elements of L. But s is strictly join-irreducible, whence s must be compact. $\square$

3.9. Lemma. *Let $a \in C_{j(0)}$, $b \in B$. Then $a \wedge b = 0$.*

Proof. By way of contradiction, suppose that $a \wedge b = a_1 > 0$. Then there exists $s \in S(L)$ such that $0 < s \leqslant a_1$. Since $b \in B$, in view of (4) we have

$$b = \bigvee_{j \in J \setminus \{j(0)\}} b_j,$$

$$s \leqslant \bigvee_{j \in J \setminus \{j(0)\}} b_j.$$

According to 3.8, the element s is compact. Thus there exists a finite subset $\{j(1), j(2), \ldots, j(n)\}$ of the set $J \setminus \{j(0)\}$ such that

$$s \leqslant b_{j(1)} \vee b_{j(2)} \vee \ldots \vee b_{j(n)}.$$

In view of 3.7 we have arrived at a contradiction. $\square$

3.10. Lemma. *For each $j \in J$ let $b^j \in C_j$. Further let $t \in L$, $t = \bigvee\limits_{j \in J} b^j$. Then for each $j \in J$ we have $t_j = b^j$.*

Proof. Let $j(0) \in J$. Since $b^{j(0)} \in C_{j(0)}$ and $b^{j(0)} \leqslant t$ we get $b^{j(0)} \leqslant t_{j(0)}$. For each $s \in S(L)$ with $s \leqslant t_{j(0)}$ we have $s \leqslant t$. In view of 3.8, s is compact, thus there are distinct elements $j(1), j(2), \ldots, j(n)$ in J such that

$$s \leqslant b^{j(1)} \vee b^{j(2)} \vee \ldots \vee b^{j(n)}.$$

Thus in view of 3.7 we must have $j(0) \in \{j(1), j(2), \ldots, j(n)\}$ and $s \leqslant b^{j(0)}$. This yields that $t_{j(0)} \leqslant b^{j(0)}$, completing the proof. $\square$

3.11. Lemma. *Let t be as above and $t \in L$, $t' = \bigvee_{j\in J} t'_j$ with $t'_j \in C_j$. Then $(t \vee t')_j = t_j \vee t'_j$ for each $j \in J$.*

Proof. We have

$$t \vee t' = \bigvee_{j\in J} (t_j \vee t'_j)$$

and in view of 2.2, $t_j \vee t'_j \in C_j$. Now it suffices to apply 3.10. $\square$

3.12. Lemma. *Let $j(0)$ and B be as above. Then B is a convex sublattice of L and $C_{j(0)} \cap B = \{0\}$.*

Proof. If $b \in B$, $x \in L$ and $x \leqslant b$, then in view of the definition of B the relation $x \in B$ is valid; hence B is convex in L. From this and from 3.11 we conclude that B is a sublattice of L. From 3.9 we obtain that $C_{j(0)} \cap B = \{0\}$. $\square$

Let $t \in L$ and consider the relation (4). Since L is conditionally complete there exists $x \in L$ such that

$$x = \bigvee_{j\in J\setminus\{j(0)\}} t_j.$$

Then in view of 3.10 we have $x \in B$. Put

$$\psi(t) = (t_{j(0)}, x).$$

Thus ψ is a mapping of L into $C_{j(0)} \times B$.

We apply the following convention. The pair $(t_{j(0)}, 0)$ or $(0, x)$ will be identified with $t_{j(0)}$ or with x, respectively.

In view of this convention we have $\psi(t) = t$ for each $t \in C_{j(0)} \cup B$.

3.13. Lemma. *Let $t^0 \in C_{j(0)}$, $b \in B$, $t = t^0 \vee b$. Then $\psi(t) = (t^0, b)$.*

Proof. This is a consequence of 3.11. $\square$

3.14. Lemma. *Let $t^1, t^2 \in L$. Then*

$$t^1 \leqslant t^2 \Longleftrightarrow \psi(t^1) \leqslant \psi(t^2).$$

Proof. Let $t^1 \leqslant t^2$. Then $t^1_j \leqslant t^2_j$ for each $j \in J$, whence $\psi(t^1) \leqslant \psi(t^2)$. Conversely, let $\psi(t^1) \leqslant \psi(t^2)$. Put $\psi(t^i) = (t^i_{j(0)}, b^i)$ $(i = 1, 2)$. Hence $t^1_{j(0)} \leqslant t^2_{j(0)}$ and $b^1 \leqslant b^2$. From the last relation and by applying 3.6 we obtain that $t^1_j \leqslant t^2_j$ is valid for each $j \in J \setminus \{j(0)\}$. Therefore in view of (4) we have $t^1 \leqslant t^2$. $\square$

3.15. Proposition. *Let $j(0)$ and B be as above. Then $L = C_{j(0)} \times B$.*

Proof. This is a consequence of 3.13 and 3.14. $\square$

3.16. Lemma. *Let $0 < s \in S(L)$. Then $[s]^0$ is a direct factor of L. Moreover, $[s]^0$ is an atom of $D(L)$.*

Proof. There exists $j(0) \in J$ such that $[s]^0 = C_{j(0)}$. Hence according to 3.15, $[s]^0$ is a direct factor of L. Then each direct factor of $[s]^0$ is, at the same time, a direct factor of L. Now from the definition of $[s]^0$ and from 3.5 we conclude that $[s]^0$ is directly indecomposable. Hence in view of 1.1, $[s]^0$ is an atom of $D(L)$. $\square$

Proof of (B):

Let (1) be valid, $A \neq \{0\}$. Hence there are $0 < a \in A$ and $0 < s \in S(L)$ with $s \leqslant a$. Then $s \in A$, thus $[s]^0 \subseteq A$. In view of 3.16, $[s]^0$ is an atom of $D(L)$. Therefore $D(L)$ is atomistic.

4. The condition (α)

In this section we assume that L is a lattice having the least element 0. We suppose that L satisfies the condition (α) and the conditions (i), (iii) from (B).

Let us remark that (α) implies the validity of the following condition:

(α_1) If $\{x_i\}_{i\in I}$ is a nonempty subset of L, $y = \bigvee_{i\in I} x_i$, $x \in S(L)$, $x \wedge x_i = 0$ for each $i \in I$, then $x \wedge y = 0$.

We apply the method from Section 3 with the distinction that we modify those parts where the condition (ii) from (B) was used. Hence 3.1–3.7 remain without change.

4.1. Lemma. *The assertion of 3.9 is valid.*

Proof. We begin as in the proof of 3.9; let a_1, s, b_j ($j \in J \setminus \{j(0)\}$) be as in this proof. Hence we have

$$s \leqslant \bigvee_{j\in J\setminus\{j(0)\}} b_j. \tag{4.1}$$

If $j \in J$, then there is a set K_j and a system $\{s_{vj}\}_{v\in K_j}$ such that this system is a subset of $S(L)$ and

$$b_j = \bigvee_{v\in K_j} s_{vj}.$$

In view of 3.6,

$$s \wedge s_{vj} = 0 \tag{4.2}$$

for each $j \in J \setminus \{j(0)\}$ and each s_{vj} $(v \in K_j)$. According to (4.1) we get

$$s \leqslant \bigvee_{j \in J \setminus \{j(0)\}} \bigvee_{v \in K_j} s_{vj}. \tag{4.3}$$

Then in view of (4.2) and (4.3) we have arrived at a contradiction with the condition (α_1). □

4.2. Lemma. *Let t, $j(0)$ and B be as above, $j \in J$, $j \neq j(0)$. Then $t_j \in B$.*

Proof. The element t_j is a join of some elements s of $S(L)$ and these elements belong to C_j, hence for each such s and each $a \in C_{j(0)}$ we have $a \wedge s = 0$. Then (α_1) yields that $a \wedge t_j = 0$. Thus $(t_j)_{j(0)} = 0$ and therefore $t_j \in B$. □

4.3. Lemma. *The assertion of* 3.10 *is valid.*

Proof. Similarly as in the proof of 3.10 we have $b^{j(0)} \leqslant t_{j(0)}$. Further,

$$t_{j(0)} \leqslant t = \bigvee_{j \in J} b^j. \tag{4.4}$$

From $b^j \in C_j$ we infer that $(b^j)_j = b_j$ and hence according to 4.2 we have $b^j \in B$. Thus 4.1 yields that

$$s \wedge b^j = 0$$

for each $s \in S(L)$ belonging to $C_{j(0)}$. Hence 4.4 and (α) imply that $t_{j(0)} \leqslant b^{j(0)}$. Therefore $t_{j(0)} = b^{j(0)}$. □

Now by the same method as in Section 3 we verify that 3.11–3.16 are valid under the present assumptions.

Hence we obtain:

4.4. Theorem. *Let L be a lattice such that*

(i) *it is conditionally complete and has the least element 0;*
(ii) *it satisfies the condition (α);*
(iii) *it is a V_1-lattice.*

Then $D(L)$ is atomistic.

From Examples 1 and 2 in Section 1 we infer that neither 4.4 is a corollary of (B), nor (B) is a corollary of 4.4.

Again, let L be a lattice with the least element 0. An indexed system $(x_i)_{i\in I}$ of elements of L is called disjoint if $x_{i(1)} \wedge x_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

5.1. Definition. The lattice L is said to be *orthogonally complete* if each nonempty disjoint indexed system of elements of L has the supremum in L.

The analogous notions of orthogonal completeness of lattice ordered groups or of vector lattices have been frequently applied in literature.

Example. Let A be the set of all non-negative reals with the natural linear order, $B = A$, $L = A \times B$. Then L is conditionally complete and orthogonally complete, but it fails to be complete.

5.2. Theorem. *Let L be a lattice. Suppose that it is orthogonally complete and satisfies the conditions* (i), (ii) *and* (iii) *from* (B). *Then L is a direct product of directly indecomposable lattices.*

Proof. We apply the notation as in Section 3. For each $t \in L$ we put

$$\psi_1(t) = (t_j)_{j\in J}.$$

Then in view of 3.15, ψ_1 is a homomorphism of L into the direct product

$$C = \prod_{j\in J} C_j.$$

Let $t_1, t_2 \in L$ and suppose that $\psi_1(t_1) = \psi_2(t_2)$. Then $(t_1)_j = (t_2)_j$ for each $j \in J$, whence in view of (4) we obtain that $t_1 = t_2$. Thus ψ_1 is an isomorphism of L into C. Choose $c^j \in C_j$ for each $j \in J$. Then $(c^j)_{j\in J}$ is a disjoint indexed system of elements of L (cf. 3.12); hence there exists $c \in L$ with

$$c = \bigvee_{j\in J} c^j.$$

According to 3.10 we have $\psi_1(c) = (c^j)_{j\in J}$. Thus ψ_1 is a surjection. We obtain that $C = L$. In view of 3.16 and 1.1, all C_j are directly indecomposable. $\square$

The above theorem generalizes (A).

By applying the results of Section 4 we can verify that it is possible to replace the condition (ii) in 5.2 by the condition (α).

The following example shows that the assumption of orthogonal completeness cannot be omitted in 4.2.

Let N be the set of all positive integers and let B be the Boolean algebra of all subsets of N. Further let L be the sublattice of B consisting of all finite subsets of N. Then L is a lattice satisfying the conditions (i), (ii) and (iii) from (B). The lattice L cannot be represented as a direct product of directly indecomposable lattices.

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