α-CONTINUOUS AND α-IRRESOLUTE MULTIFUNCTIONS

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Abstract. Recently Popa and Noiri [10] established some new characterizations and basic properties of α-continuous multifunctions. In this paper, we improve some of their results and examine further properties of α-continuous and α-irresolute multifunctions. We also make corrections to some theorems of Neubrunn [7].

Keywords: upper (lower) α-continuous, upper (lower) α-irresolute, strongly α-closed graph, almost compact, almost paracompact

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1. Introduction

In 1965, Njåstad [8] introduced a weak form of open sets called α-sets. Some kinds of generalized continuous functions were defined in terms of α-sets by several authors. For example, Maheshwari and Thakur [4] defined a function $f\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ to be *α-irresolute* if $f^{-1}(V)$ is an α-set for every α-set V of $(Y,\mathcal{U})$. Mashhour et al [6] defined a function $f\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ to be *α-continuous* if $f^{-1}(V)$ is an α-set for every open set V of $(Y,\mathcal{U})$. In 1986, Neubrunn [7] extended these concepts to multifunctions. Recently Popa and Noiri [10] obtained several new characterizations and properties of α-continuous multifunctions. The purpose of this paper is to improve some results of [4] and [10], to exploit further properties of α-continuous and α-irresolute multifunctions, and to make corrections to some theorems of Neubrunn [7].

Throughout this paper, $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ are always topological spaces. The closure (resp. interior) of a subset A in $(X,\mathcal{T})$ is denoted by $\mathrm{Cl}(A)$ (resp. $\mathrm{Int}(A)$). Then A is called *α-open* [8] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, and A is *α-closed* if $X - A$ is α-open. Note that α-closed sets are called *coα-sets* in [4]. Let $\mathcal{T}_\alpha$ denote the family of all α-open subsets of $(X,\mathcal{T})$. It was shown in [8] that $\mathcal{T}_\alpha$ is a topology on X. Let $\alpha\,\mathrm{Cl}(A)$ (resp. $\alpha\,\mathrm{Int}(A)$) denote the closure (resp. interior) of A with respect to $\mathcal{T}_\alpha$. A subset U of $(X,\mathcal{T})$ is called an *α-neighborhood* of a point $x \in X$ if there exists a $V \in \mathcal{T}_\alpha$ such that $x \in V \subset U$. By a multifunction $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$, we mean a point-to-set correspondence from $(X,\mathcal{T})$ into $(Y,\mathcal{U})$, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For each $B \subset Y$, $F^+(B) = \{x \in X \mid F(x) \subset B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subset X$, $F(A) = \bigcup_{x\in A} F(x)$. Then F is said to be a *surjection* if $F(X) = Y$, or equivalently, if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$. Moreover $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is called *upper semicontinuous*, abbreviated as u.s.c. (resp. *lower semicontinuous*, abbreviated as l.s.c.) if $F^+(V)$ (resp. $F^-(V)$) is open in $(X,\mathcal{T})$ for every open set V of $(Y,\mathcal{U})$. The *graph* $G(F)$ of F is defined by $G(F) = \{(x,y) \mid x \in X, y \in F(x)\}$. We say that F has a *closed (resp. α-closed) graph* if $G(F)$ is closed (resp. α-closed) in $(X \times Y, \mathcal{T} \times \mathcal{U})$. The *graph multifunction* $G_F\colon (X,\mathcal{T}) \to (X \times Y, \mathcal{T} \times \mathcal{U})$ of F is defined by $G_F(x) = \{x\} \times F(x)$ for each $x \in X$. Other basic concepts and terminology about multifunctions are as in [2] and [3].

2. α-continuous multifunctions

Following Neubrunn [7], we define the fundamental concepts.

Definition 2.1. ([7]) A multifunction $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is called
(a) *upper α-continuous*, abbreviated as u.α.c., if $F\colon (X,\mathcal{T}_\alpha) \to (Y,\mathcal{U})$ is u.s.c.,
(b) *lower α-continuous*, abbreviated as l.α.c., if $F\colon (X,\mathcal{T}_\alpha) \to (Y,\mathcal{U})$ is l.s.c.
Now $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is *α-continuous* if it is both u.α.c. and l.α.c.

The following characterizations of upper α-continuity and lower α-continuity are due to Popa and Noiri [10].

Theorem 2.2. ([10]) *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a multifunction. Then the following statements are equivalent.*
(a) *$F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is u.α.c.*
(b) *$F^+(V) \in \mathcal{T}_\alpha$ for any $V \in \mathcal{U}$.*
(c) *$F^-(V)$ is α-closed in $(X,\mathcal{T})$ for any closed V of $(Y,\mathcal{U})$.*
(d) *For each point $x \in X$ and each neighborhood V of $F(x)$, there exists an α-neighborhood U of x such that $F(U) \subset V$.*
(e) *$\alpha\,\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ for any $B \subset Y$.*

Theorem 2.3. ([10]) *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a multifunction. Then the following statements are equivalent.*
(a) *$F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is l.α.c.*
(b) *$F^-(V) \in \mathcal{T}_\alpha$ for any $V \in \mathcal{U}$.*
(c) *$F^+(V)$ is α-closed in $(X,\mathcal{T})$ for any closed V of $(Y,\mathcal{U})$.*
(d) *$\alpha\,\mathrm{Cl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ for any $B \subset Y$.*

(e) $F(\alpha\,\mathrm{Cl}(A)) \subset \mathrm{Cl}(F(A))$ *for any* $A \subset X$.

In our next result, we provide a simple and direct proof of Theorem 3.9 of [10].

Theorem 2.4. ([10]) *A multifunction* $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ *is l.α.c. if and only if its graph multifunction* G_F *is l.α.c.*

Proof. Suppose that G_F is l.α.c. Then for any open subset V of $(Y,\mathcal{U})$, $F^-(V) = G_F^-(X \times V) \in \mathcal{T}_\alpha$. Hence F is l.α.c. Conversely, suppose that F is l.α.c. For each $U \in \mathcal{T}$ and each $V \in \mathcal{U}$, we have $G_F^-(U \times V) = U \cap F^-(V) \in \mathcal{T}_\alpha$. Therefore G_F is l.α.c. from Proposition 6.3.5 of [3]. □

Definition 2.5. A multifunction $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is said to have a *strongly α-closed graph* if for each pair $(x,y) \notin G(F)$ there exist $U \in \mathcal{T}_\alpha$ and $V \in \mathcal{U}_\alpha$ containing x and y respectively such that $(U \times V) \cap G(F) = \emptyset$.

From this definition, we see that $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ has a strongly α-closed graph if and only if $F\colon (X,\mathcal{T}_\alpha) \to (Y,\mathcal{U}_\alpha)$ has a closed graph. Moreover, if $G(F)$ is strongly α-closed, then it is α-closed. The following example will show that the converse is not true in general.

Example 2.6. Let X be an infinite set, let $x_i \in X$ $(i = 1,2,3)$ be three different points and $\mathcal{T} = \{G \subset X \mid x_i \notin G, i = 1,2,3\} \cup \{G \subset X \mid X - G \text{ is finite}\}$. Then it is easy to verify that $\mathcal{T}$ is a topology on X and $\mathcal{T}_\alpha = \mathcal{T}$. Choose an infinite subset P of X such that $x_i \notin P$ $(i = 1,2,3)$ and $X - P$ is also infinite. Define a multifunction $F\colon (X,\mathcal{T}) \to (X,\mathcal{T})$ by

$$F(x) = \begin{cases} \{x_1, x_2\}, & \text{if } x \in P; \\ \{x_2, x_3\}, & \text{if } x \in X - P. \end{cases}$$

The graph $G(F) = P \times \{x_1, x_2\} \cup (X - P) \times \{x_2, x_3\}$ of F is α-closed, since $\emptyset = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(G(F)))) \subset G(F)$. But for any two α-neighborhoods U and V of x_1, we have $(U \times V) \cap G(F) \neq \emptyset$. Therefore $G(F)$ is not strongly α-closed.

Recall that a subset A of a space $(X,\mathcal{T})$ is called *α-paracompact* [1] if for every open cover $\mathcal{V}$ of A in $(X,\mathcal{T})$ there exists a locally finite open cover $\mathcal{W}$ of A which refines $\mathcal{V}$. Our next several results concern the relationship between upper α-continuity and strongly α-closed graphs.

Theorem 2.7. *Let* $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ *be a u.α.c. multifunction from a space* $(X,\mathcal{T})$ *into a Hausdorff space* $(Y,\mathcal{U})$*. If* $F(x)$ *is α-paracompact for each* $x \in X$*, then* $G(F)$ *is strongly α-closed.*

Proof. Suppose that $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Since $(Y,\mathcal{U})$ is a Hausdorff space, for each $y \in F(x_0)$ there exist open sets $V(y)$ and $W(y)$ containing

y and y_0 respectively such that $V(y) \cap W(y) = \emptyset$. The family $\{V(y) \mid y \in F(x_0)\}$ is an open cover of $F(x_0)$. Thus, by α-paracompactness of $F(x_0)$, there is a locally finite open cover $\mathcal{V} = \{U_\beta \mid \beta \in I\}$ which refines $\{V(y) \mid y \in F(x_0)\}$. Therefore there exists an open neighborhood W_0 of y_0 such that W_0 intersects only finitely many members $U_{\beta_1}, U_{\beta_2}, \ldots, U_{\beta_n}$ of $\mathcal{V}$. Choose $y_1, y_2, \ldots, y_n$ in $F(x_0)$ such that $U_{\beta_i} \subset V(y_i)$ for each $1 \leqslant i \leqslant n$, and set $W = W_0 \cap \big(\bigcap_{i=1}^n W(y_i)\big)$. Then W is an open neighborhood of y_0 such that $W \cap \big(\bigcup_{\beta \in I} V_\beta\big) = \emptyset$. By the upper α-continuity of F, there is a $U \in \mathcal{T}_\alpha$ such that $x_0 \in U \subset F^+\big(\bigcup_{\beta \in I} V_\beta\big)$. It follows that $(U \times W) \cap G(F) = \emptyset$. Therefore $G(F)$ is strongly α-closed. □

Corollary 2.8. ([10]) *If $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is a u.α.c. multifunction into a Hausdorff space $(Y,\mathcal{U})$ such that $F(x)$ is compact for each $x \in X$, then the graph $G(F)$ is α-closed.*

Theorem 2.9. *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a multifunction from a space $(X,\mathcal{T})$ into an α-compact space $(Y,\mathcal{U})$. If $G(F)$ is strongly α-closed, then F is u.α.c.*

P r o o f. Suppose that F is not u.α.c. By Theorem 2.2, there exists a nonempty closed subset C of $(Y,\mathcal{U})$ such that $F^-(C)$ is not α-closed in $(X,\mathcal{T})$. We may assume $F^-(C) \neq \emptyset$. Then there exists a point $x_0 \in \alpha\,\mathrm{Cl}(F^-(C)) - F^-(C)$. Hence for each point $y \in C$, we have $(x_0, y) \notin G(F)$. Since F has a strongly α-closed graph, there are α-open subsets $U(y)$ and $V(y)$ containing x_0 and y respectively such that $(U(y) \times V(y)) \cap G(F) = \emptyset$. Then $\{Y - C\} \cup \{V(y) \mid y \in C\}$ is an α-open cover of $(Y,\mathcal{U})$, and thus it has a subcover $\{Y - C\} \cup \{V(y_i) \mid y_i \in C, 1 \leqslant i \leqslant n\}$. Let $U = \bigcap_{i=1}^n U(y_i)$ and $V = \bigcup_{i=1}^n V(y_i)$. It is easy to verify that $C \subset V$ and $(U \times V) \cap G(F) = \emptyset$. Since U is an α-neighborhood of x_0, $U \cap F^-(C) \neq \emptyset$. It follows that $\emptyset \neq (U \times C) \cap G(F) \subset (U \times V) \cap G(F)$. This is a contradiction. Hence the proof is completed. □

Corollary 2.10. *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a multifunction into an α-compact Hausdorff space $(Y,\mathcal{U})$ such that $F(x)$ is α-closed for each $x \in X$. Then F is u.α.c. if and only if it has a strongly α-closed graph.*

3. α-irresolute multifunctions

In this section, we discuss some properties of upper (lower) α-irresolute multifunctions and generalize the main results of [4] to multifunctions.

Definition 3.1. ([7]) A multifunction $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is called

(a) *upper α-irresolute*, abbreviated as u.α.i., if $F\colon (X,\mathcal{T}_\alpha) \to (Y,\mathcal{U}_\alpha)$ is u.s.c.,

(b) *lower α-irresolute*, abbreviated as l.α.i., if $F\colon (X,\mathcal{T}_\alpha) \to (Y,\mathcal{U}_\alpha)$ is l.s.c.

Now $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is *α-irresolute* if it is both u.α.i. and l.α.i.

It follows from the definitions that a u.α.i. (resp. l.α.i.) multifunction is u.α.c. (resp. l.α.c.). In [4], the authors introduced the concept of α-Hausdorff spaces in order to ensure the graph of an α-irresolute function to be α-closed. It was shown by Reilly and Vamanamurthy [13] that α-Hausdorff spaces are precisely Hausdorff spaces. Therefore, as corollaries of Theorem 2.7, we have the following results.

Theorem 3.2. *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a u.α.i. multifunction into a Hausdorff space $(Y,\mathcal{U})$. If $F(x)$ is α-paracompact for each $x \in X$, then $G(F)$ is strongly α-closed.*

Corollary 3.3. ([4]) *If $f\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is an α-irresolute function and $(Y,\mathcal{U})$ is α-Hausdorff, then $G(f)$ is α-closed.*

Let A be a subset of a space $(X,\mathcal{T})$. Then $F\colon (X,\mathcal{T}) \to (A,\mathcal{T}_A)$ is called a *retracting multifunction* [16] if $x \in F(x)$ for each $x \in A$. By using the same technique as in the proof of Theorem 2.7, we can obtain the following results.

Theorem 3.4. *Let F be a u.α.i. multifunction of a Hausdorff space $(X,\mathcal{T})$ into itself. If $F(x)$ is α-paracompact for each $x \in X$, then the set $A = \{x \mid x \in F(x)\}$ is an α-closed subset.*

Proof. Let $x_0 \in \alpha\operatorname{Cl}(A)$. Suppose that $x_0 \notin A$, i.e. $x_0 \notin F(x_0)$. Since $(X,\mathcal{T})$ is Hausdorff, for each $x \in F(x_0)$ there exist open sets $U(x)$ and $V(x)$ containing x_0 and x respectively such that $U(x) \cap V(x) = \emptyset$. Then $\{V(x) \mid x \in F(x_0)\}$ is an open cover of $F(x_0)$. By the α-paracompactness of $F(x_0)$, $\{V(x) \mid x \in F(x_0)\}$ has a locally finite open refinement $\mathcal{W} = \{W_\beta \mid \beta \in I\}$ which covers $F(x_0)$. Therefore we can choose an open neighborhood U_0 of x_0 such that U_0 intersects only finitely many members $W_{\beta_1}, W_{\beta_2}, \ldots, W_{\beta_n}$ of $\mathcal{W}$. Choose $x_1, x_2, \ldots, x_n$ in $F(x_0)$ such that $W_{\beta_i} \subset V(x_i)$ for each $1 \leqslant i \leqslant n$, and let $U = U_0 \cap \bigl(\bigcap_{i=1}^n U(x_i)\bigr)$. Then U is an open neighborhood of x_0 such that $U \cap \bigl(\bigcup_{\beta\in I} W_\beta\bigr) = \emptyset$. Since F is u.α.i., there is an α-neighborhood G of x_0 such that $F(G) \subset \bigcup_{\beta\in I} W_\beta$. It follows that $G \cap U$ is an α-neighborhood of x_0 and satisfies $(G \cap U) \cap A = \emptyset$. This contradicts the fact that $x_0 \in \alpha\operatorname{Cl}(A)$. $\square$

Corollary 3.5. ([4]) *If f is an α-irresolute function of an α-Hausdorff space $(X,\mathcal{T})$ into itself, then the set $A = \{x \mid x = f(x)\}$ is an α-closed subset.*

Corollary 3.6. *Let A be a subset of $(X,\mathcal{T})$ and $F\colon (X,\mathcal{T}) \to (A,\mathcal{T}_A)$ a u.α.i. retracting multifunction such that $F(x)$ is α-paracompact for each $x \in A$. If $(X,\mathcal{T})$ is Hausdorff, then A is α-closed.*

Corollary 3.7. ([4]) *Let A be a subset of $(X,\mathcal{T})$ and $f\colon (X,\mathcal{T}) \to (A,\mathcal{T}_A)$ an α-irresolute retraction. If $(X,\mathcal{T})$ is Hausdorff, then A is α-closed.*

R e m a r k. From the proof of Theorem 3.4, it is easy to see that Theorem 3.4 and Corollary 3.6 are still valid if the upper α-irresolution of F is replaced by upper α-continuity.

In considering when a u.α.c. (resp. l.α.c.) multifunction is u.α.i. (resp. l.α.i.), Neubrunn [7] introduced the concepts of upper and lower somewhat openness. A multifunction $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is said to be *upper somewhat open* if $\mathrm{Int}(F(U)) \neq \emptyset$ for any open set $U \in \mathcal{T}$ with $F(U) \neq \emptyset$. It is said to be *lower somewhat open* if for any subset $U \in \mathcal{T}$ and $V \in \mathcal{U}$ such that $F(x) \cap V \neq \emptyset$ for any $x \in U$, we have $\mathrm{Int}(F(U) \cap V) \neq \emptyset$. Neubrunn (Theorem 5, [7]) claimed to prove that a u.α.c. and upper almost open multifunction $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is u.α.i. Unfortunately, this result is false as is shown in the following example.

E x a m p l e 3.8. Let $X = \{a,b,c,d\}$ and $Y = \{p,q,r\}$. Define a topology $\mathcal{T} = \{\emptyset, X, \{a\}, \{b,c\}, \{a,b,c\}\}$ on X and a topology $\mathcal{U} = \{\emptyset, Y, \{p\}\}$ on Y. A multifunction $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is defined as follows:

$$F(x) = \begin{cases} \{p\}, & \text{if } x = a; \\ Y, & \text{if } x = b, \text{ or } c; \\ \{p,q\}, & \text{if } x = d. \end{cases}$$

Then F is upper somewhat open and u.α.c. Since $\{p,q\}$ is α-open in $(Y,\mathcal{U})$ and $F^{+}(\{p,q\}) = \{a,d\}$ is not α-open in $(X,\mathcal{T})$, F is not u.α.i.

Neubrunn also claimed that there is no essential difference between the proofs of Theorem 6 and Theorem 5 of [7]. Since there is a gap in the proof of Theorem 5 of [7], we conclude this section by providing a complete proof to Theorem 6 of [7].

Theorem 3.9. ([7]) *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a multifunction. If F is both l.α.c. and lower somewhat open, then F is l.α.i.*

P r o o f. Suppose that F is not l.α.i. Then there is a nonempty $V \in \mathcal{U}_\alpha$ such that $F^{-}(V) \notin \mathcal{T}_\alpha$. We may assume $F^{-}(V) \neq \emptyset$. Let $U = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(V)))$. Since F is l.α.c., $F^{-}(U) \in \mathcal{T}_\alpha$. Then $F^{-}(V) \subset F^{-}(U) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^{-}(U))))$. It follows that $F^{-}(U) \not\subset \mathrm{Cl}(\mathrm{Int}(F^{-}(V)))$. Indeed, suppose this is not the case. Then $F^{-}(V)$ is α-open. Thus there exists a point $p \in F^{-}(U)$ and an open neighborhood G of p such that $G \cap \mathrm{Int}(F^{-}(V)) = \emptyset$. Since $G \cap F^{-}(U)$ is a nonempty α-open subset of $(X,\mathcal{T})$, $\mathrm{Int}(G \cap F^{-}(U)) \neq \emptyset$. Let $W = \mathrm{Int}(G \cap F^{-}(U))$. Clearly, W is open in $(X,\mathcal{T})$ and $W \cap \mathrm{Int}(F^{-}(V)) = \emptyset$. By the lower somewhat openness of F, we have $\emptyset \neq \mathrm{Int}(F(W) \cap U) = \mathrm{Int}(F(W)) \cap U$, which implies that $\emptyset \neq \mathrm{Int}(F(W) \cap V) \subset F(W) \cap \mathrm{Int}(V)$. Then $W \cap F^{-}(\mathrm{Int}(V)) \neq \emptyset$. By the lower α-continuity of F again, $W \cap F^{-}(\mathrm{Int}(V))$ is a nonempty α-open set. Hence $\emptyset \neq \mathrm{Int}(W \cap F^{-}(\mathrm{Int}(V))) \subset W \cap \mathrm{Int}(F^{-}(V))$. This contradicts the fact that $W \cap \mathrm{Int}(F^{-}(V)) = \emptyset$. $\square$

4. Mapping theorems

In this section, we will establish some mapping theorems by using the method of change of topology. A subset A of a space $(X,\mathcal{T})$ is called *α-compact* if every α-open cover of A in $(X,\mathcal{T})$ has a finite subcover. Hence the concept of an α-compact space in [5] can be restated as: A space X is α-compact if and only if X is an α-compact subset of itself. From the definition, a subset A of $(X,\mathcal{T})$ is α-compact if and only if A is compact in $(X,\mathcal{T}_\alpha)$.

Theorem 4.1. *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a u.α.c. multifunction such that $F(x)$ is compact for each point $x \in X$. Then $F(K)$ is compact for each α-compact subset K of $(X,\mathcal{T})$.*

Proof. It follows directly from Definition 2.1 (a) and Theorem 7.4.2 of [3]. □

Corollary 4.2. ([10]) *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a u.α.c. surjective multifunction such that $F(x)$ is compact for each point $x \in X$. If $(X,\mathcal{T})$ is α-compact, then $(Y,\mathcal{U})$ is compact.*

Theorem 4.3. *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a multifunction from a connected space $(X,\mathcal{T})$ onto $(Y,\mathcal{U})$ such that $F(x)$ is connected for each point $x \in X$. If F is either u.α.c. or l.α.c., then $(Y,\mathcal{U})$ is connected.*

Proof. It follows from Theorem 2 of [11], Theorem 7.4.4 of [3], Theorem 2.2 and Theorem 2.3. □

Corollary 4.4. ([11]) *If $(X,\mathcal{T})$ is connected and $f\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is an α-continuous surjection, then $(Y,\mathcal{U})$ is connected.*

Recall that a space $(X,\mathcal{T})$ is *almost compact* [14] if each open cover has a finite subfamily whose union is dense in $(X,\mathcal{T})$. And $(X,\mathcal{T})$ is called *almost paracompact* [15] if every open cover of $(X,\mathcal{T})$ has a locally finite open refinement whose union is dense in $(X,\mathcal{T})$. To obtain more mapping theorems, we first establish the following two lemmas. The proof of first lemma is not difficult, so we omit it.

Lemma 4.5. *$(X,\mathcal{T})$ is almost compact if and only if $(X,\mathcal{T}_\alpha)$ is almost compact.*

Lemma 4.6. *For a space $(X,\mathcal{T})$, the following statements are equivalent.*

(a) *$(X,\mathcal{T})$ is almost paracompact.*
(b) *Every open cover of $(X,\mathcal{T})$ has a $\mathcal{T}_\alpha$-locally finite α-open refinement whose union is dense in $(X,\mathcal{T})$.*
(c) *Every open cover of $(X,\mathcal{T})$ has a $\mathcal{T}_\alpha$-locally finite α-open one-to-one refinement whose union is dense in $(X,\mathcal{T})$.*

(d) *Every α-open cover of $(X,\mathcal{T})$ has a $\mathcal{T}_\alpha$-locally finite α-open refinement whose union is dense in $(X,\mathcal{T}_\alpha)$.*
(e) *$(X,\mathcal{T}_\alpha)$ is almost paracompact.*
(f) *Every α-open cover of $(X,\mathcal{T})$ has a $\mathcal{T}$-locally finite open refinement whose union is dense in $(X,\mathcal{T})$.*

Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (d) $\Rightarrow$ (e) and (f) $\Rightarrow$ (a) are straightforward.

(c) $\Rightarrow$ (d): Suppose that $\{U_\beta \mid \beta \in I\}$ is an α-open cover of $(X,\mathcal{T})$. Then $\{\mathrm{Int}(\mathrm{Cl}(U_\beta)) \mid \beta \in I\}$ is an open cover of $(X,\mathcal{T})$, thus it has a $\mathcal{T}_\alpha$-locally finite α-open one-to-one refinement $\{V_\beta \mid \beta \in I\}$ such that $X = \mathrm{Cl}\left(\bigcup_{\beta\in I} V_\beta\right) = \bigcup_{\beta\in I}\mathrm{Cl}(V_\beta)$. Now let $W_\beta = U_\beta \cap \mathrm{Int}(V_\beta)$ for each $\beta \in I$. Then $\{W_\beta \mid \beta \in I\}$ is a $\mathcal{T}_\alpha$-locally finite α-open refinement of $\{U_\beta \mid \beta \in I\}$. For each $\beta \in I$, it is easy to verify that

$$\alpha\,\mathrm{Cl}(W_\beta) = \mathrm{Cl}(U_\beta \cap \mathrm{Int}(V_\beta)) = \mathrm{Cl}(\mathrm{Cl}(U_\beta) \cap \mathrm{Int}(V_\beta)) = \mathrm{Cl}(\mathrm{Int}(V_\beta)) = \mathrm{Cl}(V_\beta).$$

Therefore $X = \bigcup_{\beta\in I}\alpha\,\mathrm{Cl}(W_\beta) = \alpha\,\mathrm{Cl}\left(\bigcup_{\beta\in I} W_\beta\right)$.

(e) $\Rightarrow$ (f): Let $\mathcal{V} = \{V_\beta \mid \beta \in I\}$ be an α-open cover of $(X,\mathcal{T})$. Then there exists a $\mathcal{T}_\alpha$-locally finite α-open refinement $\mathcal{W} = \{W_\lambda \mid \lambda \in \Lambda\}$ of $\mathcal{V}$ such that $X = \bigcup_{\lambda\in\Lambda}\alpha\,\mathrm{Cl}(W_\lambda)$. Then $\{\mathrm{Int}(W_\lambda) \mid \lambda \in \Lambda\}$ is an open refinement of $\mathcal{V}$. Since $\mathcal{W}$ is $\mathcal{T}_\alpha$-locally finite, for each $x \in X$ there is an α-open set G containing x such that G intersects only finitely many members of $\mathcal{W}$. Thus $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(G)))$ is an open neighborhood of x and intersects only finitely many members of $\{\mathrm{Int}(W_\lambda) \mid \lambda \in \Lambda\}$, which says that $\{\mathrm{Int}(W_\lambda) \mid \lambda \in \Lambda\}$ is $\mathcal{T}$-locally finite. For each $\lambda \in \Lambda$, $\alpha\,\mathrm{Cl}(W_\lambda) = \mathrm{Cl}(\mathrm{Int}(W_\lambda))$, hence we have $X = \bigcup_{\lambda\in\Lambda}\mathrm{Cl}(\mathrm{Int}(W_\lambda)) = \mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda}\mathrm{Int}(W_\lambda)\right)$. Therefore $\{\mathrm{Int}(W_\lambda) \mid \lambda \in \Lambda\}$ is a $\mathcal{T}$-locally finite open refinement of $\mathcal{V}$ and its union is dense in $(X,\mathcal{T})$. So the proof is completed. $\square$

Theorem 4.7. *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be an α-continuous surjection such that $F(x)$ is compact for each point $x \in X$. If $(X,\mathcal{T})$ is almost compact, then $(Y,\mathcal{U})$ is almost compact.*

Proof. Let $\mathcal{V} = \{V_\beta \mid \beta \in I\}$ be an open cover of $(Y,\mathcal{U})$. For each $x \in X$, there exists a finite subset $I(x) \subset I$ such that $F(x) \subset \bigcup\{V_\beta \mid \beta \in I(x)\} = V(x)$. Since F is u.α.c., there exists a $U(x) \in \mathcal{T}_\alpha$ containing x such that $F(U(x)) \subset V(x)$. We obtain an α-open cover $\{U(x) \mid x \in X\}$ of $(X,\mathcal{T})$. By Lemma 4.5, there are finitely many points $x_1, x_2, \ldots, x_n$ of X such that $X = \bigcup_{i=1}^n \alpha\,\mathrm{Cl}(U(x_i))$. Since F is l.α.c., we have

$$\begin{aligned} Y = F\left(\bigcup_{i=1}^n \alpha\,\mathrm{Cl}(U(x_i))\right) &= \bigcup_{i=1}^n F(\alpha\,\mathrm{Cl}(U(x_i))) \subset \bigcup_{i=1}^n \mathrm{Cl}(F(U(x_i))) \\ &\subset \bigcup_{i=1}^n \mathrm{Cl}(V(x_i)) = \bigcup_{i=1}^n \bigcup_{\beta\in I(x_i)} \mathrm{Cl}(V_\beta). \end{aligned}$$

This shows that $(Y,\mathcal{U})$ is almost compact. $\square$

Definition 4.8. *A multifunction $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is called α-open (resp. α-closed) if $F(G)$ is α-open (resp. α-closed) in $(Y,\mathcal{U})$ for each open (resp. closed) subset G of $(X,\mathcal{T})$.*

The proof of the following lemma is straightforward, so we omit it.

Lemma 4.9. *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a multifunction. Then the following statements are equivalent.*

(a) *$F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ is α-closed.*

(b) *For each $U \in \mathcal{T}$ and $B \subset Y$ with $F^-(B) \subset U$, there exists a $V \in \mathcal{U}_\alpha$ such that $B \subset V$ and $F^-(V) \subset U$.*

(c) *For each $U \in \mathcal{T}$ and each point $y \in Y$ with $F^-(y) \subset U$, there exists an α-neighborhood V of y such that $F^-(V) \subset U$.*

(d) *$F\colon (X,\mathcal{T}) \to (Y,\mathcal{U}_\alpha)$ is closed.*

Theorem 4.10. *Let $F\colon (X,\mathcal{T}) \to (Y,\mathcal{U})$ be an α-continuous, α-open and α-closed surjection from an almost paracompact space $(X,\mathcal{T})$ onto a space $(Y,\mathcal{U})$ such that $F(x)$ is α-paracompact for each $x \in X$ and $F^-(y)$ is compact for each $y \in Y$. Then $(Y,\mathcal{U})$ is almost paracompact.*

Proof. Let $\{U_\beta \mid \beta \in I\}$ be an open cover of $(Y,\mathcal{U})$. Since $F(x)$ is α-paracompact for each $x \in X$, there exists a $\mathcal{U}$-locally finite open cover $\mathcal{V}_x$ of $F(x)$ such that $\mathcal{V}_x$ refines $\{U_\beta \mid \beta \in I\}$. Then $\{F^+\bigl(\bigcup \mathcal{V}_x\bigr) \mid x \in X\}$ is an α-open cover of $(X,\mathcal{T})$, thus it has a $\mathcal{T}$-locally finite open refinement $\{W_\lambda \mid \lambda \in \Lambda\}$ such that $X = \bigcup_{\lambda\in\Lambda} \mathrm{Cl}(W_\lambda)$, following from Lemma 4.6. Hence for each $\lambda \in \Lambda$, there exists an $x_\lambda \in X$ such that $F(W_\lambda) \subset \bigcup \mathcal{V}_{x_\lambda}$. Let $\mathcal{G}_\lambda = \{F(W_\lambda) \cap V \mid V \in \mathcal{V}_{x_\lambda}\}$ for each $\lambda \in \Lambda$, and $\mathcal{G} = \{G \mid G \in \mathcal{G}_\lambda$ for some $\lambda \in \Lambda\}$. It is easy to see that $\mathcal{G}$ is an α-open refinement of $\{U_\beta \mid \beta \in I\}$, since F is α-open.

We now show that $\mathcal{G}$ is $\mathcal{U}_\alpha$-locally finite. For each $y \in Y$ and each $x \in F^-(y)$, we can choose an open neighborhood H_x such that H_x intersects only finitely many members of $\{W_\lambda \mid \lambda \in \Lambda\}$. Since $F^-(y)$ is compact, there are finitely many points $x_1, x_2, \ldots, x_n$ in $F^-(y)$ such that $F^-(y) \subset \bigcup_{i=1}^n H_{x_i} = H$. Then H intersects only finitely many members of $\{W_\lambda \mid \lambda \in \Lambda\}$, namely $W_{\lambda_1}, W_{\lambda_2}, \ldots, W_{\lambda_k}$. By the α-closedness of F and Lemma 4.9, there exists an α-open subset Q containing y such that $F^-(Q) \subset H$. It follows that Q intersects at most finitely many members $F(W_{\lambda_1}), F(W_{\lambda_2}), \ldots, F(W_{\lambda_n})$ of $\{F(W)_\lambda \mid \lambda \in \Lambda\}$. On the other hand, $\mathcal{V}_{x_{\lambda_i}}$ is $\mathcal{U}$-locally finite, thus we can choose an open neighborhood Q_i of y such that Q_i intersects only finitely many members of $\mathcal{V}_{x_{\lambda_i}}$. Then $\bigl(\bigcap_{i=1}^k Q_i\bigr) \cap Q$ is an α-open set containing y and meeting at most finitely many members of $\mathcal{G}$.

From Theorem 2.3, we have $F(\mathrm{Cl}(W_\lambda)) = F(\alpha\,\mathrm{Cl}(W_\lambda)) \subset \mathrm{Cl}(F(W_\lambda))$ for each $\lambda \in \Lambda$. Therefore

$$Y = F\Big(\bigcup_{\lambda\in\Lambda} \mathrm{Cl}(W_\lambda)\Big) \subset \bigcup_{\lambda\in\Lambda} \mathrm{Cl}(F(W_\lambda)) = \bigcup\{\mathrm{Cl}(G) \mid G \in \mathcal{G}\ .$$

By virtue of Lemma 4.6 (b), $(Y,\mathcal{U})$ is almost paracompact. $\square$

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