THE DIVERGENCE THEOREM

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(Received March 14, 1996)

Abstract. This is an expository paper dealing with Jan Mařík's results concerning perimeter and the divergence theorem of Gauss-Green-Ostrogradski.

Keywords: perimeter, the divergence theorem of Gauss-Green-Ostrogradski

MSC 1991: 28A75, 28–03, 01A70

The present text represents the English version of a lecture delivered on November 13, 1995 at a meeting commemorating the 75th birthday of the late Professor Jan Mařík. Scientific research of Professor Mařík was to a large extent inspired by his teaching activities. This is in particular true of his investigation concerning validity of the divergence theorem; he lectured on this topic in Prague in the mid-fifties. It is to be noted that nowadays widely recognized De Giorgi's papers [2], [3] were not known to J. Mařík at that time and his publications [6], [7], [8] arose independently of [2], [3]. What follows is a slight modification of Mařík's ideas.

Let us denote by $\mathcal{C}_0^{(1)}$ the class of all continuously differentiable functions with a compact support on the Euclidean m-space $\mathbb{R}^m$. Accordingly, $[\mathcal{C}_0^{(1)}]^m$ will stand for the class of all m-dimensional vector-valued functions $v = [v_1, \ldots, v_m]$ with components $v_j \in \mathcal{C}_0^{(1)}$ $(j = 1, \ldots, m)$. The symbol ∂_j will denote the partial derivative with respect to the j-th variable, λ_m is the Lebesgue measure on $\mathbb{R}^m$. If the boundary ∂A of a set $A \subset \mathbb{R}^m$ is formed by a smooth regular hypersurface, then the surface area measure σ is naturally defined on ∂A and the unit exterior normal vector $\nu(y) = \bigl[\nu_1(y), \ldots, \nu_m(y)\bigr]$ is available at each $y \in \partial A$. For each

Support of GAČR grant No. 201/96/0431 is gratefully acknowledged.

$v = [v_1, \ldots, v_m] \in [\mathcal{C}_0^{(1)}]^m$ the integral of the scalar product

$$v \cdot \nu := \sum_{j=1}^{m} v_j \nu_j$$

with respect to σ over ∂A can be transformed into the integral of the divergence

$$\operatorname{div} v := \sum_{j=1}^{m} \partial_j v_j$$

over A by means of the formula

$$\int_{\partial A} v \cdot \nu \,\mathrm{d}\sigma = \int_A \operatorname{div} v \,\mathrm{d}\lambda_m, \tag{1}$$

which is usually connected with the names of Gauss, Green and Ostrogradski. If we consider only vector-valued functions $v \in [\mathcal{C}_0^{(1)}]^m$, then the right-hand side of (1) is well defined for general Lebesgue measurable sets $A \subset \mathbb{R}^m$; on the other hand, the left-hand side in (1) (where the exterior normal ν and the surface area measure σ appear) has an obvious natural meaning for special smoothly bounded sets A only which do not form a sufficiently broad class of sets from the point of view of mathematical operations (not being closed w.r. to intersections, differences, ...) and from the point of view of applications (where bodies A with non-smooth boundaries like those built of cubes and their deformations frequently occur). The question naturally emerges whether one can characterize geometrically those Lebesgue measurable sets $A \subset \mathbb{R}^m$ on the boundary of which it is possible to define a finite Borel measure σ and a Borel measurable vector-valued function $\nu = [\nu_1, \ldots, \nu_m]$ with unit norm

$$|\nu| := \left(\sum_{j=1}^{m} \nu_j^2 \right)^{1/2} = 1$$

such that the formula (1) holds for all $v \in [\mathcal{C}_0^{(1)}]^m$.

Having fixed a Lebesgue measurable $A \subset \mathbb{R}^m$ we put in Mařík's notation

$$P(A, v) := \int_A \operatorname{div} v \,\mathrm{d}\lambda_m, \quad v \in [\mathcal{C}_0^{(1)}]^m$$

(which gives rise to a linear functional $v \mapsto P(A, v)$ on $[\mathcal{C}_0^{(1)}]^m$), and

$$P_j(A, f) = \int_A \partial_j f \,\mathrm{d}\lambda_m, \quad f \in \mathcal{C}_0^{(1)}$$

(which gives rise to a linear functional $f \mapsto P_j(A, f)$ on $\mathcal{C}_0^{(1)}$), $j = 1, \dots, m$. It is easy to see that $P_j(A, f)$ actually depends only on the restriction $f\big|_{\partial A}$ of f to the boundary of A. In other words, $P_j(A, f) = 0$ whenever $f \in \mathcal{C}_0^{(1)}$ vanishes on ∂A. In order to indicate this for $j = m$ let us agree to write points $x = [x_1, \dots, x_m] \in \mathbb{R}^m$ in the form $[\widehat{x}, x_m]$, where $\widehat{x} = [x_1, \dots, x_{m-1}] \in \mathbb{R}^{m-1}$ denotes the projection of x into $\mathbb{R}^{m-1}$ in the direction of the m-th coordinate axis. If $A \subset \mathbb{R}^m$ and $\widehat{x} \in \mathbb{R}^{m-1}$ are fixed, put

$$A_{\widehat{x}} := \big\{t \in \mathbb{R};\ [\widehat{x}, t] \in A\big\}.$$

Given $f \in \mathcal{C}_0^{(1)}$ define $f_{\widehat{x}}$ on $\mathbb{R}$ by

$$f_{\widehat{x}}(t) := f\big([\widehat{x}, t]\big), \quad t \in \mathbb{R}.$$

We have thus by Fubini's theorem

$$P_m(A, f) = \int_{\mathbb{R}^{m-1}} \left(\int_{A_{\widehat{x}}} (f_{\widehat{x}})' \,\mathrm{d}\lambda_1 \right) \mathrm{d}\lambda_{m-1}(\widehat{x}). \tag{2}$$

Observe now that $\int_B g' \,\mathrm{d}\lambda_1 = 0$ for any Lebesgue measurable $B \subset \mathbb{R}$ and any continuously differentiable function g with compact support on $\mathbb{R}$ vanishing on ∂B. Applying this to $B = A_{\widehat{x}}$ and $g = f_{\widehat{x}}$ one gets from (2) that $P_m(A, f) = 0$ for any $f \in \mathcal{C}_0^{(1)}$ vanishing on ∂A, as asserted. Consequently, $P(A, v)$ depends on $v|_{\partial A}$ only. The problem consists in representability of the functional $P(A, .)$ in the form

$$P(A, v) = \int_{\partial A} v \cdot \nu \,\mathrm{d}\sigma, \quad \forall v \in [\mathcal{C}_0^{(1)}]^m, \tag{3}$$

with a suitable finite Borel measure σ on ∂A and an appropriate Borel measurable vector-valued function $\nu = [\nu_1, \dots, \nu_m]$ on ∂A with $|\nu| = 1$ on ∂A. If the representation (3) is possible then for any $v \in [\mathcal{C}_0^{(1)}]^m$ with a norm $|v| \leqslant 1$ the inequality

$$\big|P(A, v)\big| \leqslant \int_{\partial A} \big|v \cdot \nu\big| \,\mathrm{d}\sigma \leqslant \sigma(\partial A)$$

holds, which shows that the perimeter of A defined by

$$\|A\| := \sup\big\{|P(A, v)|;\ v \in [\mathcal{C}_0^{(1)}]^m, |v| \leqslant 1\big\} \tag{4}$$

must necessarily be finite. Defining

$$\|A\|_j := \sup\big\{|P_j(A, f)|;\ f \in \mathcal{C}_0^{(1)}, |f| \leqslant 1\big\}$$

we have

$$\|A\|_j \leqslant \|A\|, \quad j \in \{1, \ldots, m\};$$

to see this observe that $P_j(A, f) = P(A, w)$ with $w = [\delta_{j1} f, \ldots, \delta_{jm} f] \in [\mathcal{C}_0^{(1)}]^m$ where δ_{ji} (=1 or 0 according as $j = i$ or $j \neq i$) is the Kronecker symbol. On the other hand, the relation

$$P(A, v) = \sum_{j=1}^{m} P_j(A, v_j), \quad v = [v_1, \ldots, v_m] \in [\mathcal{C}_0^{(1)}]^m \tag{5}$$

implies

$$\|A\| \leqslant \sum_{j=1}^{m} \|A\|_j$$

so that, in particular,

$$\|A\| < \infty \Longleftrightarrow \sum_{j=1}^{m} \|A\|_j < \infty. \tag{6}$$

Assume now

$$\|A\| < \infty. \tag{7}$$

Then any functional

$$P_j \colon f\big|_{\partial A} \mapsto P_j(A, f)$$

is continuous (with respect to the uniform convergence on ∂A) in the space

$$\mathcal{C}_0^{(1)}(\partial A) := \{f\big|_{\partial A};\, f \in \mathcal{C}_0^{(1)}\}$$

consisting of all restrictions to ∂A of functions in $\mathcal{C}_0^{(1)}$. It follows that P_j extends uniquely to a continuous linear functional on the space $\mathcal{C}_0(\partial A)$ of all continuous functions g on ∂A satisfying, in case ∂A is unbounded, the additional requirement

$$\lim_{\substack{|x| \to \infty \\ x \in \partial A}} g(x) = 0;$$

$\mathcal{C}_0(\partial A)$ is equipped with the norm

$$\|g\| = \max_{\partial A} |g|.$$

The Riesz representation theorem guarantees the existence of a uniquely determined finite signed Borel measure μ_j on ∂A such that

$$P_j(g) = \int_{\partial A} g \, \mathrm{d}\mu_j, \quad g \in \mathcal{C}_0(\partial A).$$

The m-tuple of Borel measures μ_j defines the vector-valued Borel measure $\mu = [\mu_1, \ldots, \mu_m]$ on ∂A representing the functional $P(A, .)$ in the sense that

$$P(A, v) = \int_{\partial A} v \, \mathrm{d}\mu \Big(\equiv \sum_{j=1}^{m} \int_{\partial A} v_j \, \mathrm{d}\mu_j \Big), \quad v = [v_1, \ldots, v_m] \in [\mathcal{C}_0^{(1)}]^m.$$

If $\sigma \equiv |\mu|$ denotes the variation measure of μ and $B \subset \partial A$ is a Borel set, then

$$\Big(\sum_{j=1}^{m} \big[\mu_j(B)\big]^2 \Big)^{1/2} \leqslant \sigma(B),$$

which shows that each μ_j is absolutely continuous w.r. to σ. The Radon-Nikodym theorem yields a Borel measurable function ν_j on ∂A such that

$$\int_{\partial A} g \, \mathrm{d}\mu_j = \int_{\partial A} g\nu_j \, \mathrm{d}\sigma, \quad g \in \mathcal{C}_0(\partial A).$$

Writing $\nu = [\nu_1, \ldots, \nu_m]$ we have

$$P(A, v) = \int_{\partial A} v \cdot \nu \, \mathrm{d}\sigma, \quad v \in [\mathcal{C}_0^{(1)}]^m.$$

It is not difficult to prove that $|\nu| = 1$ almost everywhere with respect to σ; changing ν on a set of σ-measure zero we can assume that $|\nu| = 1$ everywhere on ∂A which gives us the required representation (3). The corresponding measure σ on ∂A is uniquely determined and ν is determined almost uniquely w.r. to σ.

The analytical definition of the perimeter $\|A\|$ presented in (4) does not offer concrete geometrical criteria guaranteeing (7). In order to get such criteria, J. Mařík used (6) and derived a geometrical interpretation for $\|A\|_j$. We are going to describe his result corresponding to $j = m$. It is based on the observation that for any Lebesgue measurable $B \subset \mathbb{R}$ the perimeter

$$\|B\| = \sup \Big\{ \Big| \int_B f' \, \mathrm{d}\lambda_1 \Big| ; \, f \in \mathcal{C}_0^{(1)}, |f| \leqslant 1 \Big\}$$

coincides with the total number (possibly zero or $+\infty$) of all points $t \in \mathbb{R}$ enjoying the property

$$\lambda_1 \big[(t - \delta, t + \delta) \cap B\big] > 0 \quad \text{and} \quad \lambda_1 \big[(t - \delta, t + \delta) \setminus B\big] > 0$$

for each $\delta > 0$; such a point t will be called a significant boundary point of B. With the notation described above and $\widehat{x} \in \mathbb{R}^{m-1}$ let $n_m(A, \widehat{x})$ denote the total number of all significant boundary points of $A_{\widehat{x}}$. J. Mařík proved that the function $\widehat{x} \mapsto n_m(A, \widehat{x})$ is λ_{m-1}-measurable and

$$\|A\|_m = \int_{\mathbb{R}^{m-1}} n_m(A, \widehat{x}) \, \mathrm{d}\lambda_{m-1}(\widehat{x}).$$

In a similar way one defines $n_j(A, \widehat{x})$ corresponding to the projection in the direction of the j-th coordinate axis (so that $\widehat{x} \in \mathbb{R}^{m-1}$ arises from $x \in \mathbb{R}^m$ by omitting j-th coordinate) for $j \in \{1, \ldots, m\}$; again, $\widehat{x} \mapsto n_j(A, \widehat{x})$ is λ_{m-1}-measurable and the formula

$$\|A\|_j = \int_{\mathbb{R}^{m-1}} n_j(A, \widehat{x}) \, \mathrm{d}\lambda_{m-1}(\widehat{x}), \quad j \in \{1, \ldots, m\} \tag{8}$$

holds. Let $\mathfrak{A}$ be the class of all Lebesgue measurable sets $A \subset \mathbb{R}^m$ with finite perimeter (7); then a Lebesgue measurable $A \subset \mathbb{R}^m$ belongs to $\mathfrak{A}$ iff the integrals occuring in (8) are finite ($j = 1, \ldots, m$). This geometrical characterization of sets in $\mathfrak{A}$ represents the main result of [7]. As a consequence it follows that $\mathfrak{A}$ is an algebra of sets $\subset \mathbb{R}^m$. Let us note that $\lambda_m(\partial A) > 0$ is possible for an $A \in \mathfrak{A}$. In order to get such examples it is sufficient to consider Swiss-cheese type sets of the form

$$A = B_1(0) \setminus \bigcup_{n=1}^{\infty} B_{r_n}(z_n),$$

where $B_r(z)$ denotes the open ball of center $z \in \mathbb{R}^m$ and radius $r > 0$, $z_n \in B_1(0)$ form a dense set in $B_1(0)$ and r_n satisfy $\sum\limits_{n=1}^{\infty} r_n^{m-1} < 1$. Nevertheless, as observed by Mařík, the surface area measure σ corresponding to an $A \in \mathfrak{A}$ is always carried by a Borel set $B \subset \partial A$ with $\lambda_m(B) = 0$.

$\mathfrak{A}$ consists precisely of those Lebesgue measurable sets $A \subset \mathbb{R}^m$ for which the characteristic function χ_A has distributional partial derivatives $\partial_j \chi_A \quad (j = 1, \ldots, m)$ representable by finite signed Borel measures. Under the simplifying assumption $\lambda_m(\partial A) = 0$ such sets were treated later in [5].

Useful modifications of the formula (8) are due to M. Chlebík [1]. Let us recall the concept of the essential boundary $\partial_e A$ (occurring in [9]) which is formed by those $x \in \mathbb{R}^m$ for which both A and $\mathbb{R}^m \setminus A$ have positive upper m-dimensional density at x:

$$\limsup_{r \downarrow 0} \frac{\lambda_m\big[B_r(x) \cap A\big]}{\lambda_m\big[B_r(x)\big]} > 0 \quad \text{and} \quad \limsup_{r \downarrow 0} \frac{\lambda_m\big[B_r(x) \setminus A\big]}{\lambda_m\big[B_r(x)\big]} > 0.$$

It can be proved that, for λ_{m-1}-a.e. $\widehat{x} \in \mathbb{R}^{m-1}$, $n_j(A, \widehat{x})$ coincides with the total number of points in $(\partial_e A)_{\widehat{x}}$; it is remarkable that this holds for any measurable $A \subset \mathbb{R}^m$ (even if $A \notin \mathfrak{A}$).

Mařík's papers do not give a geometric characterization of the corresponding normal vector ν on ∂A ($A \in \mathfrak{A}$). Such a characterization of ν and also the complete description of the surface area measure σ were made possible by results of E. De Giorgi [2], [3] complemented by H. Federer (cf. section 4.5 in [4]). Let us recall that a unit vector $n_A(z) \in \mathbb{R}^m$ is termed Federer's exterior normal of a measurable $A \subset \mathbb{R}^m$ at $z \in \mathbb{R}^m$ provided the half-space

$$H = \{x \in \mathbb{R}^m \,;\, (x - z) \cdot n_A(z) < 0\}$$

approximates A closely near z in the sense that the symmetric difference of A and H, to be denoted by

$$D := (A \setminus H) \cup (H \setminus A),$$

has vanishing m-dimensional density at z:

$$\lim_{r \downarrow 0} \frac{\lambda_m\big[B_r(z) \cap D\big]}{\lambda_m\big[B_r(z)\big]} = 0;$$

the set of all $z \in \mathbb{R}^m$ for which such a vector $n_A(z)$ (which is then uniquely determined) exists is called the reduced boundary of A and denoted by $\partial_* A$. Clearly, $\partial_* A \subset \partial_e A \subset \partial A$. Let us agree to denote by λ_{m-1} the $(m-1)$-dimensional Hausdorff measure (with the appropriate normalization, so that λ_{m-1} coincides with the Lebesgue measure on subsets of $\mathbb{R}^{m-1}$). Then a Lebesgue measurable $A \subset \mathbb{R}^m$ belongs to $\mathfrak{A}$ iff $\lambda_{m-1}(\partial_e A) < \infty$. If this is the case, then $\lambda_{m-1}(\partial_e A \setminus \partial_* A) = 0$ and the corresponding surface area measure σ coincides with the restriction of λ_{m-1} to $\partial_* A$:

$$\sigma(B) = \lambda_{m-1}(B \cap \partial_* A) \quad \text{for all Borel sets } B \subset \partial A;$$

besides, the normal vector $\nu(z)$ coincides with Federer's exterior normal $n_A(z)$ for σ-a.e. $z \in \partial A$.

Let us mention that E. De Giorgi and his collaborators showed that sets of finite perimeter (called also Cacciopoli's sets in honour of this author's approach to surface integration) represent a handy tool for general formulation and treatment of the Plateau problem in the theory of minimal surfaces, but these topics remained beyond the scope of Mařík's interests. Mařík's paper [8] contains a proof of the fact that the perimeter of any complementary domain of a Jordan curve C in the plane coincides with the length of C.

In higher dimensional spaces the situation is much more complicated. It may happen for $m \geqslant 3$ that one complementary domain of a topological sphere $S \subset \mathbb{R}^m$ has finite perimeter while the perimeter of the other complementary domain of the same S is infinite; of course, this is possible only if $\lambda_m(S) > 0$. (A method of construction of such topological spheres is due to A. S. Besicovitch.)

Lectures and papers of J. Mařík influenced considerably research of Czech mathematicians in area theory, in potential theory and in the theory of non-absolutely convergent integrals; this last topic will be treated in a separate lecture.

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