A GROWTH ESTIMATE FOR CONTINUOUS RANDOM FIELDS

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Abstract. We prove a polynomial growth estimate for random fields satisfying the Kolmogorov continuity test. As an application we are able to estimate the growth of the solution to the Cauchy problem for a stochastic diffusion equation.

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1. Introduction

The main purpose of this paper is to give a growth estimate for random fields satisfying the famous Kolmogorov test for the existence of a continuous version. Growth estimates for this version (for parameters tending to infinity) are important in many applications. As an example we will consider the growth of the solution to the Cauchy problem for a linear stochastic diffusion equation. This behaviour determines a function space in which the solution of the corresponding semilinear equation will exist. The paper is divided into four sections. In the next section we prove the main result. Then we discuss the Gaussian case. The final section is devoted to applications.

[1] Partially supported by DFG

2. Main result

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space. Denote by $\mathbb{N}$ the set $\{1, 2, \ldots\}$. We shall prove the following assertion.

Theorem 2.1. *Let $Z = (Z(r))_{r \in \mathbb{R}^d}$ be a random field with the property that there exist constants $\nu > 1$, $\kappa \geqslant 0$, $\alpha > d$ and $c > 0$ such that*

$$\mathbb{E}|Z(r) - Z(s)|^\nu \leqslant c \cdot n^\kappa \cdot |r - s|^\alpha$$

for all $n \in \mathbb{N}$ and any $r, s \in [-n, n]^d$. Then the following holds:
(i) *There exists a pathwise locally H lder continuous version Y of Z.*
(ii) *For any $\delta > 1$ this version satisfies the inequality*

$$|Y(r)| \leqslant \eta_\delta \cdot \left(1 + |r|^{\frac{\alpha+\delta+\kappa}{\nu}}\right)$$

for any $r \in \mathbb{R}^d$ where η_δ is a random variable with $\mathbb{P}(\{0 < \eta_\delta < \infty\}) = 1$.

Proof. (i) The conditions of the Kolmogorov continuity test are satisfied on $[-n, n]^d$, $n \in \mathbb{N}$, which by a standard argument implies the existence of a pathwise locally Hölder continuous version Y of Z on $\mathbb{R}^d$. From now on we will deal only with Y. $\square$

(ii) In order to show the growth estimate for Y we will use the following auxilary result.

Lemma 2.2 (Garsia [3]). *Let $\psi\colon \mathbb{R} \to \mathbb{R}_+$ and $p\colon [-\sqrt{d}, \sqrt{d}] \to \mathbb{R}_+$ be continuous even functions, where p is increasing for $u \in [0, \sqrt{d}]$ and $p(0) = 0$, while ψ is convex and $\lim\limits_{x \to \infty} \psi(x) = \infty$. If $f\colon [0,1]^d \to \mathbb{R}$ is a continuous function with the property*

$$\int_{[0,1]^d} \int_{[0,1]^d} \psi\left(\frac{f(r) - f(s)}{p(|r - s|/\sqrt{d})}\right) \mathrm{d}s\, \mathrm{d}r =: B(f) < \infty,$$

then

$$|f(t) - f(s)| \leqslant 8 \cdot \int_0^{|t-s|} \psi^{-1}\left(\frac{B(f)}{u^{2d}}\right) \mathrm{d}p(u) \tag{2.1}$$

holds for all $t, s \in [0,1]^d$, where ψ^{-1} is defined by

$$\psi^{-1}(u) := \begin{cases} \sup\{r \in \mathbb{R}\colon \psi(r) \leqslant u\}, & u \geqslant \psi(0), \\ \\ 0, & \textit{else.} \end{cases}$$

Put now $a = (1/2, \dots, 1/2) \in \mathbb{R}^d$ and define random fields $Y^{(n)} \colon \Omega \times [0,1]^d \to \mathbb{R}$, $n \in \mathbb{N}$, by

$$Y^{(n)}(r) := Y(2n(r-a)),\ r \in [0,1]^d.$$

Set $\varepsilon := \alpha - d$. We obtain

$$\begin{aligned}\mathbb{E}|Y^{(n)}(r) - Y^{(n)}(s)|^\nu &= \mathbb{E}|Y(2n(r-a)) - Y(2n(s-a))|^\nu \\ &\leqslant c \cdot n^\kappa \cdot |2n(r-s)|^{d+\varepsilon} \\ &= c \cdot 2^{d+\varepsilon} \cdot n^{\kappa+d+\varepsilon} \cdot |r-s|^{d+\varepsilon}.\end{aligned}$$

In order to apply Lemma 2.2 we put

$$\psi(x) := |x|^\nu,\ x \in \mathbb{R}^d$$

and

$$p(x) := |x|^{\frac{2d+\varepsilon}{\nu}} \cdot \left(\log \frac{\gamma}{|x|}\right)^{2/\nu},\ |x| \leqslant \sqrt{d},$$

where

$$\gamma := \sqrt{d} \cdot \exp(\nu/d).$$

Furthermore, define

$$B(Y^{(n)}) := \int_{[0,1]^d} \int_{[0,1]^d} \psi\left(\frac{Y^{(n)}(r) - Y^{(n)}(s)}{p(|r-s|/\sqrt{d})}\right) \mathrm{d}s\, \mathrm{d}r$$

and

$$Q := \sum_{n=1}^{\infty} (2n)^{-(d+\delta+\varepsilon+\kappa)} \cdot B(Y^{(n)}),$$

where $\delta > 1$. As in Walsh [9], proof of Corollary 1.2, we get

$$\mathbb{E}B(Y^{(n)}) \leqslant c_0(d,\varepsilon,\nu) \cdot (2n)^{d+\varepsilon+\kappa},$$

which implies

$$0 \leqslant \mathbb{E}Q \leqslant c_0(d,\varepsilon,\nu) \cdot \sum_{n=1}^{\infty} (2n)^{-\delta} < \infty$$

and hence $0 \leqslant Q < \infty$ $\mathbb{P}$ a.s. Now we shall proceed pathwise. For this purpose we fix an $\omega \in \Omega$ such that $Q(\omega) < \infty$. Obviously, we have

$$B(Y^{(n)}) \leqslant (2n)^{d+\delta+\varepsilon+\kappa} \cdot Q < \infty$$

for every $n \in \mathbb{N}$. Applying now Lemma 2.2 to $B(Y^{(n)})$ and $Y^{(n)}$ we arrive at

$$|Y^{(n)}(r) - Y^{(n)}(s)| \leqslant c_1(d, \varepsilon, \nu) \cdot (2n)^{\frac{d+\delta+\varepsilon+\kappa}{\nu}} \cdot Q^{1/\nu} \cdot |r-s|^{\varepsilon/\nu} \cdot \Big(\log \frac{\gamma}{|r-s|} \Big)^{2/\nu}$$

for any $r, s \in [0,1]^d$, compare again Walsh [9]. This yields

$$\begin{aligned} |Y(r) - Y(s)| &= \Big| Y^{(n)}\Big(\frac{r}{2n} + a\Big) - Y^{(n)}\Big(\frac{s}{2n} + a\Big) \Big| \\ &\leqslant c_2(d, \delta, \varepsilon, \kappa, \nu, \omega) \cdot n^{\frac{d+\delta+\kappa}{\nu}} \cdot |r-s|^{\varepsilon/\nu} \cdot \Big(\log \frac{2\gamma n}{|r-s|} \Big)^{2/\nu} \end{aligned}$$

for $r, s \in [-n, n]^d$ and hence

$$|Y(r)| \leqslant |Y(0)| + c_2(d, \delta, \varepsilon, \kappa, \nu, \omega) \cdot n^{\frac{d+\delta+\kappa}{\nu}} \cdot |r|^{\varepsilon/\nu} \cdot \Big(\log \frac{2\gamma n}{|r|} \Big)^{2/\nu}$$

for $r \in [-n, n]^d$. Consequently, if $n - 1 \leqslant |r| \leqslant n$, $n \in \mathbb{N}$, we get $n \leqslant |r| + 1$ and

$$|Y(r)| \leqslant |Y(0)| + c_2(d, \delta, \varepsilon, \kappa, \nu, \omega) \cdot (|r|+1)^{\frac{d+\delta+\kappa}{\nu}} \cdot |r|^{\varepsilon/\nu} \cdot \Big(\log \frac{2\gamma(|r|+1)}{|r|} \Big)^{2/\nu}.$$

Since $|r| \geqslant 1$ implies $|r| + 1 \leqslant 2|r|$ we can continue in this case by

$$\leqslant |Y(0)| + c_3(d, \delta, \varepsilon, \kappa, \nu, \omega) \cdot |r|^{\frac{d+\delta+\kappa+\varepsilon}{\nu}}.$$

Moreover, we know that

$$\sup_{|r| \leqslant 1} |Y(r)| < \infty.$$

Therefore, we finally have

$$|Y(r)| \leqslant c(d, \delta, \varepsilon, \kappa, \nu, \omega)\big(1 + |r|^{\frac{d+\delta+\kappa+\varepsilon}{\nu}}\big)$$

for any $r \in \mathbb{R}^d$ and almost all $\omega \in \Omega$, which completes the proof.

Intuitively, it is clear that the growth estimate should be better if we additionally know that Z is Gaussian. In fact, we obtain the following assertion.

Theorem 3.1. *Let $Z = (Z(r))_{r\in\mathbb{R}^d}$ be a centered Gaussian random field such that there exist constants $\varepsilon > 0$ and $\kappa \geqslant 0$ with*

$$\mathbb{E}|Z(r) - Z(s)|^2 \leqslant c \cdot n^\kappa \cdot |r - s|^\varepsilon$$

for any $r, s \in [-n, n]^d$, $n \in \mathbb{N}$. Then there exists a pathwise locally H lder continuous version Y of Z with the property

$$|Y(r)| \leqslant \eta \left(1 + \sqrt{|r|^{\kappa+\varepsilon} \cdot \log(1 + |r|)}\right) \ \mathbb{P}\ a.s.$$

for any $r \in \mathbb{R}^d$, where η is a random variable with $\mathbb{P}(\{0 < \eta < \infty\}) = 1$.

Proof. The existence of a locally Hölder continuous version Y of Z follows immediately from Theorem 2.1 and the well-known fact that

$$\mathbb{E}\xi^{2p} = (2p - 1)!! \cdot (\mathbb{E}\xi^2)^p$$

holds for centered Gaussian random variables ξ. In order to apply Lemma 2.2 we choose

$$\psi_n(x) = \exp\left(x^2/(3 \cdot 2^\varepsilon cn^{\kappa+\varepsilon})\right)$$

and

$$p(u) = \left(\sqrt{d} \cdot |u|\right)^{\varepsilon/2}.$$

Let $Y^{(n)}$ be the random fields defined in the proof of Theorem 2.1. By virtue of

$$\mathbb{E}\left[Y^{(n)}(r) - Y^{(n)}(s)\right]^2 \leqslant 2^\varepsilon cn^{\kappa+\varepsilon} \cdot |r - s|^\varepsilon$$

for $r, s \in [0, 1]^d$ it follows that

$$\frac{Y^{(n)}(r) - Y^{(n)}(s)}{p(|r - s|/\sqrt{d})}$$

is a Gaussian variable with mean zero and second moment less than $2^\varepsilon cn^{\kappa+\varepsilon}$, which leads to $1 \leqslant \mathbb{E}B(Y^{(n)}) \leqslant \sqrt{3}$, $n \in \mathbb{N}$, where

$$B(Y^{(n)}) := \int_{[0,1]^d} \int_{[0,1]^d} \psi_n \left(\frac{Y^{(n)}(r) - Y^{(n)}(s)}{p(|r - s|/\sqrt{d})}\right) \mathrm{d}s\, \mathrm{d}r.$$

Set

$$Q := \sum_{n=1}^{\infty} n^{-\delta} \cdot B(Y^{(n)})$$

for $\delta > 1$. Because of $\mathbb{E}Q < \infty$ we can choose an ω belonging to a set Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that $Q(\omega)$ is finite, and proceed by a pathwise consideration. We obtain $\psi_n^{-1}(u) = (3 \cdot 2^{\varepsilon} c n^{\kappa+\varepsilon} \cdot \log(u))^{1/2}$, $u \geqslant 1$. From

$$B(Y^{(n)}) \geqslant 1$$

we get

$$\begin{aligned}
|Y^{(n)}(r) - Y^{(n)}(s)| &\leqslant 8 \cdot \int_0^{|r-s|} \psi_n^{-1}\left(\frac{B(Y^{(n)})}{u^{2d}}\right) \mathrm{d}p(u) \\
&\leqslant 4\varepsilon\sqrt{3 \cdot 2^{\varepsilon} c d^{\varepsilon/2} n^{\kappa+\varepsilon}} \cdot \int_0^{|r-s|} \sqrt{|\log B(Y^{(n)}) - 2d \cdot \log(u)|} \cdot u^{\frac{\varepsilon}{2}-1}\,\mathrm{d}u \\
&\leqslant c_4(\varepsilon, d) \cdot n^{\frac{\varepsilon+\kappa}{2}} \cdot \left[\int_0^{|r-s|} \sqrt{|\log B(Y^{(n)})|} \cdot u^{\frac{\varepsilon}{2}-1}\,\mathrm{d}u + \int_0^{|r-s|} \sqrt{|\log(u)|} \cdot u^{\frac{\varepsilon}{2}-1}\,\mathrm{d}u\right] \\
&\leqslant c_5(\varepsilon, d) \cdot n^{\frac{\varepsilon+\kappa}{2}} \\
&\quad \times \left[\left(|\log Q|^{1/2} + |\delta \cdot \log(n)|^{1/2}\right)|r-s|^{\varepsilon/2} + \int_0^{|r-s|} |\log(u)|^{1/2} \cdot u^{\frac{\varepsilon}{2}-1}\,\mathrm{d}u\right].
\end{aligned}$$

The integral is bounded from above by

$$\leqslant c_6(\varepsilon) \cdot \left[|r-s|^{\varepsilon/2} \cdot \big|\log|r-s|\big|^{1/2} + 1\right].$$

Hence

$$\begin{aligned}
|Y(r) - Y(s)| &\leqslant c_7(\varepsilon, d, \delta, \kappa, \omega) \cdot n^{\frac{\varepsilon+\kappa}{2}} \\
&\times \left[\left(\frac{|r-s|}{2n}\right)^{\varepsilon/2}\left(1 + \log^{1/2}(n) + \left|\log\left(\frac{|r-s|}{2n}\right)\right|^{1/2}\right) + 1\right]
\end{aligned}$$

holds for $r, s \in [-n, n]^d$. Using the same argument as in the the general case we finally arrive at

$$|Y(r)| \leqslant c_8(\varepsilon, d, \delta, \kappa, \omega) \cdot \left[1 + (|r|^{\kappa+\varepsilon} \cdot \log(1 + |r|))^{1/2}\right],$$

proving the theorem. $\square$

4.1. Wiener fields. First let us consider the standard Wiener process $w = (w(t))_{t\geqslant 0}$. Theorem 2.1 gives $\mathbb{P}$ a.s.

$$|w(t)| \leqslant \eta_a(1+t^a),\ a\in(\tfrac{1}{2},1)$$

while Theorem 3.1 leads to

$$|w(t)| \leqslant \eta\left(1+\sqrt{t\cdot\log(1+t)}\right)\ \ \mathbb{P}\ a.s.,$$

which is quite close to the exact asymptotic behaviour. In general, if w is the Brownian sheet we obtain the same with $|t|$, $t\in\mathbb{R}^d_+$ instead of $t\geqslant 0$. In order to introduce a class of Wiener fields which is used beside the Brownian sheet as the driving term for stochastic partial differential equations we must do a little preliminary work. Choose an orthonormal basis (ONB) $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{L}_2(\mathbb{R}^d)$ consisting of elements such that the following is satisfied:

(i) The mappings $g_n\colon \mathbb{R}^d\to\mathbb{R}$ defined by

$$g_n(x) := \int_{-\infty}^{x_d}\cdots\int_{-\infty}^{x_1} h_n(y)\,\mathrm{d}y$$

are uniformly Lipschitz continuous.

(ii) We have

$$\sup_{n\in\mathbb{N}}(\|h_n\|_\infty + \|g_n\|_\infty) < \infty,$$

where $\|.\|_\infty$ denotes the sup-norm.

(iii) For any $n\in\mathbb{N}$ there exist

$$\frac{\partial}{\partial x_d}g_n,\ldots,\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_d}g_n$$

for any $x\in\mathbb{R}^d$. These derivatives are continuous and satisfy

$$\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_d}g_n(x) = h_n(x),$$

$n\in\mathbb{N}, x=(x_1,\ldots,x_d)\in\mathbb{R}^d$.

Such an ONB always exists. To see this let $\mathbb{Z}$ denote the integers and notice that

$$\mathbb{L}_2(\mathbb{R}) = \bigoplus_{k\in\mathbb{Z}}\mathbb{L}_2(k\pi,(k+1)\pi).$$

The system

$$e_{k,n}(x) := (2/\pi)^{1/2} \cdot \sin n(x - k\pi),\ x \in (k\pi, (k+1)\pi),\ n \in \mathbb{N},\ k \in \mathbb{Z},$$

is an ONB in $\mathbb{L}_2(k\pi, (k+1)\pi)$. Define for $n \in \mathbb{N}$, $k \in \mathbb{Z}$

$$e^0_{k,n}(x) := \begin{cases} e_{k,n}(x), x \in (k\pi, (k+1)\pi), \\ \\ 0, \text{else}, \end{cases}$$

and let $\beta\colon \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$ be a bijection. Then

$$e_j := e^0_{k(j),n(j)}$$

form an ONB in $\mathbb{L}_2(\mathbb{R})$, where $(k(j), n(j)) = \beta^{-1}(j)$, $j \in \mathbb{N}$. Moreover, the mappings e_j are Lipschitz continuous and

$$\sup_{j\in\mathbb{N}, x\in\mathbb{R}} |e_j(x)| \leqslant (2/\pi)^{1/2}$$

holds. Setting

$$\tilde{g}_j(x) = \int_{-\infty}^{x} e_j(y)\,\mathrm{d}y,\ x \in \mathbb{R},\ j \in \mathbb{N},$$

we get a differentiable mapping with

$$\sup_{j\in\mathbb{N}, x\in\mathbb{R}} |\tilde{g}_j(x)| \leqslant (2\pi)^{1/2}.$$

Furthermore, let

$$\bar{h}_i(x) := \prod_{l=1}^{d} e_{i(l)}(x_l)$$

and

$$\bar{g}_i(x) := \prod_{l=1}^{d} \tilde{g}_{i(l)}(x_l),$$

where $x = (x_1, \ldots, x_d)$ and $i = (i(1), \ldots, i(d)) \in \mathbb{N}^d$. Finally, for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$ put

$$h_n(x) := \bar{h}_{\kappa(n)}(x) \text{ and } g_n(x) := \bar{g}_{\kappa(n)}(x),$$

where $\kappa\colon \mathbb{N} \to \mathbb{N}^d$ is a bijection. One easily proves that

- $(h_n)_{n\in\mathbb{N}}$ *is an ONB in* $\mathbb{L}_2(\mathbb{R}^d)$,
- the mappings g_n, $n \in \mathbb{N}$, are uniformly Lipschitz continuous,

- $\sup\limits_{n\in\mathbb{N}}(\|h_n\|_\infty + \|g_n\|_\infty) < \infty$,
- $\frac{\partial^d}{\partial x_1 \ldots \partial x_d} g_n(x) = h_n(x)$.

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\sum\limits_{n=1}^{\infty} \lambda_n < \infty$. Denote by B a Wiener process on $\mathbb{L}_2(\mathbb{R}^d)$ with the covariance operator Γ given by

$$\Gamma h := \sum_{n=1}^{\infty} \lambda_n (h, h_n) h_n, \ \ h \in \mathbb{L}_2(\mathbb{R}^d),$$

where $(.,.)$ denotes the inner product in $\mathbb{L}_2(\mathbb{R}^d)$. Then

$$W(t,x) = \sum_{n=1}^{\infty} B_t(h_n) g_n(x)$$

is a centered Gaussian random field with covariance

$$\mathbb{E} W(t,x) W(s,y) = (t \wedge s) \cdot \theta(x,y),$$

where

$$\theta(x,y) = \sum_{n=1}^{\infty} \lambda_n g_n(x) g_n(y).$$

Because of

$$\begin{aligned}
&\mathbb{E}[W(t,x) - W(s,y)]^2 \\
&\quad \leqslant 2 \cdot \mathbb{E}\Big[\sum_{k=1}^{\infty} B_t(h_k)(g_k(x) - g_k(y))\Big]^2 + 2 \cdot \mathbb{E}\Big[\sum_{k=1}^{\infty} (B_t(h_k) - B_s(h_k)) g_k(y)\Big]^2 \\
&\quad \leqslant \mathrm{const} \cdot t \cdot |x-y| \cdot \sup_{k\in\mathbb{N}} \sup_{x\in\mathbb{R}^d} |g_k(x)| + \mathrm{const} \cdot |t-s| \cdot \sup_{k\in\mathbb{N}} \sup_{x\in\mathbb{R}^d} |g_k(x)|^2 \\
&\quad \leqslant \mathrm{const} \cdot [t \cdot |x-y| + |t-s|]
\end{aligned}$$

we get for $t \leqslant n$

$$\mathbb{E}[W(t,x) - W(s,y)]^2 \leqslant \mathrm{const} \cdot n |(t,x) - (s,y)|.$$

Setting $W(t,x) = W(0,x)$ for $t < 0$ and applying Theorem 3.1 we finally obtain

$$|W(t,x)| \leqslant \eta \cdot \Big(1 + \big(|(t,x)|^2 \cdot \log(1 + |(t,x)|)\big)^{1/2}\Big)$$

$\mathbb{P}$ a.s.

4.2. The Cauchy problem for a stochastic diffusion equation.

Let $L = \frac{\partial}{\partial t} - A$ be a parabolic second order differential operator with Hölder continuous and bounded coefficients and consider the formal Cauchy problem

$$\begin{aligned} (Lu)(t,x) &= \sigma(t,x)\cdot\xi(t,x), \quad (t,x)\in(0,T]\times\mathbb{R}^d, \\ u(0,x) &= 0, \quad x\in\mathbb{R}^d, \end{aligned}$$

where $0 < T < \infty$ is a fixed real number, $\sigma\colon \Omega\times[0,T]\times\mathbb{R}^d \to \mathbb{R}$ satisfies some conditions which will be specified below and ξ is a Gaussian noise which is white in time and coloured (i.e. correlated) in space. That means

$$\xi = \big(\xi(\varphi), \varphi\in\mathcal{D}((0,T)\times\mathbb{R}^d)\big)$$

is a mean zero generalized random function with

$$\mathbb{E}\xi(\varphi)\xi(\psi) = \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(t,x)\psi(t,y)\eta(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,$$

where $\mathcal{D}$ is the test function space and

$$\eta(x,y) := \sum_{n=1}^{\infty}\lambda_n h_n(x)h_n(y)$$

with $(\lambda_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ was introduced in Subsection 4.1. One easily proves

$$\xi = \frac{\partial^{d+1}}{\partial t\partial x_1\ldots\partial x_d}W$$

in distribution (i.e. the covariance functionals of the two generalized random functions coincide.) One can also consider a space-time Gaussian white noise. But in this case the space dimension d is restricted to one (cf. Manthey [6], Walsh [9]). Let $\mathbb{F} = (\mathfrak{F}_t)_{t\in[0,T]}$ be the filtration generated by W. Furthermore, let $f\colon \Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}$ be progressively measurable and such that

$$\text{(S)} \qquad \mathbb{E}\int_0^T\left[\int_{\mathbb{R}^d}|f(\omega,t,x)|\,\mathrm{d}x\right]^2\mathrm{d}t < \infty.$$

(The mapping f is called *progressively measurable*, if for every $t\in[0,T]$ the restriction of f to $\Omega\times[0,t]\times\mathbb{R}^d$ is $\mathfrak{F}_t\otimes\mathfrak{B}([0,t])\otimes\mathfrak{B}(\mathbb{R}^d)$-measurable.)

For this class of mappings the stochastic integral

$$I_t(f) = \int_0^t\int_{\mathbb{R}^d}f(s,x)\,\mathrm{d}W_{sx} := \sum_{n=1}^{\infty}\int_0^t\left[\int_{\mathbb{R}^d}f(s,x)h_n(x)\,\mathrm{d}x\right]\mathrm{d}B_s(h_n)$$

is well defined, i.e., the sum on the right hand side converges in $\mathbb{L}_2(\Omega)$. It is an $\mathbb{F}$-adapted mean zero random linear functional with covariance

$$\mathbb{E} I_t(f) I_s(g) = \mathbb{E} \sum_{n=1}^{\infty} \lambda_n \cdot \int_0^{t \wedge s} \left[\int_{\mathbb{R}^d} f(r,x) h_n(x) \, \mathrm{d}x \right] \left[\int_{\mathbb{R}^d} g(r,y) h_n(y) \, \mathrm{d}y \right] \mathrm{d}r.$$

Let $(t,x,s,y) \to G(t,x,s,y)$ denote the fundamental solution corresponding to the above Cauchy problem for $\sigma = 0$. We set $G(t,x,s,y) = 0$ if $s \in [t,T]$. Note that there exist positive constants c_1 and c_2 such that

$$\text{(G1)} \qquad 0 \leqslant G(t,x,s,y) \leqslant c_1 \cdot |t-s|^{-d/2} \cdot \exp\Big(-c_2 \frac{|x-y|}{t-s}^2 \Big)$$

for $0 \leqslant s < t$, $x, y \in \mathbb{R}^d$, compare for instance Friedman [2]. Hence

$$0 < \int_{\mathbb{R}^d} G(t,x,s,y) \, \mathrm{d}y \leqslant c_G < \infty, \; 0 \leqslant s < t \leqslant T, \; x \in \mathbb{R}^d.$$

We will essentially use

$$\text{(G2)} \qquad \begin{aligned} &\int_0^t \Big[\int_{\mathbb{R}^d} (G(t,x,\tau,z) - \mathbb{1}_{[0,s)}(\tau) G(s,y,\tau,z)) \, \mathrm{d}z \Big]^2 \mathrm{d}\tau \\ &\qquad \leqslant c(\mu,T) \cdot \big[|t-s|^{2\mu} + |x-y|^{4\mu} \big] \end{aligned}$$

for an arbitrary $\mu \in (0, \frac{1}{2})$.

This is a consequence of the mean value theorem and the following property (compare Redlinger [7]):

(G3) For $2l + m \leqslant 2$ and $0 \leqslant s < t \leqslant T$, $x, y \in \mathbb{R}^d$,

$$|D_t^l D_x^m G(t,x,s,y)| \leqslant \mathrm{const} \cdot (t-s)^{-\frac{d+m}{2} - l} \cdot \exp\Big(-\mathrm{const} \cdot \frac{|x-y|^2}{t-s} \Big)$$

holds.

Property (G3) is well-known (cf. Iljin, Kalashnikov, Olejnik [4]).

Let us make the following assumptions:

(Σ) (i) *σ is progressively measurable.*

(ii) *There exist $p > 2(d+1)$ and $m \geqslant 0$ such that*

$$\mathbb{E}|\sigma(t,x)|^{2p} \leqslant c(p,T)(1 + |x|^m)$$

for $t \in [0,T]$ and $x \in \mathbb{R}^d$.

From (Σ(i)) it follows that $(\omega, s, y) \to \mathbb{1}_{[0,t)}(s) G(t,x,s,y)\sigma(s,y)$ is a progressively measurable mapping as well. Denote the mapping $x \to 1 + |x|^m$ *by* ϱ. In the sequel we shall use the following auxiliary fact.

Lemma 4.2.1. *There exists a constant* $a(m,T)$ *such that*

$$0 \leqslant \int_{\mathbb{R}^d} G(t,x,s,y)\varrho(y)\,\mathrm{d}y \leqslant a(m,T)\varrho(x)$$

holds for $0 \leqslant s < t \leqslant T$, $x \in \mathbb{R}^d$.

Proof. Property (G1) implies

$$\begin{aligned}
0 &\leqslant \int\limits_{\mathbb{R}^d} G(t,x,s,y)\varrho(y)\,\mathrm{d}y = \int\limits_{\mathbb{R}^d} G(t,x,s,y)\,\mathrm{d}y + \int\limits_{\mathbb{R}^d} G(t,x,s,y)|y|^m\,\mathrm{d}y \\
&\leqslant c_G + c(m)\cdot \int\limits_{\mathbb{R}^d} G(t,x,s,y)(|x|^m + |x-y|^m)\,\mathrm{d}y \\
&\leqslant c_G(1 + c(m)|x|^m) + \int\limits_{\mathbb{R}^d} c_1 \cdot |t-s|^{-d/2}\cdot |x-y|^m \cdot \exp\Big(-c_2 \frac{|x-y|^2}{t-s}\Big)\,\mathrm{d}y \\
&\leqslant a(m,T)\varrho(x),
\end{aligned}$$

proving the lemma. □

The *mild solution* u to the formal Cauchy problem we are studying is defined to be

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t,x,s,y)\sigma(s,y)\,\mathrm{d}W_{sy}.$$

Now we check the correctness of this definition under the condition (Σ). Splitting $G = G^{1-\frac{1}{p}} G^{\frac{1}{p}}$ and using the Hölder inequality as well as Lemma 4.2.1 we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^d} G(t,x,s,y)|\sigma(s,y)|\,\mathrm{d}y\right]^p \\
&\quad\leqslant \left[\int_{\mathbb{R}^d} G(t,x,s,y)\,\mathrm{d}y\right]^{p-1} \cdot \int_{\mathbb{R}^d} G(t,x,s,y)\mathbb{E}|\sigma(s,y)|^p\,\mathrm{d}y \\
&\quad\leqslant c_G^{p-1}\cdot \int_{\mathbb{R}^d} G(t,x,s,y)\mathbb{E}|\sigma(s,y)|^p\,\mathrm{d}y \\
&\quad\leqslant c(T,p,m)\varrho(x) < \infty
\end{aligned}$$

for $p \geqslant 2$, which easily yields (S).

We are mainly interested in the growth of u if $|x|$ tends to infinity.

Theorem 4.2.2. *Under the condition* (Σ) *there exists a pathwise locally H lder continuous version* v *of* u. *For any* $\delta > 1$ *this version has the property*

$$|v(t,x)| \leqslant \eta_\delta \cdot \left(1 + |x|^{\frac{1}{2}\cdot(\frac{\delta+m}{p}+1)}\right),$$

$(t,x) \in [0,T] \times \mathbb{R}^d$, *where* η_δ *is a random variable with* $\mathbb{P}(\{0 < \eta_\delta < \infty\}) = 1$.

Proof. We will use Theorem 2.1. Therefore, the main point is to obtain the input inequality of this assertion. Fix $(t,x), (s,y) \in [0,T] \times \mathbb{R}^d$, $s \leqslant t$, and set

$$\begin{aligned}\mathcal{M}(r) &:= \int_0^r \int_{\mathbb{R}^d} \big(G(t,x,\tau,z) - \mathbb{1}_{[0,s)}(\tau) G(s,y,\tau,z)\big)\sigma(\tau,z)\,\mathrm{d}W_{\tau z} \\ &= \sum_{n=1}^{\infty} \int_0^r \left[\int_{\mathbb{R}^d} \big(G(t,x,\tau,z) - \mathbb{1}_{[0,s)}(\tau) G(s,y,\tau,z)\big)\sigma(\tau,z) h_n(z)\,\mathrm{d}z \right] \mathrm{d}B_\tau(h_n)\end{aligned}$$

and

$$\mathcal{N}(r) := \int_0^r \int_{\mathbb{R}^d} G(t,x,\tau,z)\sigma(\tau,z)\,\mathrm{d}W_{\tau z}, \ \ r \in [0,t].$$

□

These processes are square integrable martingales with respect to the filtration $\mathbb{F}^t = (\mathfrak{F}_s)_{s\in[0,t]}$, possessing a pathwise continuous version which is considered in the sequel (cf. [1], Theorem 4.12 and 4.3.3).

Next we prove that the conditions of Theorem 2.1 are satisfied under the assumption required above. Consider

$$\begin{aligned}&\mathbb{E}\,[u(t,x) - u(s,y)]^{2p} \\ &\quad = \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \big(G(t,x,\tau,z) - \mathbb{1}_{[0,s)}(\tau) G(s,y,\tau,z)\big)\sigma(\tau,z)\,\mathrm{d}W_{\tau z} \right]^{2p}.\end{aligned}$$

Applying the Burkholder-Gundy inequality (compare for instance Liptser and Shiryayev [5]) to $\mathcal{M}$ we obtain

$$\begin{aligned}&\leqslant c(p) \cdot \mathbb{E}\Bigg[\sum_{n=1}^{\infty} \lambda_n \cdot \int_0^t \Big[\int_{\mathbb{R}^d} (G(t,x,\tau,z)) \\ &\qquad - \mathbb{1}_{[0,s)}(\tau) G(s,y,\tau,z))\sigma(\tau,z) h_n(z)\,\mathrm{d}z \Big]^2 \mathrm{d}\tau \Bigg]^p \\ &\leqslant c_1(p) \cdot \mathbb{E}\Bigg[\int_0^t \Big[\int_{\mathbb{R}^d} (G(t,x,\tau,z) \\ &\qquad - \mathbb{1}_{[0,s)}(\tau) G(s,y,\tau,z))\sigma(\tau,z)\,\mathrm{d}z \Big]^2 \mathrm{d}\tau \Bigg]^p\end{aligned}$$

To abbreviate the expressions we will use the notation

$$H(\tau,z) := |G(t,x,\tau,z) - \mathbb{1}_{[0,s)}(\tau)G(s,y,\tau,z)|$$

as well as

$$H^+(\tau,z) := G(t,x,\tau,z) + \mathbb{1}_{[0,s)}(\tau)G(s,y,\tau,z)$$

for fixed $(t,x),(s,y)$. We continue the estimate with the help of the Cauchy-Schwarz inequality by

$$\leqslant c_1(p)\cdot\mathbb{E}\left[\int_0^t\left[\int_{\mathbb{R}^d}H(\tau,z)\,\mathrm{d}z\right]\left[\int_{\mathbb{R}^d}H^+(\tau,z)\sigma^2(\tau,z)\,\mathrm{d}z\right]\mathrm{d}\tau\right]^p$$
$$\leqslant c_1(p)\cdot\mathbb{E}\left[\int_0^t\left[\int_{\mathbb{R}^d}H(\tau,z)\,\mathrm{d}z\right]^2\mathrm{d}\tau\right]^{p/2}\cdot\left[\int_0^t\left[\int_{\mathbb{R}^d}H^+(\tau,z)\sigma^2(\tau,z)\,\mathrm{d}z\right]^2\mathrm{d}\tau\right]^{p/2}.$$

To handle the first integral we use (G2):

$$\leqslant c_2(p,T,\mu)\cdot\left[|t-s|^{2\mu}+|x-y|^{4\mu}\right]^{p/2}\cdot\mathbb{E}\left[\int_0^t\left[\int_{\mathbb{R}^d}H^+(\tau,z)\sigma^2(\tau,z)\,\mathrm{d}z\right]^2\mathrm{d}\tau\right]^{p/2}.$$

Let us seperately estimate the last expression by Hölder's inequality:

$$\begin{aligned}&\mathbb{E}\left[\int_0^t\left[\int_{\mathbb{R}^d}H^+(\tau,z)\sigma^2(\tau,z)\,\mathrm{d}z\right]^2\mathrm{d}\tau\right]^{p/2}\\ &\quad\leqslant c_3(T,p)\cdot\mathbb{E}\int_0^t\left[\int_{\mathbb{R}^d}H^+(\tau,z)\sigma^2(\tau,z)\,\mathrm{d}z\right]^p\mathrm{d}\tau\\ &\quad\leqslant c_3(T,p)\cdot\int_0^t\left[\int_{\mathbb{R}^d}H^+(\tau,z)\,\mathrm{d}z\right]^{p-1}\mathrm{d}\tau\cdot\int_0^t\int_{\mathbb{R}^d}H^+(\tau,z)\mathbb{E}\sigma^{2p}(\tau,z)\,\mathrm{d}z\,\mathrm{d}\tau.\end{aligned}$$

Using (G1) we obtain

$$\leqslant c_4(T,p)\cdot\int_0^t\int_{\mathbb{R}^d}H^+(\tau,z)\mathbb{E}\sigma^{2p}(\tau,z)\,\mathrm{d}z\,\mathrm{d}\tau,$$

and (Σ(ii)) together with Lemma 4.2.1 leads to

$$\leqslant c_5(T,p)\cdot(1+|x|^m+|y|^m).$$

Summing up we have

$$\mathbb{E}\left[u(t,x)-u(s,y)\right]^{2p}\leqslant c_6(p,T,\mu)\cdot\left[|t-s|^{2\mu}+|x-y|^{4\mu}\right]^{p/2}\cdot(1+|x|^m+|y|^m)$$

for an arbitrary $\mu \in (0, \frac{1}{2})$. Put $u(t,.) := u(T,.)$ for $t > T$ and $u(t,.) := 0$ for $t \leqslant 0$. Hence u is defined on the whole $\mathbb{R}^{d+1}$. Moreover, we observe that

$$u(t,x) - u(s,y) = u(0 \vee t \wedge T, x) - u(0 \vee s \wedge T, y).$$

Consequently, if $(t,x), (s,y) \in [-n,n]^{d+1}$, the above estimate implies

$$\mathbb{E}\left[u(t,x) - u(s,y)\right]^{2p} \leqslant c_7(p,T,\mu) \cdot n^m \cdot \left[|t-s|^{2\mu} + |x-y|^{4\mu}\right]^{p/2}.$$

By $(\Sigma(\text{ii}))$ there exist $\varepsilon > 0$ and $\mu \in (0, \frac{1}{2})$ such that

$$p > \frac{d+1+\varepsilon}{\mu}.$$

Thus we can find a $b > 0$ ensuring the representation

$$p = (\mu^{-1} + b)(d + 1 + \varepsilon).$$

This leads to

$$\begin{aligned}
\left[|t-s|^{2\mu} + |x-y|^{4\mu}\right]^{p/2} &\leqslant c_8(p)\left[|t-s|^{2+2\mu b} + |x-y|^{4+4\mu b}\right]^{\frac{d+1+\varepsilon}{2}} \\
&\leqslant c_8(p)\left[(|t-s|^{2\mu b} \vee |x-y|^{2+4\mu b})(|t-s|^2 + |x-y|^2)\right]^{\frac{d+1+\varepsilon}{2}} \\
&\leqslant c_9(p) n^{(1+2\mu b)(d+1+\varepsilon)} \cdot |(t,x) - (s,y)|^{d+1+\varepsilon}
\end{aligned}$$

if $(t,x), (s,y) \in [-n,n]^{d+1}$. Therefore, Theorem 2.1 implies the existence of a locally Hölder continuous version v of u. Moreover, Theorem 2.1 gives the growth estimate

$$|v(t,x)| \leqslant c_\delta(p,T,d) \cdot \left(1 + |(t,x)|^{\frac{2(d+\varepsilon+1)(1+\mu b)+\delta+m}{2p}}\right),$$

$t \in [0,T], x \in \mathbb{R}^d$, from which we easily obtain

$$|v(t,x)| \leqslant \eta_\delta(p,T,d)\left(1 + |x|^{\frac{1}{2}(\frac{\delta+m}{p}+1)}\right),$$

$t \in [0,T]$, $x \in \mathbb{R}^d$, where $\mathbb{P}(\{0 < \eta_\delta(p,T,d) < \infty\}) = 1$. This proves Theorem 4.2.2.

Finally, we shall discuss some particular cases. First we assume instead of $(\Sigma(\text{ii}))$ the more restrictive condition that there exists $p > 2(d+1)$ such that

(BM) $$\sup_{\substack{t\in[0,T] \\ x\in\mathbb{R}^d}} \mathbb{E}|\sigma(t,x)|^p \leqslant c < \infty.$$

Obviously, there exists a pathwise Hölder continuous version v of u. From the previous consideration we easily see that

$$|v(t,x)| \leqslant \eta_\delta(p,T,\mu)\Big(1+|x|^{\frac{1}{2}(\frac{\delta}{p}+1)}\Big),$$

$t\in[0,T]$, $x\in\mathbb{R}^d$.

In the case when σ is deterministic and satisfies $\sigma^2(t,x)\leqslant \mathrm{const}(1+|x|^m)$, $x\in\mathbb{R}^d$ we observe ($p=1$)

$$\mathbb{E}\left[u(t,x)-u(s,y)\right]^2 \leqslant c_{10}(T,\mu)\cdot n^m\cdot\left[|t-s|^{2\mu}+|x-y|^{4\mu}\right]^{1/2},$$

$(t,x),(s,y)\in[-n,n]^{d+1}$, $\mu\in(0,\frac{1}{2})$, which leads to

$$\leqslant c_{11}(T,\mu)n^{\mu+m}\cdot|(t,x)-(s,y)|^{\mu}.$$

Therefore, Theorem 3.1 guarantees the existence of a locally Hölder continuous version v of u for which we get ($2\mu+m\leqslant m+1$)

$$|v(t,x)|\leqslant\eta\left(1+\sqrt{|x|^{m+1}\cdot\log(1+|x|)}\right)$$

for any $t\in[0,T]$, $x\in\mathbb{R}^d$, where η is a random variable with $\mathbb{P}(\{0<\eta<\infty\})=1$. In the case that

$$\sup_{\substack{t\in[0,T]\\ x\in\mathbb{R}^d}}|\sigma(t,x)|\leqslant c<\infty$$

holds, we have even

$$|v(t,x)|\leqslant\eta\left(1+\sqrt{|x|\cdot\log(1+|x|)}\right)$$

for any $t\in[0,T]$, $x\in\mathbb{R}^d$, where η is a random variable with $\mathbb{P}(\{0<\eta<\infty\})=1$.

So we have proved

Theorem 4.2.3. *Let $\sigma\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$ be a nonrandom function with the property*

$$|\sigma(t,x)|^2\leqslant c(T)(1+|x|^m)$$

for $t\in[0,T]$, $x\in\mathbb{R}^d$. Then there exists a H lder continuous version v of u such that

$$|v(t,x)|\leqslant\eta\left(1+\sqrt{|x|^{m+1}\log(1+|x|)}\right)$$

for any $t\in[0,T]$, $x\in\mathbb{R}^d$, where η is a random variable with $\mathbb{P}(\{0<\eta<\infty\})=1$.

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