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ABSTRACT PERRON-STIELTJES INTEGRAL

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Abstract. Fundamental results concerning Stieltjes integrals for functions with values in Banach spaces are presented. The background of the theory is the Kurzweil approach to integration, based on Riemann type integral sums (see e.g. [4]). It is known that the Kurzweil theory leads to the (non-absolutely convergent) Perron-Stieltjes integral in the finite dimensional case. In [3] Ch. S. Hönig presented a Stieltjes integral for Banach space valued functions. For Hönig's integral the Dushnik interior integral presents the background.

It should be mentioned that abstract Stieltjes integration was recently used by O. Diekmann, M. Gyllenberg and H. R. Thieme in [1] and [2] for describing the behaviour of some evolutionary systems originating in problems concerning structured population dynamics.

Keywords: bilinear triple, Perron-Stieltjes integral

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Bilinear triples

Assume that X, Y and Z are Banach spaces and that there is a bilinear mapping $B\colon X\times Y\to Z$. We use the short notation $xy = B(x,y)$ for the value of the bilinear form B for $x\in X$, $y\in Y$ and assume that

$$\|xy\|_Z \leqslant \|x\|_X\|y\|_Y.$$

By $\|\cdot\|_X$ the norm in the Banach space X is denoted (and similarly for the other ones).

Triples of Banach spaces X, Y, Z with these properties are called *bilinear triples* and they are denoted by $\mathcal{B} = (X,Y,Z)$ or shortly $\mathcal{B}$.

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E x a m p l e s. If X and Y are Banach spaces, let us denote by $L(X,Y)$ the Banach space of all bounded linear operators $A\colon X \to Y$ with the uniform operator topology. Defining $B(A,x) = Ax \in Y$ for $A \in L(X,Y)$ and $x \in X$, we obtain in a natural way the bilinear triple $\mathcal{B} = (L(X,Y),X,Y)$.

Similarly if X, Y and Z are Banach spaces, then $\mathcal{B} = (L(X,Y),L(Y,Z),L(X,Z))$ forms a bilinear triple with the natural bilinear form given by the composition $AB \in L(X,Z)$ of operators $A \in L(X,Y)$ and $B \in L(Y,Z)$.

The usual operator norm is used in both examples given above.

If X' is the dual to the Banach space X then (X,X',C) is a bilinear triple with $B(x,x') = x'(x)$ for $x \in X$ and $x' \in X'$.

Also $(\mathbb{R},X,X)$ and $(X,\mathbb{R},X)$ are bilinear triples with the bilinear map $B(r,x) = rx$ and $B(x,r) = rx$, respectively, where $r \in \mathbb{R}$ and $x \in X$.

Variation of functions with values in a Banach space

Assume that $[a,b] \subset \mathbb{R}$ is a bounded interval and that X is a Banach space. Given $x\colon [a,b] \to X$, the function x is of *bounded variation on* $[a,b]$ if

$$\operatorname*{var}_a^b(x) = \sup\left\{\sum_{j=1}^k \|x(\alpha_j) - x(\alpha_{j-1})\|_X\right\} < \infty,$$

where the supremum is taken over all finite partitions

$$D\colon a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

of the interval $[a,b]$. The set of all functions $x\colon [a,b] \to X$ with $\operatorname{var}_a^b(x) < \infty$ will be denoted by $BV([a,b];X)$ or shortly $BV([a,b])$ if it is clear which Banach space X we have in mind.

Assume now that $\mathcal{B} = (X,Y,Z)$ is a bilinear triple of Banach spaces.

For $x\colon [a,b] \to X$ and a partition D of the interval $[a,b]$ define

$$V_a^b(x,D) = \sup\left\{\left\|\sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})]y_j\right\|_Z\right\},$$

where the supremum is taken over all possible choices of $y_j \in Y$, $j = 1,\ldots,k$ with $\|y_j\| \leqslant 1$ and set

$$(\mathcal{B})\operatorname*{var}_a^b(x) = \sup V_a^b(x,D),$$

where the supremum is taken over all finite partitions

$$D\colon a=\alpha_0<\alpha_1<\ldots<\alpha_{k-1}<\alpha_k=b$$

of the interval $[a,b]$.

A function $x\colon [a,b]\to X$ with $(\mathcal{B})\operatorname{var}_a^b(x)<\infty$ is called a *function with bounded $\mathcal{B}$-variation on* $[a,b]$ (sometimes also *a function of bounded semi-variation* [2], [3]).

For a given bilinear triple $\mathcal{B}=(X,Y,Z)$ the set of all functions $x\colon [a,b]\to X$ with $(\mathcal{B})\operatorname{var}_a^b(x)<\infty$ will be denoted by $(\mathcal{B})BV([a,b];X)$ or shortly by $(\mathcal{B})BV([a,b])$ if it is clear which bilinear triple (X,Y,Z) we have in mind.

1. Proposition. *If $\mathcal{B}=(X,Y,Z)$ is a bilinear triple then*

$$BV([a,b];X)\subset(\mathcal{B})BV([a,b];X)$$

and if $x\in BV([a,b];X)$, then

$$(\mathcal{B})\operatorname*{var}_a^b(x)\leqslant\operatorname*{var}_a^b(x).$$

P r o o f. For a given function $x\colon [a,b]\to X$ with $x\in BV([a,b];X)$, a partition D of $[a,b]$ and arbitrary $y_j\in Y$, $j=1,\ldots,k$ with $\|y_j\|\leqslant 1$ we have

$$\begin{aligned}\Big\|\sum_{j=1}^{k}(x(\alpha_j)-x(\alpha_{j-1}))y_j\Big\|_Z&\leqslant\sum_{j=1}^{k}\|x(\alpha_j)-x(\alpha_{j-1})\|_X\|y_j\|_Y\\&\leqslant\sum_{j=1}^{k}\|x(\alpha_j)-x(\alpha_{j-1})\|_X\leqslant\operatorname*{var}_a^b(x).\end{aligned}$$

Passing to the suprema corresponding to the definition of $(\mathcal{B})\operatorname{var}_a^b(x)$ in this inequality we immediately obtain the inclusion as well as the inequality claimed in the statement. □

R e m a r k. It is easy to show that if $x\colon [a,b]\to\mathbb{R}$ and $\mathcal{B}=(\mathbb{R},\mathbb{R},\mathbb{R})$ then $x\in(\mathcal{B})BV([a,b])$ if and only if $x\in BV([a,b])$.

Indeed, in this case we have

$$V_a^b(x,D)=\sup\left\{\Big|\sum_{j=1}^{k}[x(\alpha_j)-x(\alpha_{j-1})]y_j\Big|\right\}=\sum_{j=1}^{k}|x(\alpha_j)-x(\alpha_{j-1})|$$

because we can take $y_j=1$ if $x(\alpha_j)-x(\alpha_{j-1})\geqslant 0$ and $y_j=-1$ if $x(\alpha_j)-x(\alpha_{j-1})<0$.

The same is true also if $x\colon [a,b] \to X$ and $\mathcal{B} = (X, \mathbb{R}, X)$, where the Banach space X is finite-dimensional.

This shows that the concept of $\mathcal{B}$-variation of a function $x\colon [a,b] \to X$ is relevant only for infinite-dimensional Banach spaces X.

Regulated functions and step functions with values in a Banach space

Assume that $[a,b] \subset \mathbb{R}$ is a bounded interval and that X is a Banach space. Given $x\colon [a,b] \to X$, the function x is called *regulated on* $[a,b]$ if it has one-sided limits at every point of $[a,b]$, i.e. if for every $s \in [a,b)$ there is a value $x(s+) \in X$ such that

$$\lim_{t \to s+} \|x(t) - x(s+)\|_X = 0$$

and if for every $s \in (a,b]$ there is a value $x(s-) \in X$ such that

$$\lim_{t \to s-} \|x(t) - x(s-)\|_X = 0.$$

The set of all regulated functions $x\colon [a,b] \to X$ will be denoted by $G([a,b];X)$ or shortly $G([a,b])$ if it is clear which Banach space X we have in mind.

Assume now that $\mathcal{B} = (X,Y,Z)$ is a bilinear triple of Banach spaces.

A function $x\colon [a,b] \to X$ is called *$\mathcal{B}$-regulated on* $[a,b]$ if for every $y \in Y$, $\|y\|_Y \leqslant 1$ the function $xy\colon [a,b] \to Z$ given by $t \mapsto x(t)y \in Z$ for $t \in [a,b]$ is regulated, i.e. $xy \in G([a,b],Z)$ for every $y \in Y$, $\|y\|_Y \leqslant 1$.

For a given bilinear triple $\mathcal{B} = (X,Y,Z)$ the set of all $\mathcal{B}$-regulated functions $x\colon [a,b] \to X$ will be denoted by $(\mathcal{B})G([a,b];X)$ or shortly by $(\mathcal{B})G([a,b])$ if it is clear which bilinear triple (X,Y,Z) we have in mind.

A function $x\colon [a,b] \to X$ is called a *(finite) step function on* $[a,b]$ if there exists a finite partition

$$D\colon a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

of the interval $[a,b]$ such that x has a constant value on (α_{j-1}, α_j) for every $j = 1, \ldots, k$.

The following result is well known for regulated functions.

2. Proposition. (see e.g. [3 Theorem 3.1, p. 16]) *A function $x\colon [a,b] \to X$ is regulated ($x \in G([a,b];X)$) if and only if x is the uniform limit of step functions.*

3. Proposition. *If $\mathcal{B} = (X,Y,Z)$ is a bilinear triple and $x \in G([a,b];X)$ then $x \in (\mathcal{B})G([a,b];X)$, i.e.*

$$G([a,b];X) \subset (\mathcal{B})G([a,b];X).$$

P r o o f. For any $y \in Y$ with $\|y\|_Y \leqslant 1$ and $s, t \in [a, b]$ we have

$$\|x(t)y - x(s)y\|_Z \leqslant \|x(t) - x(s)\|_X \|y\|_Y \leqslant \|x(t) - x(s)\|_X$$

and this implies the statement (e.g. by the Bolzano-Cauchy condition for the existence of onesided limits of the function x). $\square$

R e m a r k. If the bilinear triple $\mathcal{B} = (X, \mathbb{R}, X)$, with a Banach space X is given, then it is easy to check that a function $x\colon [a, b] \to X$ is $\mathcal{B}$-regulated if and only if it is regulated, the bilinear form $B(x, r)$ is given by the product xr.

Stieltjes integration of vector valued functions

A finite system of points

$$\{\alpha_0, \tau_1, \alpha_1, \tau_2, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

such that

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

and

$$\tau_j \in [\alpha_{j-1}, \alpha_j] \quad \text{for} \quad j = 1, \ldots, k$$

is called a *P-partition* of the interval $[a, b]$.

Any positive function $\delta\colon [a, b] \to (0, \infty)$ is called a *gauge on* $[a, b]$.

For a given gauge δ on $[a, b]$ a P-partition $\{\alpha_0, \tau_1, \alpha_1, \tau_2, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ of $[a, b]$ is called *δ-fine* if

$$[\alpha_{j-1}, \alpha_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)) \text{ for } j = 1, \ldots, k.$$

4. Cousin's Lemma. *Given an arbitrary gauge δ on $[a, b]$ there is a δ-fine P-partition of $[a, b]$.*

(See e.g. [4].)

5. Definition. Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x\colon [a, b] \to X$ and $y\colon [a, b] \to Y$ are given.

We say that the Stieltjes integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists if there is an element $I \in Z$ such that to every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that for

$$S(\mathrm{d}x, y, D) = \sum_{j=1}^{k} \big[x(\alpha_j) - x(\alpha_{j-1})\big] y(\tau_j)$$

we have

$$\big\|S(\mathrm{d}x, y, D) - I\big\|_Z < \varepsilon$$

provided D is a δ-fine P-partition of $[a,b]$. We denote $I = \int_a^b \mathrm{d}[x(s)]y(s)$. For the case $a = b$ it is convenient to set $\int_a^b \mathrm{d}[x(s)]y(s) = 0$ and if $b < a$, then $\int_a^b \mathrm{d}[x(s)]y(s) = -\int_b^a \mathrm{d}[x(s)]y(s)$.

Similarly we can define the Stieltjes integral $\int_a^b x(s)\,\mathrm{d}[y(s)]$ using Stieltjes integral sums of the form

$$S(x,\, \mathrm{d}y, D) = \sum_{j=1}^{k} x(\tau_j)\big[y(\alpha_j) - y(\alpha_{j-1})\big].$$

R e m a r k. Note that Cousin's Lemma 4 is essential for this definition. The Stieltjes integral introduced in this way is determined uniquely and has the following elementary properties.

6. Proposition. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x\colon [a,b] \to X$ and $y_i\colon [a,b] \to Y$ are such that the Stieltjes integrals $\int_a^b \mathrm{d}[x(s)]y_i(s)$, $i = 1, 2$ exist.*

Then for every $c_1, c_2 \in \mathbb{R}$ the integral $\int_a^b \mathrm{d}[x(s)](c_1 y_1(s) + c_2 y_2(s))$ exists and

$$\int_a^b \mathrm{d}[x(s)]\big(c_1 y_1(s) + c_2 y_2(s)\big) = c_1 \int_a^b \mathrm{d}[x(s)]y_1(s) + c_2 \int_a^b \mathrm{d}[x(s)]y_2(s).$$

If functions $x_i\colon [a,b] \to X$ and $y\colon [a,b] \to Y$ are such that the Stieltjes integrals $\int_a^b \mathrm{d}[x_i(s)]y(s)$, $i = 1, 2$ exist then for every $c_1, c_2 \in \mathbb{R}$ the integral $\int_a^b \mathrm{d}[c_1 x_1(s) + c_2 x_2(s)]y(s)$ exists and

$$\int_a^b \mathrm{d}[c_1 x_1(s) + c_2 x_2(s)]y(s) = c_1 \int_a^b \mathrm{d}[x_1(s)]y(s) + c_2 \int_a^b \mathrm{d}[x_2(s)]y(s).$$

P r o o f. The statements are easy consequences of the equalities holding for the corresponding integral sums, i.e.

$$S(\mathrm{d}x, c_1 y_1 + c_2 y_2, D) = c_1 S(\mathrm{d}x, y_1, D) + c_2 S(\mathrm{d}x, y_2, D)$$

and

$$S\big(\mathrm{d}(c_1 x_1 + c_2 x_2), y, D\big) = c_1 S(\mathrm{d}x_1, y, D) + c_2 S(\mathrm{d}x_2, y, D).$$

□

7. Proposition. (Bolzano-Cauchy condition) *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x\colon [a,b] \to X$ and $y\colon [a,b] \to Y$ are given.*

Then the Stieltjes integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists if and only if for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\text{(BC)} \qquad \big\|S(\mathrm{d}x, y, D_1) - S(\mathrm{d}x, y, D_2)\big\|_Z < \varepsilon$$

provided D_1, D_2 are δ-fine P-partitions of $[a,b]$.

Proof. Clearly, if the integral in question exists, the Bolzano-Cauchy condition is satisfied.

Assume on the contrary that the Bolzano-Cauchy condition (BC) holds. For a certain $\varepsilon > 0$ define

$$I(\varepsilon) = \big\{S(\mathrm{d}x, y, D); D \text{ an arbitrary } \delta\text{-fine } P\text{-partition of } [a,b]\big\} \subset Z$$

where $\delta = \delta_\varepsilon$ is the corresponding gauge. By Cousin's Lemma 4 the set $I(\varepsilon)$ is nonempty. By the condition (BC) we have

$$\operatorname{diam} I(\varepsilon) < \varepsilon$$

and also

$$I(\varepsilon_1) \subset I(\varepsilon_2)$$

for $\varepsilon_1 < \varepsilon_2$. Hence the intersection

$$\bigcap_{\varepsilon > 0} \overline{I(\varepsilon)} = \{I\};\ I \in Z$$

consists of a single point because the space Z is complete ($\overline{I(\varepsilon)}$ denotes the closure of $I(\varepsilon)$ in Z). Therefore for an arbitrary δ-fine P-partition D of $[a,b]$ we get

$$\big\|S(\mathrm{d}x, y, D) - I\big\| \leqslant \varepsilon.$$

□

8. Proposition. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x\colon [a,b] \to X$ and $y\colon [a,b] \to Y$ are given. If the Stieltjes integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists, then for every interval $[c,d] \subset [a,b]$ the integral $\int_c^d \mathrm{d}[x(s)]y(s)$ exists.*

Proof. Given $\varepsilon > 0$ assume that δ is the gauge on $[a,b]$ such that

$$\big\|S(\mathrm{d}x, y, D_1) - S(\mathrm{d}x, y, D_2)\big\|_Z < \varepsilon$$

provided D_1, D_2 are δ-fine P-partitions of $[a,b]$ (see the Bolzano-Cauchy condition for the existence of the integral).

Assume that D_1^*, D_2^* are arbitrary δ-fine P-partitions of $[c,d]$. Let D_- be a δ-fine P-partition of $[a,c]$ and D_+ a δ-fine P-partition of $[d,b]$. The union of D_-, D_1^* and D_+ forms a partition D_1 of $[a,b]$ and similarly D_-, D_2^* and D_+ gives a partition D_2 of $[a,b]$ and both partitions D_1 and D_2 are δ-fine. It is easy to check that

$$S(\,\mathrm{d}x, y, D_1) - S(\,\mathrm{d}x, y, D_2) = S(\,\mathrm{d}x, y, D_1^*) - S(\,\mathrm{d}x, y, D_2^*).$$

Hence

$$\big\|S(\,\mathrm{d}x, y, D_1^*) - S(\,\mathrm{d}x, y, D_2^*)\big\|_Z < \varepsilon$$

and by the Bolzano-Cauchy condition the integral $\int_c^d \mathrm{d}[x(s)]y(s)$ exists. □

9. Proposition. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x\colon [a,b] \to X$ and $y\colon [a,b] \to Y$ are such that for $c \in [a,b]$ the Stieltjes integrals $\int_a^c \mathrm{d}[x(s)]y(s)$ and $\int_c^b \mathrm{d}[x(s)]y(s)$ exist.*

Then the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and

$$\int_a^b \mathrm{d}[x(s)]y(s) = \int_a^c \mathrm{d}[x(s)]y(s) + \int_c^b \mathrm{d}[x(s)]y(s).$$

Proof. If $c = a$ or $c = b$ then the statement is clear because we have $\int_c^c \mathrm{d}[x(s)]y(s) = 0$ by definition. Assume therefore that $c \in (a,b)$.

For every $\varepsilon > 0$ there exist gauges δ_- and δ_+ on $[a,c]$ and $[c,b]$ respectively such that by the definition we have

$$\left\|S(\,\mathrm{d}x, y, D_-) - \int_a^c \mathrm{d}[x(s)]y(s)\right\|_Z < \varepsilon$$

provided D_- is a δ_--fine P-partition of $[a,c]$ and

$$\left\|S(\,\mathrm{d}x, y, D_+) - \int_c^b \mathrm{d}[x(s)]y(s)\right\|_Z < \varepsilon$$

provided D_+ is a δ_+-fine P-partition of $[c,b]$.

Let us choose a gauge δ on $[a,b]$ such that

$$0 < \delta(s) < \min\big(\delta_-(s), \operatorname{dist}(s,c)\big) \text{ for } s \in [a,c),$$
$$0 < \delta(s) < \min\big(\delta_+(s), \operatorname{dist}(s,c)\big) \text{ for } s \in (c,b]$$

and

$$0 < \delta(c) < \min\big(\delta_-(c), \delta_+(c)\big).$$

Let us assume that $D = \{\alpha_0, \tau_1, \alpha_1, \tau_2, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$ is a δ-fine P-partition of the interval $[a,b]$. It is easy to check that by the choice of the gauge δ there is an index $l \in \{1,\dots,k\}$ such that $\tau_l = c$ and that $D_- = \{\alpha_0, \tau_1, \alpha_1, \tau_2, \dots, \alpha_{l-1}, \tau_l = \alpha_l = c\}$ and $D_+ = \{c = \alpha_l = \tau_l, \alpha_{l+1}, \tau_{l+1}, \dots, \alpha_{k-1}, \tau_k = \alpha_k\}$ are δ_-- and δ_+-fine P-partitions of $[a,c]$ and $[c,b]$, respectively. Then we have $S(\,\mathrm{d}x, y, D) = S(\,\mathrm{d}x, y, D_-) + S(\,\mathrm{d}x, y, D_+)$ and

$$\begin{aligned}
&\left\| S(\,\mathrm{d}x, y, D) - \int_a^c \mathrm{d}[x(s)]y(s) - \int_c^b \mathrm{d}[x(s)]y(s) \right\|_Z \\
&= \left\| S(\,\mathrm{d}x, y, D_-) + S(\,\mathrm{d}x, y, D_+) - \int_a^c \mathrm{d}[x(s)]y(s) - \int_c^b \mathrm{d}[x(s)]y(s) \right\|_Z \\
&\leqslant \left\| S(\,\mathrm{d}x, y, D_-) - \int_a^c \mathrm{d}[x(s)]y(s) \right\|_Z + \left\| S(\,\mathrm{d}x, y, D_+) - \int_c^b \mathrm{d}[x(s)]y(s) \right\|_Z < 2\varepsilon.
\end{aligned}$$

This inequality yields by definition the existence of the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ as well as the equality

$$\int_a^b \mathrm{d}[x(s)](y(s) = \int_a^c \mathrm{d}[x(s)]y(s) + \int_c^b \mathrm{d}[x(s)]y(s).$$

□

R e m a r k. In the opposite direction we evidently have:

If $c \in [a,b]$ and the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists, then the Stieltjes integrals $\int_a^c \mathrm{d}[x(s)]y(s)$ and $\int_c^b \mathrm{d}[x(s)]y(s)$ exist and

$$\int_a^b \mathrm{d}[x(s)]y(s) = \int_a^c \mathrm{d}[x(s)]y(s) + \int_c^b \mathrm{d}[x(s)]y(s).$$

10. Proposition. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x\colon [a,b] \to X$ and $y\colon [a,b] \to Y$ are given. If the Stieltjes integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and $(\mathcal{B}) \operatorname{var}_a^b(x) < \infty$ then*

$$\left\| \int_a^b \mathrm{d}[x(s)]y(s) \right\|_Z \leqslant \sup_{s\in[a,b]} \|y(s)\|_Y . (\mathcal{B}) \operatorname*{var}_a^b(x).$$

P r o o f. Assume that $\varepsilon > 0$ is given. Since the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists, there is a gauge δ on $[a,b]$ such that we have

$$\left\| \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})]y(\tau_j) - \int_a^b \mathrm{d}[x(s)]y(s) \right\|_Z < \varepsilon$$

provided

$$D = \{\alpha_0, \tau_1, \alpha_1, \tau_2, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

is a δ-fine P-partition of $[a,b]$. Hence

$$\begin{aligned}
&\left\| \int_a^b \mathrm{d}[x(s)]y(s) \right\|_Z \\
&\leqslant \left\| \int_a^b \mathrm{d}[x(s)]y(s) - \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})]y(\tau_j) \right\|_Z + \left\| \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})]y(\tau_j) \right\|_Z \\
&< \varepsilon + \left\| \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})]y(\tau_j) \right\|_Z .
\end{aligned}$$

Further we have

$$\begin{aligned}
&\left\| \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})]y(\tau_j) \right\|_Z = \left\| \sum_{\substack{j=1,\\ y(\tau_j)\neq 0}}^k [x(\alpha_j) - x(\alpha_{j-1})]y(\tau_j) \right\|_Z \\
&= \left\| \sum_{\substack{j=1,\\ y(\tau_j)\neq 0}}^k [x(\alpha_j) - x(\alpha_{j-1})] \frac{y(\tau_j)}{\|y(\tau_j)\|_Y} \|y(\tau_j)\|_Y \right\|_Z \\
&\leqslant \sup_{s\in[a,b]} \|y(s)\|_Y \cdot \left\| \sum_{\substack{j=1,\\ y(\tau_j)\neq 0}}^k [x(\alpha_j) - x(\alpha_{j-1})] \frac{y(\tau_j)}{\|y(\tau_j)\|_Y} \right\|_Z \leqslant \sup_{s\in[a,b]} \|y(s)\|_Y . (\mathcal{B}) \operatorname*{var}_a^b(x).
\end{aligned}$$

This yields the inequality

$$\left\| \int_a^b \mathrm{d}[x(s)]y(s) \right\|_Z < \varepsilon + \sup_{s\in[a,b]} \|y(s)\|_Y .(\mathcal{B}) \operatorname*{var}_a^b(x)$$

and the statement is proved because $\varepsilon > 0$ can be arbitrarily small. □

11. Uniform convergence theorem. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x\colon [a,b] \to X$ and y, $y_n\colon [a,b] \to Y$, $n = 1, 2, \dots$ are given. If $(\mathcal{B}) \operatorname{var}_a^b(x) < \infty$, the Stieltjes integrals $\int_a^b \mathrm{d}[x(s)]y_n(s)$ exist and the sequence y_n converges on $[a,b]$ uniformly to y, i.e.*

$$\lim_{n\to\infty} \|y_n(s) - y(s)\|_Y = 0 \text{ uniformly on } [a,b],$$

then the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and

$$\int_a^b \mathrm{d}[x(s)]y(s) = \lim_{n\to\infty} \int_a^b \mathrm{d}[x(s)]y_n(s).$$

P r o o f. Let $\varepsilon > 0$ be given arbitrarily.

Since the sequence y_n converges on $[a,b]$ uniformly to y, there is a positive integer n_0 such that for any $n > n_0$ and $s \in [a,b]$ we have

$$\|y_n(s) - y(s)\|_Y < \frac{\varepsilon}{6((\mathcal{B}) \operatorname{var}_a^b(x) + 1)}.$$

Hence for any $m, n > n_0$ and $s \in [a,b]$ we have

$$\begin{aligned} \|y_n(s) - y_m(s)\|_Y &\leqslant \|y_n(s) - y(s)\|_Y + \|y_m(s) - y(s)\|_Y \\ &< \frac{2\varepsilon}{6((\mathcal{B}) \operatorname{var}_a^b(x) + 1)} = \frac{\varepsilon}{3((\mathcal{B}) \operatorname{var}_a^b(x) + 1)}. \end{aligned}$$

By Proposition 10 we get

$$\begin{aligned} &\left\| \int_a^b \mathrm{d}[x(s)]y_n(s) - \int_a^b \mathrm{d}[x(s)]y_m(s) \right\|_Z = \left\| \int_a^b \mathrm{d}[x(s)](y_n(s) - y_m(s)) \right\|_Z \\ &\leqslant \sup_{s\in[a,b]} \|y_n(s) - y_m(s)\|_Y (\mathcal{B}) \operatorname*{var}_a^b(x) < \frac{(\mathcal{B}) \operatorname{var}_a^b(x)}{3((\mathcal{B}) \operatorname{var}_a^b(x) + 1)}\varepsilon \leqslant \tfrac{\varepsilon}{3} \end{aligned}$$

for $m, n > n_0$. Since Z is a Banach space this inequality implies that the limit

$$\lim_{n\to\infty} \int_a^b \mathrm{d}[x(s)]y_n(s) = I \in Z$$

exists. Let $n_1 \in \mathbb{N}$ be such that for $m > n_0$ we have

$$\left\| \int_a^b \mathrm{d}[x(s)]y_m(s) - I \right\|_Z < \tfrac{\varepsilon}{3}.$$

Let now $m > n_2 = \max(n_0, n_1)$ be fixed. Since the integral $\int_a^b \mathrm{d}[x(s)]y_m(s)$ exists, there is a gauge δ on $[a,b]$ such that

$$\left\| \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})]y_m(\tau_j) - \int_a^b \mathrm{d}[x(s)]y_m(s) \right\|_Z < \tfrac{\varepsilon}{3}$$

provided $D = \{\alpha_0, \tau_1, \alpha_1, \tau_2, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ is a δ-fine P-partition of $[a,b]$.

For such a partition we have

$$\begin{aligned}
\|S(\mathrm{d}x, y, D) - I\|_Z &= \left\| \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})]y(\tau_j) - I \right\|_Z \\
&\leqslant \left\| \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})](y(\tau_j) - y_m(\tau_j)) \right\|_Z \\
&\quad + \left\| \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})]y_m(\tau_j) - \int_a^b \mathrm{d}[x(s)]y_m(s) \right\|_Z \\
&\quad + \left\| \int_a^b \mathrm{d}[x(s)]y_m(s) - I \right\|_Z \\
&< \tfrac{2\varepsilon}{3} + \left\| \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})](y(\tau_j) - y_m(\tau_j)) \right\|_Z.
\end{aligned}$$

We have further

$$\begin{aligned}
&\left\| \sum_{j=1}^k [x(\alpha_j) - x(\alpha_{j-1})](y(\tau_j) - y_m(\tau_j)) \right\|_Z \\
&= \left\| \sum_{\substack{j=1, \\ y(\tau_j) \neq y_m(\tau_j)}}^k [x(\alpha_j) - x(\alpha_{j-1})](y(\tau_j) - y_m(\tau_j)) \right\|_Z \\
&= \left\| \sum_{\substack{j=1, \\ y(\tau_j) \neq y_m(\tau_j)}}^k [x(\alpha_j) - x(\alpha_{j-1})] \tfrac{y(\tau_j) - y_m(\tau_j)}{\|y(\tau_j) - y_m(\tau_j)\|_Y} \|y(\tau_j) - y_m(\tau_j)\|_Y \right\|_Z \\
&\leqslant \max_j \|y(\tau_j) - y_m(\tau_j)\|_Y \cdot (\mathcal{B}) \operatorname*{var}_a^b(x) < \tfrac{\varepsilon (\mathcal{B}) \operatorname{var}_a^b(x)}{6((\mathcal{B}) \operatorname{var}_a^b(x) + 1)} \leqslant \tfrac{\varepsilon}{6} < \tfrac{\varepsilon}{3}.
\end{aligned}$$

Therefore we get

$$\|S(\mathrm{d}x,y,D)-I\|_Z=\left\|\sum_{j=1}^{k}[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j)-I\right\|_Z<\tfrac{2\varepsilon}{3}+\tfrac{\varepsilon}{3}=\varepsilon$$

and this means that the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and

$$\int_a^b \mathrm{d}[x(s)]y(s)=I=\lim_{n\to\infty}\int_a^b \mathrm{d}[x(s)]y_n(s).$$

□

12. Lemma. *Assume that $\mathcal{B}=(X,Y,Z)$ is a bilinear triple and that $x\colon[a,b]\to X$ is $\mathcal{B}$-regulated on $[a,b]$ ($x\in(\mathcal{B})G([a,b],X)$). Let $y^*\in Y$ be a given fixed element in Y.*

For $c\in[a,b]$ let us define a function $y\colon[a,b]\to Y$ such that $y(c)=y^*$ and $y(t)=0$ for $t\in[a,b]$, $t\neq c$. Then the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and

$$\int_a^b \mathrm{d}[x(s)]y(s)=\lim_{t\to a+}x(t)y^*-x(a)y^*\quad \text{if } c=a,$$

$$\int_a^b \mathrm{d}[x(s)]y(s)=x(b)y^*-\lim_{t\to b-}x(t)y^*\quad \text{if } c=b$$

and

$$\int_a^b \mathrm{d}[x(s)]y(s)=\lim_{t\to c+}x(t)y^*-\lim_{t\to c-}x(t)y^*\quad \text{if } c\in(a,b).$$

Proof. By the assumption we have $x\in(\mathcal{B})G([a,b],X)$ and therefore the onesided limits $\lim_{t\to c+}x(t)y^*=z_c^+$, $\lim_{t\to c-}x(t)y^*=z_c^-$ of the function $t\mapsto x(t)y^*\in Z$ exist if $c\in[a,b)$ or $c\in(a,b]$.

Note that if the assumption $x\in(\mathcal{B})G([a,b],X)$ is replaced by the stronger requirement $x\in G([a,b],X)$ then the limit $\lim_{t\to c+}x(t)=x(c+)\in X$ exists and $\lim_{t\to c+}x(t)y^*=x(c+)y^*$ and similarly also the limit $\lim_{t\to c-}x(t)=x(c-)\in X$ exists and $\lim_{t\to c-}x(t)y^*=x(c-)y^*$.

We will show the result for the case $c\in(a,b)$ only; the proof for the cases $c=a$ and $c=b$ is similar.

Let $\varepsilon>0$ be given and let $\Delta>0$ be such that

$$\|x(t)y^*-z_c^-\|_Z<\varepsilon\quad \text{for } t\in(c-\Delta,c)$$

and

$$\|x(t)y^* - z_c^+\|_Z < \varepsilon \quad \text{ for } t \in (c, c+\Delta).$$

Define a gauge δ such that $0 < \delta(c) < \Delta$ and $0 < \delta(t) < |t-c|$ for $t \in [a,b]$, $t \neq c$.

Assume that $D = \{\alpha_0, \tau_1, \alpha_1, \tau_2, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ is a δ-fine P-partition of $[a,b]$. From the properties of the gauge given above it follows that there is an index $l \in \{1, \ldots, k\}$ such that $\tau_l = c$ and

$$c - \Delta < \alpha_{l-1} < \tau_l = c < \alpha_l < c + \Delta.$$

For the integral sum $S(\,\mathrm{d}x, y, D)$ we have by the properties of the function y and of the partition D the equality

$$S(\,\mathrm{d}x, y, D) = [x(\alpha_l) - x(\alpha_{l-1})]y(\tau_l) = [x(\alpha_l) - x(\alpha_{l-1})]y^*$$

and

$$\begin{aligned}\|S(\,\mathrm{d}x, y, D) - z_c^+ + z_c^-\|_Z &= \|[x(\alpha_l) - x(\alpha_{l-1})]y^* - (z_c^+ - z_c^-)\|_Z \\ &\leqslant \|x(\alpha_l)y^* - z_c^+\|_Z + \|x(\alpha_{l-1})y^* - z_c^-\|_Z < 2\varepsilon.\end{aligned}$$

Hence the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and

$$\int_a^b \mathrm{d}[x(s)]y(s) = z_c^+ - z_c^- = \lim_{t \to c+} x(t)y^* - \lim_{t \to c-} x(t)y^*.$$

□

13. Lemma. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that $x\colon [a,b] \to X$ is $\mathcal{B}$-regulated on $[a,b]$ ($x \in (\mathcal{B})G([a,b], X)$). Let $y^* \in Y$ be a given fixed element in Y.*

For $c, d \in [a,b]$, $c < d$ let us define the function $y\colon [a,b] \to Y$ such that $y(t) = y^*$ for $t \in (c,d)$ and $y(t) = 0$ for $t \in [a,b] \setminus (c,d)$. Then the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and

$$\int_a^b \mathrm{d}[x(s)]y(s) = \lim_{t \to d-} x(t)y^* - \lim_{t \to c+} x(t)y^*.$$

P r o o f. Let $\varepsilon > 0$ be given and let $\Delta > 0$ be such that

$$\|x(t)y^* - \lim_{t \to d-} x(t)y^*\|_Z < \varepsilon \quad \text{ for } t \in (d - \Delta, d)$$

and

$$\|x(t)y^* - \lim_{t \to c+} x(t)y^*\|_Z < \varepsilon \quad \text{ for } t \in (c, c+\Delta).$$

Define a gauge δ such that $0 < \delta(c) < \Delta$, $0 < \delta(d) < \Delta$ and $0 < \delta(t) < \min(|t-c|, |t-d|)$ for $t \in [a,b]$, $t \neq c, t \neq d$.

Assume that $D = \{\alpha_0, \tau_1, \alpha_1, \tau_2, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ is a δ-fine P-partition of $[a,b]$. From the properties of the gauge δ it follows that there are indices $l, m \in \{1, \ldots, k\}$ such that $\tau_l = c$, $\tau_m = d$ and

$$\begin{aligned} &\tau_l = c < \alpha_l < c + \Delta, \\ &d - \Delta < \alpha_{m-1} < \tau_m = d. \end{aligned}$$

Since $y(t) = 0$ for $t \leqslant c$ and $t \geqslant d$ we have for the integral sum $S(\,\mathrm{d}x, y, D)$

$$\begin{aligned} S(\,\mathrm{d}x, y, D) &= \sum_{j=1}^{k} [x(\alpha_j) - x(\alpha_{j-1})] y(\tau_j) \\ &= \sum_{j=l+1}^{m-1} [x(\alpha_j) - x(\alpha_{j-1})] y(\tau_j) \\ &= \sum_{j=l+1}^{m-1} [x(\alpha_j) - x(\alpha_{j-1})] y^* = [x(\alpha_{m-1}) - x(\alpha_l)] y^* \end{aligned}$$

and therefore

$$\begin{aligned} &\big\| S(\,\mathrm{d}x, y, D) - (\lim_{t \to d-} x(t) y^* - \lim_{t \to c+} x(t) y^*) \big\|_Z \\ &= \big\| [x(\alpha_{m-1}) - x(\alpha_l)] y^* - (\lim_{t \to d-} x(t) y^* - \lim_{t \to c+} x(t) y^*) \big\|_Z \\ &\leqslant \big\| x(\alpha_{m-1}) y^* - \lim_{t \to d-} x(t) y^* \big\|_Z + \big\| x(\alpha_l) y^* - \lim_{t \to c+} x(t) y^* \big\|_Z < 2\varepsilon. \end{aligned}$$

Hence the integral $\int_a^b \mathrm{d}[x(s)] y(s)$ exists and

$$\int_a^b \mathrm{d}[x(s)] y(s) = \lim_{t \to d-} x(t) y^* - \lim_{t \to c+} x(t) y^*.$$

□

14. Proposition. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that $x\colon [a,b] \to X$ is $\mathcal{B}$-regulated on $[a,b]$ ($x \in (\mathcal{B})G([a,b], X)$). Let $y\colon [a,b] \to Y$ be a step function, i.e. there is a finite partition*

$$a = \beta_0 < \beta_1 < \ldots < \beta_{k-1} < \beta_k = b$$

of the interval $[a,b]$ such that y has a constant value $y_j^* \in Y$ on (β_{j-1}, β_j) for every $j = 1, \ldots, k$.

Then the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ *exists and*

$$\begin{aligned}\int_a^b \mathrm{d}[x(s)]y(s) = &\lim_{t\to a+} x(t)y(a) - x(a)y(a)\\ &+\sum_{j=1}^{k-1}[\lim_{t\to\beta_j+} x(t)y(\beta_j) - \lim_{t\to\beta_j-} x(t)y(\beta_j)] + x(b)y(b) - \lim_{t\to b-} x(t)y(b)\\ &+\sum_{j=1}^{k}[\lim_{t\to\beta_j-} x(t)y_j^* - \lim_{t\to\beta_{j-1}+} x(t)y_j^*].\end{aligned}$$

Proof. Every step function $y\colon [a,b]\to Y$ is clearly a finite linear combination of functions of the type given in Lemma 12 and 13.

Hence the existence of the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ easily follows from the linearity of the integral and from Lemmas 12 and 13. The value of the integral can be calculated by the values of integrals given in those lemmas. □

15. Proposition. *Assume that* $\mathcal{B}=(X,Y,Z)$ *is a bilinear triple and that* $x\colon [a,b]\to X$ *is* $\mathcal{B}$*-regulated on* $[a,b]$ *(*$x\in(\mathcal{B})G([a,b],X)$*) and* $(\mathcal{B})\operatorname{var}_a^b(x)<\infty$*. Let* $y\colon [a,b]\to Y$ *be a regulated function.*

Then the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ *exists.*

Proof. Since $y\colon [a,b]\to Y$ is assumed to be a regulated function, it is the uniform limit of a sequence y_n of Y-valued step functions (see Proposition 2). By Proposition 14 the integrals $\int_a^b \mathrm{d}[x(s)]y_n(s)$ exist and the existence of the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ immediately follows from the Uniform Convergence Theorem 11. □

The following statement provides an operative tool in the theory of generalized Perron integral. Its original version belongs to S. Saks and it was formulated for generalized integrals using Riemann-like sums by R. Henstock.

16. Lemma (Saks-Henstock). *Assume that* $\mathcal{B}=(X,Y,Z)$ *is a bilinear triple and that functions* $x\colon [a,b]\to X$ *and* $y\colon [a,b]\to Y$ *are such that the Stieltjes integral* $\int_a^b \mathrm{d}[x(s)]y(s)$ *exists.*

Given $\varepsilon>0$ *assume that the gauge* δ *on* $[a,b]$ *is such that*

$$\Bigl\|\sum_{j=1}^{k}[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j) - \int_a^b \mathrm{d}[x(s)]y(s)\Bigr\|_Z < \varepsilon$$

for every δ*-fine* P*-partition* $D=\{\alpha_0,\tau_1,\alpha_1,\dots,\alpha_{k-1},\tau_k,\alpha_k\}$ *of* $[a,b]$.

If $\{(\xi_j,[\beta_j,\gamma_j]),\ j=1,\dots,m\}$ *is a* δ*-fine system, i.e.*

$$a\leqslant\beta_1\leqslant\xi_1\leqslant\gamma_1\leqslant\beta_2\leqslant\xi_2\leqslant\gamma_2\leqslant\dots\leqslant\beta_m\leqslant\xi_m\leqslant\gamma_m\leqslant b$$

and

$$\xi_j\in[\beta_j,\gamma_j]\subset\bigl[\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j)\bigr],\quad j=1,\dots,m$$

then

$$\Bigl\|\sum_{j=1}^m\Bigl[[x(\gamma_j)-x(\beta_j)]y(\xi_j)-\int_{\beta_j}^{\gamma_j}\mathrm{d}[x(s)]y(s)\Bigr]\Bigr\|_Z\leqslant\varepsilon.$$

P r o o f. Without any loss of generality it can be assumed that $\beta_j<\gamma_j$ for every $j=1,\dots,m$. Denote $\gamma_0=a$ and $\beta_{m+1}=b$. If $\gamma_j<\beta_{j+1}$ for some $j=0,1,\dots,m$ then Proposition 8 yields the existence of the integral $\int_{\gamma_j}^{\beta_{j+1}}\mathrm{d}[x(s)]y(s)$ and therefore for every $\eta>0$ there exists a gauge δ_j on $[\gamma_j,\beta_{j+1}]$ such that $\delta_j(\tau)<\delta(\tau)$ for $\tau\in[\gamma_j,\beta_{j+1}]$ and for every δ_j-fine partition D^j of $[\gamma_j,\beta_{j+1}]$ we have

$$\Bigl\|S(\mathrm{d}x,y,D^j)-\int_{\gamma_j}^{\beta_{j+1}}\mathrm{d}[x(s)]y(s)\Bigr\|_Z<\tfrac{\eta}{m+1}.$$

If $\gamma_j=\beta_{j+1}$ then we set $S(\mathrm{d}x,y,D^j)=0$.

The expression

$$\sum_{j=1}^m[x(\gamma_j)-x(\beta_j)]y(\xi_j)+\sum_{j=1}^m S(\mathrm{d}x,y,D^j)$$

represents an integral sum which corresponds to a certain δ-fine P-partition of $[a,b]$ and consequently

$$\Bigl\|\sum_{j=1}^m[x(\gamma_j)-x(\beta_j)]y(\xi_j)+\sum_{j=1}^m S(\mathrm{d}x,y,D^j)-\int_a^b\mathrm{d}[x(s)]y(s)\Bigr\|_Z<\varepsilon.$$

Hence

$$\begin{aligned}&\Bigl\|\sum_{j=1}^m\bigl[[x(\gamma_j)-x(\beta_j)]y(\xi_j)-\int_{\beta_j}^{\gamma_j}\mathrm{d}[x(s)]y(s)\bigr]\Bigr\|_Z\\&\leqslant\Bigl\|\sum_{j=1}^m[x(\gamma_j)-x(\beta_j)]y(\xi_j)+\sum_{j=1}^m S(\mathrm{d}x,y,D^j)-\int_a^b\mathrm{d}[x(s)]y(s)\Bigr\|_Z\\&\quad+\sum_{j=1}^m\bigl\|S(\mathrm{d}x,y,D^j)-\int_{\gamma_j}^{\beta_{j+1}}\mathrm{d}[x(s)]y(s)\bigr\|_Z<\varepsilon+(m+1)\tfrac{\eta}{m+1}=\varepsilon+\eta.\end{aligned}$$

Since this inequality holds for every $\eta>0$ we immediately obtain the inequality from the statement. $\square$

17. Theorem. *Assume that $\mathcal{B}=(X,Y,Z)$ is a bilinear triple and that functions $x\colon[a,b]\to X$ and $y\colon[a,b]\to Y$ are such that the Stieltjes integral $\int_a^c \mathrm{d}[x(s)]y(s)$ exists for every $c\in[a,b)$ and let the limit*

$$\lim_{c\to b-}\left[\int_a^c \mathrm{d}[x(s)]y(s)+[x(b)-x(c)]y(b)\right]=I\in Z \tag{1}$$

exist. Then the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and

$$\int_a^b \mathrm{d}[x(s)]y(s)=I.$$

P r o o f. Assume that $\varepsilon>0$ is given. By (1) for every $\varepsilon>0$ we can find a $B\in[a,b)$ such that for every $c\in[B,b)$ the inequality

$$\left\|\int_a^c \mathrm{d}[x(s)]y(s)+[x(b)-x(c)]y(b)-I\right\|_Z<\varepsilon \tag{2}$$

is satisfied. Assume that $a=c_0<c_1<\ldots<b$ with $\lim\limits_{p\to\infty}c_p=b$. By the assumption the integral
$\int_a^{c_p} \mathrm{d}[x(s)]y(s)$ exists for every $p=1,2,\ldots$ and therefore for every $p=1,2,\ldots$ there exists a gauge $\delta_p\colon[a,c_p]\to(0,+\infty)$ such that for any δ_p-fine P-partition D of $[a,c_p]$ we have

$$\left\|S(\mathrm{d}x,y,D)-\int_a^{c_p} \mathrm{d}[x(s)]y(s)\right\|_Z<\frac{\varepsilon}{2^{p+1}},\quad p=1,2,\ldots. \tag{3}$$

For any $\tau\in[a,b)$ there is exactly one $p(\tau)=1,2,\ldots$ for which $\tau\in[c_{p(\tau)-1},c_{p(\tau)})$. Given $\tau\in[a,b)$ let us choose $\widehat{\delta}(\tau)>0$ such that $\widehat{\delta}(\tau)\leqslant\delta_{p(\tau)}(\tau)$ and

$$[\tau-\widehat{\delta}(\tau),\tau+\widehat{\delta}(\tau)]\cap[a,b)\subset[a,c_{p(\tau)}).$$

Assume that $c\in[a,b)$ is given and that

$$\widehat{D}=\{\alpha_0,\tau_1,\alpha_1,\ldots,\alpha_{k-2},\tau_{k-1},\alpha_{k-1}\}$$

is a $\widehat{\delta}$-fine P-partition of $[a,c]$. If $p(\tau_j)=p$ then $[\alpha_{j-1},\alpha_j]\subset[\tau_j-\widehat{\delta}(\tau_j),\tau_j+\widehat{\delta}(\tau_j)]\subset[a,c_p]$ and also $[\alpha_{j-1},\alpha_j]\subset[\tau_j-\delta_p(\tau_j),\tau_j+\delta_p(\tau_j)]$. Let

$$\sum_{\substack{j=1,\\ p(\tau_j)=p}}^{k-1}\left[[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j)-\int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[x(s)]y(s)\right]$$

be the sum of those terms in the corresponding "total" sum

$$\sum_{j=1}^{k-1}\Big[[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j)-\int_{\alpha_{j-1}}^{\alpha_j}\mathrm{d}[x(s)]y(s)\Big]$$

for which the tags τ_j satisfy the relation $\tau_j\in[c_{p-1},c_p)$. Since (3) holds we obtain by the Saks-Henstock Lemma 16 the inequality

$$\Big\|\sum_{\substack{j=1,\\ p(\tau_j)=p}}^{k-1}\Big[[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j)-\int_{\alpha_{j-1}}^{\alpha_j}\mathrm{d}[x(s)]y(s)\Big]\Big\|<\frac{\varepsilon}{2^{p+1}}$$

and finally

$$\begin{aligned}
&\Big\|\sum_{j=1}^{k-1}[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j)-\int_a^c\mathrm{d}[x(s)]y(s)\Big\|\\
&=\Big\|\sum_{j=1}^{k-1}\Big[[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j)-\int_{\alpha_{j-1}}^{\alpha_j}\mathrm{d}[x(s)]y(s)\Big]\Big\|\\
&\leqslant\sum_{p=1}^{\infty}\Big\|\sum_{\substack{j=1,\\ p(\tau_j)=p}}^{k-1}\Big[[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j)-\int_{\alpha_{j-1}}^{\alpha_j}\mathrm{d}[x(s)]y(s)\Big]\Big\|\leqslant\sum_{p=1}^{\infty}\frac{\varepsilon}{2^{p+1}}=\varepsilon.
\end{aligned}$$

Define now a gauge δ on the interval $[a,b]$ as follows. For $\tau\in[a,b)$ set

$$0<\delta(\tau)<\min\{b-\tau,\widehat{\delta}(\tau)\},$$

while

$$0<\delta(b)<b-B.$$

If $D=\{\alpha_0,\tau_1,\alpha_1,\ldots,\alpha_{k-1},\tau_k,\alpha_k\}$ is an arbitrary δ-fine P-partition of $[a,b]$ then by the choice of the gauge δ we have $\tau_k=\alpha_k=b$ and $\alpha_{k-1}\in(B,b)$. Using (2) we get

$$\begin{aligned}
\|S(\mathrm{d}x,y,D)-I\|_Z&=\Big\|\sum_{j=1}^{k-1}[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j)+[x(\alpha_k)-x(\alpha_{k-1})]y(\tau_k)-I\Big\|_Z\\
&\leqslant\Big\|\sum_{j=1}^{k-1}[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j)-\int_a^{\alpha_{k-1}}d[x(s)]y(s)\Big\|_Z\\
&\quad+\Big\|\int_a^{\alpha_{k-1}}d[x(s)]y(s)+[x(b)-x(\alpha_{k-1})]y(b)-I\Big\|_Z\\
&<\varepsilon+\Big\|\sum_{j=1}^{k-1}[x(\alpha_j)-x(\alpha_{j-1})]y(\tau_j)-\int_a^{\alpha_{k-1}}\mathrm{d}[x(s)]y(s)\Big\|_Z.
\end{aligned}$$

Since $\alpha_{k-1} < b$ and $\widehat{D} = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-2}, \tau_{k-1}, \alpha_{k-1}\}$ is a $\widehat{\delta}$-fine partition of $[a, \alpha_{k-1}]$, the second term on the right hand side of the last inequality can be esimated by ε as above. In this way we finally obtain

$$\left\|S(\,\mathrm{d}x, y, D) - I\right\| < 2\varepsilon$$

and this yields the existence of the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ as well as the equality

$$\int_a^b \mathrm{d}[x(s)]y(s) = I.$$

□

18. Remark. The "left endpoint" analog of Theorem 17 can be proved in a completely similar manner:

Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x\colon [a, b] \to X$ and $y\colon [a, b] \to Y$ are such that the Stieltjes integral $\int_a^c \mathrm{d}[x(s)]y(s)$ exists for every $c \in (a, b]$ and let the limit

$$\lim_{c \to a+} \left[\int_c^b \mathrm{d}[x(s)]y(s) + [x(c) - x(a)]y(a) \right] = I \in Z$$

exist. Then the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and

$$\int_a^b \mathrm{d}[x(s)]y(s) = I.$$

19. Theorem. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x\colon [a, b] \to X$ and $y\colon [a, b] \to Y$ are such that the Stieltjes integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and $c \in [a, b]$. Then*

$$\lim_{\substack{r \to c \\ r \in [a,b]}} \left[\int_a^r \mathrm{d}[x(s)]y(s) + [x(c) - x(r)]y(c) \right] = \int_a^c \mathrm{d}[x(s)]y(s).$$

Proof. Let $\varepsilon > 0$ be given and let δ be a gauge on $[a, b]$ which corresponds to ε by the definition of the integral $\int_a^b \mathrm{d}[x(s)]y(s)$, i.e. the inequality

$$\left\| S(\,\mathrm{d}x, y, D) - \int_a^b \mathrm{d}[x(s)]y(s) \right\| < \varepsilon$$

holds for every δ-fine P-partition D of $[a,b]$. If $r \in [c-\delta(c), c+\delta(c)] \cap [a,b]$ then the Saks-Henstock lemma 16 yields

$$\left\| [x(r)-x(c)]y(c) - \int_c^r \mathrm{d}[x(s)]y(s) \right\| < \varepsilon,$$

that is

$$\begin{aligned} &\left\| \int_a^r \mathrm{d}[x(s)]y(s) + [x(c)-x(r)]y(c) - \int_a^c \mathrm{d}[x(s)]y(s) \right\| \\ &= \left\| \int_c^r \mathrm{d}[x(s)]y(s) - [x(r)-x(c)]y(c) \right\| < \varepsilon, \end{aligned}$$

and this yields the relation given in the statement. □

20. R e m a r k. Theorem 19 shows that the function given by

$$r \in [a,b] \mapsto \int_a^r \mathrm{d}[x(s)]y(s) \in Z,$$

i.e. the *indefinite Stieltjes integral* is not continuous in general. The indefinite integral is continuous at a point $c \in [a,b]$ if and only if $\lim\limits_{r\to c}[x(c)-x(r)]y(c) = 0$.

21. Corollary. *Assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple and that functions $x\colon [a,b] \to X$ and $y\colon [a,b] \to Y$ are such that the Stieltjes integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and $c \in [a,b]$. If $x \in (\mathcal{B})G([a,b], X)$, then*

$$\begin{aligned} \lim_{r\to c_-^+} \int_a^r \mathrm{d}[x(s)]y(s) &= \lim_{r\to c_-^+} [x(r)-x(c)]y(c) + \int_a^c \mathrm{d}[x(s)]y(s) \\ &= \lim_{r\to c_-^+} x(r)y(c) - x(c)y(c) + \int_a^c \mathrm{d}[x(s)]y(s). \end{aligned}$$

P r o o f. Since $x \in (\mathcal{B})G([a,b], X)$ is assumed, the limits $\lim\limits_{r\to c_-^+} x(r)u$ exist for every $u \in Y$ and therefore also the limits $\lim\limits_{r\to c_-^+} x(r)y(c)$ exist. The equality given in the statement is now a consequence of the equality given in Theorem 19. □

22. Proposition. *Assume that X, Y are Banach spaces and consider the bilinear triple $\mathcal{B} = (L(X,Y), X, Y)$. If $A\colon [a,b] \to L(X,Y)$ is $\mathcal{B}$-regulated ($A \in (\mathcal{B})G([a,b], L(X,Y))$) then for every $c \in [a,b)$ there exists $A(c+) \in L(X,Y)$ such that $\lim\limits_{t\to c+} A(t)x = A(c+)x$ for every $x \in X$ and for every $c \in (a,b]$ there exists $A(c-) \in L(X,Y)$ such that $\lim\limits_{t\to c-} A(t)x = A(c-)x$ for every $x \in X$.*

P r o o f. If A is $\mathcal{B}$-regulated then for every $x \in X$ the limit $\lim_{t \to c+} A(t)x = y_{c+}(x) \in Y$ exists and by $A(c+)x = y_{c+}(x)$ a linear operator from X to Y is defined. By the Banach-Steinhaus theorem (see e.g. [5]) the operator $A(c+)$ is bounded, i.e. $A(c+) \in L(X,Y)$. A similar argument holds for $A(c-)$, too. $\square$

23. R e m a r k. In the special case considered in Proposition 22 the formulae given in Lemma 12, 13 and Proposition 14 can be written in a more explicit form.

For example Proposition 14 assumes the following form.

Assume that X and Y are Banach spaces and consider the bilinear triple $\mathcal{B} = (L(X,Y), X, Y)$.

If $x\colon [a,b] \to L(X,Y)$ is $\mathcal{B}$-regulated on $[a,b]$ ($x \in (\mathcal{B})G([a,b], L(X,Y))$) and $y\colon [a,b] \to X$ is a step function, i.e. there is a finite partition

$$a = \beta_0 < \beta_1 < \ldots < \beta_{k-1} < \beta_k = b$$

of the interval $[a,b]$ such that y has a constant value $y_j^* \in X$ on (β_{j-1}, β_j) for every $j = 1, \ldots, k$, then the integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and

$$\begin{aligned}\int_a^b \mathrm{d}[x(s)]y(s) = x(a+)y(a) - x(a)y(a) \\ + \sum_{j=1}^{k-1}[x(\beta_j+)y(\beta_j) - x(\beta_j-)y(\beta_j)] + x(b)y(b) - x(b-)y(b) \\ + \sum_{j=1}^{k}[x(\beta_j-)y_j^* - x(\beta_{j-1}+)y_j^*]\end{aligned}$$

where for $x(c+) \in L(X,Y), c \in [a,b)$, $x(c-) \in L(X,Y), c \in (a,b]$ is given by

$$\lim_{r \to c+} x(r)y = x(c+)y, \ \lim_{r \to c-} x(r)y = x(c-)y,$$

respectively.

24. Corollary. *Assume that X, Y are Banach spaces and consider the bilinear triple $\mathcal{B} = (L(X,Y), X, Y)$. Suppose that functions $x\colon [a,b] \to L(X,Y)$ and $y\colon [a,b] \to X$ are such that the Stieltjes integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists and let $c \in [a,b]$. If $x \in (\mathcal{B})G([a,b], L(X,Y))$ then*

$$\lim_{r \to c^{+}_{-}} \int_a^r \mathrm{d}[x(s)]y(s) = [x(c^{+}_{-}) - x(c)]y(c) + \int_a^c \mathrm{d}[x(s)]y(s)$$

where $x(c_-^+) \in L(X,Y)$ *is given by the relation*

$$\lim_{r \to c_-^+} x(r)y = x(c_-^+)y.$$

25. Remark. In the situation of Corollary 24, i.e. if $x \in (\mathcal{B})G([a,b], L(X,Y))$ and $y\colon [a,b] \to X$ is such a function that the Stieltjes integral $\int_a^b \mathrm{d}[x(s)]y(s)$ exists, the indefinite integral given by

$$F(r) = \int_a^r \mathrm{d}[x(s)]y(s) \quad \text{ for } r \in [a,b]$$

is a function $F\colon [a,b] \to Y$ which is regulated, i.e. $F \in G([a,b], Y)$.

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