TRANSITIVITY AND PARTIAL ORDER

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Abstract. In this paper we find a one-to-one correspondence between transitive relations and partial orders. On the basis of this correspondence we deduce the recurrence formula for enumeration of their numbers. We also determine the number of all transitive relations on an arbitrary n-element set up to $n = 14$.

Keywords: enumeration, transitivity, partial order

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1. Introduction

Let A be a finite n-element set. By $\exp A$ we shall denote the set of all subsets of A. As usual a binary relation ϱ is called a quasi-order if it is reflexive and transitive and ϱ is called a partial order or an ordering if it is reflexive, antisymmetric and transitive. Let $T(A)$ denote the set of all transitive relations on A and $P(A)$ the set of all partial orders on A. The number of all transitive relations on A will be denoted by T_n and the number of all partial orders by P_n. Further, let Q_n denote the number of all quasi-orders on A. If $n = 0$, then we put $T_0 = P_0 = Q_0 = 1$. Moreover, let $\mathscr{P}(A)$ denote the set of all partitions of a set A and let $\mathbb{P} \in \mathscr{P}(A)$ be an arbitrary partition of A. The block of $\mathbb{P}$ containing an element $x \in A$ will be denoted by $[x]$. Now we recall the important notion of Stirling's number of the second kind and also several known results. Stirling's number of the second kind $S(n,k)$ is the number of partitions of an n-set into k blocks. By convention, we put $S(0,0) = 1$. Clearly, we have $S(n,k) = 0$ if $k > n$, $S(n,0) = 0$, $S(n,1) = 1$, $S(n,2) = 2^{n-1} - 1$, $S(n,n-1) = \binom{n}{2}$, $S(n,n) = 1$. In addition, there are a number of other possible methods in counting these numbers. Stirling's numbers of the second kind satisfy the recursions

$$S(n,k) = kS(n-1,k) + S(n-1,k-1), \tag{1}$$

$$S(n,k) = \sum_{i=1}^{n-1} \binom{n-1}{i} S(i,k-1). \tag{2}$$

We can also count $S(n,k)$ by means of the explicit identities

$$S(n,k) = (k!)^{-1} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n, \tag{3}$$

$$S(n,k) = \sum_{a_1+\ldots+a_k=n} 1^{a_1-1} 2^{a_2-1} \cdots k^{a_k-1}, \tag{4}$$

where the sum extends over all $\binom{n-1}{k-1}$ k-compositions of an integer n.

It is also well known that there is a one-to-one correspondence between topologies on A and quasi-orders on A, and a one-to-one correspondence between partial orders on A and T_0-topologies on A. Evans, Harary and Lynn derived in [5] a formula relating the number of all quasi-orders on a set of n elements and the number of all partial orders. They proved that

$$Q_n = \sum_{k=1}^{n} S(n,k) P_k. \tag{5}$$

Erné showed in [3] that

$$\frac{Q_n}{P_n} \to 1 \quad \text{for} \quad n \to \infty. \tag{6}$$

Further, transitive ternary relations and quasiorderings were studied by Novák and Novotný, see e.g. [9] and [10].

The structure of the paper is as follows. First we draw our attention to the correspondence between transitive relations and partial orders. Our correspondence turns out to be important for finding recurrence formulas for enumeration of the numbers of such relations. Then we introduce the number of all transitive relations for $n \leqslant 14$. These numbers will be obtained from our formula and from results in [4]. Furthermore we shall be concerned with the asymptotic value for T_n and in the end we shall present one interesting property of the sequence P_n.

2. The correspondence between transitive relations and partial orders

In this section we derive a one-to-one correspondence between the transitive relations and partial orders. First we introduce the following definition. Let $T_a(A)$ denote the set of all transitive and antisymmetric relations on A. Let X be an arbitrary subset of A and let $\mathbb{P}$ be a partition of the set $A - X$. Then we define $T_a^*(X \cup \mathbb{P}) = \{\varrho \in T_a(X \cup \mathbb{P}); \forall z \in X \cup \mathbb{P}\colon z \varrho z \Leftrightarrow z \in \mathbb{P}\}$. The following assertion shows the connection between the transitive relations and the antisymmetric and transitive relations.

Lemma 1. *There exists a bijection*

$$f\colon T(A) \to \bigcup_{X \in \exp A} \Bigg(\bigcup_{\mathbb{P} \in \mathbb{P}(A-X)} T_a^*(X \cup \mathbb{P}) \Bigg). \tag{7}$$

P r o o f. For every $\varrho \in T(A)$ we define the set $A_\varrho = \{x \in A; [x,x] \in \varrho\}$ and for all $x, y \in A_\varrho$ we put $x \sim y$ iff $[x,y] \in \varrho$ and $[y,x] \in \varrho$. It is evident that $\sim$ is an equivalence relation on A_ϱ. Now we set $X = A - A_\varrho$ and $\mathbb{P} = A_\varrho/{\sim}$ and for every $\varrho \in T(A)$ we define $\sigma \in T_a^*(X \cup \mathbb{P})$ as follows. Let $[x,y] \in \varrho$. We put $[x,y] \in \sigma$ or $\big[[x],y\big] \in \sigma$ or $\big[x,[y]\big] \in \sigma$ or $\big[[x],[y]\big] \in \sigma$ if $x, y \in X$ or $x \in A - X, y \in X$ or $x \in X, y \in A - X$ or $x, y \in A - X$, respectively. Further, setting $f(\varrho) = \sigma$ we have defined the mapping f. The verification that f is a bijection is elementary. $\square$

Lemma 2. *There exists a bijection*

$$f\colon T_a^*(X \cup \mathbb{P}) \to P(X \cup \mathbb{P}). \tag{8}$$

P r o o f. For every $\varrho \in T_a^*(X \cup \mathbb{P})$ we put $f(\varrho) = \varrho \cup \triangle_{X \cup \mathbb{P}}$, where $\triangle_{X \cup \mathbb{P}} = \{[z,z]; z \in X \cup \mathbb{P}\}$ is the diagonal relation on $X \cup \mathbb{P}$. The bijectivity of (8) is now evident. $\square$

By means of the above lemmas we have shown how the transitive relations on A correspond bijectively to the partial orders on sets $X \cup \mathbb{P}$. Now we demonstrate this correspondence by a suitable example.

E x a m p l e 1. Let us consider a set $A = \{a, b, c, d, e, f\}$ and the transitive relation $\varrho = \{[a,a], [b,b], [d,d], [e,e], [f,f], [a,b], [b,a], [a,c], [b,c], [d,e], [e,d], [d,c], [e,c], [d,f], [e,f]\}$ on A. This relation can be expressed in a simple way by means of a graph, see Figure 1.

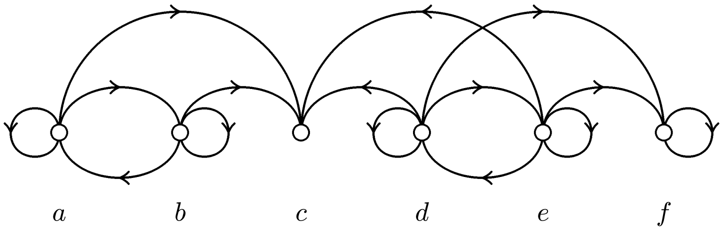

Figure 1

By Lemma 1 we have $A_\varrho = \{a, b, d, e, f\}$ and $X = \{c\}$. The equivalence relation $\sim$ is $\big\{[a,a],[b,b],[d,d],[e,e],[f,f],[a,b],[b,a],[d,e],[e,d]\big\}$ and the partition of the set A_ϱ is $\mathbb{P} = \big\{\{a,b\},\{d,e\},\{f\}\big\}$. Furthermore, by Lemma 1 we have $\sigma = \big\{[\{a,b\},\{a,b\}],[\{d,e\},\{d,e\}],[\{f\},\{f\}],[\{a,b\},c],[\{d,e\},c],[\{d,e\},\{f\}]\big\}$. The graph of the relation σ is shown in Figure 2.

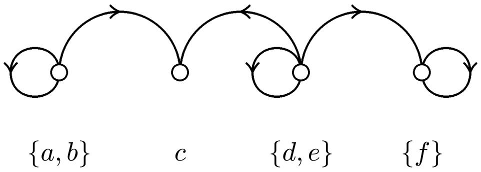

Figure 2

The relation σ is antisymmetric and transitive, but it is not reflexive. Further, by Lemma 2 the binary relation $\sigma \cup \triangle_{X\cup\mathbb{P}}$ is a partial order on $X \cup \mathbb{P}$. The graph of the relation $\sigma \cup \triangle_{X\cup\mathbb{P}}$ is presented in Figure 3. In this way we can associate a partial order with every transitive relation. We remark that removing the loops and orientations of the edges from Figure 3 we obtain the Hasse diagram.

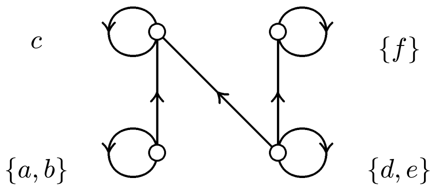

Figure 3

3. The recurrence formula

In this section we introduce the main result of this paper. Our formula makes the recurrent enumeration of the values P_n or T_n possible provided the members of the second sequence are known.

Theorem 1. *For each positive integer n we have the formula for T_n*

$$T_n = \sum_{k=1}^{n} N_k(n) P_k, \quad \text{where} \quad N_k(n) := \sum_{s=0}^{k} \binom{n}{s} S(n-s, k-s). \tag{9}$$

Proof. First by our convention $T_n = |T(A)|$ and from Lemma 1 and 2 we have

$$T_n = \Big| \bigcup_{X \in \exp A} \Big(\bigcup_{\mathbb{P} \in \mathbb{P}(A-X)} P(X \cup \mathbb{P}) \Big) \Big|.$$

Since $X \cap \mathbb{P} = \emptyset$ we get by the rule of the sum and by the rule of the product

$$\Big| \bigcup_{X \in \exp A} \Big(\bigcup_{\mathbb{P} \in \mathbb{P}(A-X)} P(X \cup \mathbb{P}) \Big) \Big| = \sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{n-k} S(n-k, m) P_{k+m}.$$

Now expanding this sum, factoring all P_k out and then using a different way of summation we obtain

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{n-k} S(n-k, m) P_{k+m} = \sum_{k=1}^{n} \Big(\sum_{s=0}^{k} S(n-s, k-s) \binom{n}{s} \Big) P_k,$$

which is nothing but (9). This completes the proof. □

Corollary 1. *For each positive integer n we have the following formula for P_n:*

$$P_n = \frac{1}{2^n} \Big(T_n - \sum_{k=1}^{n-1} N_k(n) P_k \Big). \tag{10}$$

Proof. Clearly, we have $\sum_{s=0}^{n} S(n-s, n-s) \binom{n}{s} = \sum_{s=0}^{n} \binom{n}{s} = 2^n$. Now (10) immediately follows from (9). □

Example 2. In this example we show the enumeration of the value T_4 by means of our formula (9). To find Stirling's numbers of the second kind we can use the formulas (1)–(4). First we determine the values $N_1(4) = 1$, $N_2(4) = 11$, $N_3(4) = 24$ and $N_4(4) = 16$. Further we suppose that the numbers $P_1 = 1$, $P_2 = 3$, $P_3 = 19$ and $P_4 = 219$ are already known. From this and from (9) we obtain $T_4 = P_1 + 11P_2 + 24P_3 + 16P_4 = 1 + 33 + 456 + 3504 = 3994$.

Now we introduce the table of the numbers P_n and T_n up to $n = 14$. Currently the number T_{14} constitutes the greatest known value of the sequence T_n.

Table. Initial values of P_n and T_n for $n \leqslant 14$.

$P_1 = 1$	$T_1 = 2$
$P_2 = 3$	$T_2 = 13$
$P_3 = 19$	$T_3 = 171$
$P_4 = 219$	$T_4 = 3\,994$
$P_5 = 4\,231$	$T_5 = 154\,303$
$P_6 = 130\,023$	$T_6 = 9\,415\,189$
$P_7 = 6\,129\,859$	$T_7 = 878\,222\,530$
$P_8 = 431\,723\,379$	$T_8 = 122\,207\,703\,623$
$P_9 = 44\,511\,042\,511$	$T_9 = 24\,890\,747\,921\,947$
$P_{10} = 6\,611\,065\,248\,783$	$T_{10} = 7\,307\,450\,299\,510\,288$
$P_{11} = 1\,396\,281\,677\,105\,899$	$T_{11} = 3\,053\,521\,546\,333\,103\,057$
$P_{12} = 414\,864\,951\,055\,853\,499$	$T_{12} = 1\,797\,003\,559\,223\,770\,324\,237$
$P_{13} = 171\,850\,728\,381\,587\,059\,351$	$T_{13} = 1\,476\,062\,693\,867\,019\,126\,073\,312$
$P_{14} = 98\,484\,324\,257\,128\,207\,032\,183$	$T_{14} = 1\,679\,239\,558\,149\,570\,229\,156\,802\,997$

Remark. In 1977 Kim and Rush (see [6]) showed that the number of transitive relations on a set of n elements is asymptotically 2^n times the number of partial orders. It may be noteworthy to mention here that the asymptotic value for T_n, where $n = 14$, differs from this number by about 3.91 percent. Now we shall deduce this theorem by means of our formula (10).

Theorem 2. *The sequences* $\{T_n\}_{n=1}^{\infty}$ *and* $\{2^n P_n\}_{n=1}^{\infty}$ *are asymptotically equal, i.e.*

$$\frac{T_n}{2^n P_n} \to 1 \quad \text{for} \quad n \to \infty. \tag{11}$$

P r o o f. It follows from (10) that $2^n P_n \leqslant T_n$, and moreover, we have

$$1 \leqslant \lim_{n\to\infty} \frac{T_n}{2^n P_n}. \tag{12}$$

Now by virtue of (1) we have $S(n-s,k-s) \leqslant S(n,k)$ for $0 \leqslant s \leqslant k$. Combining this fact with (5) we obtain

$$\sum_{k=1}^{n-1} N_k(n) P_k \leqslant 2^n (Q_n - P_n). \tag{13}$$

Further, the preceeding estimate together with (10) immediately gives

$$\lim_{n\to\infty} \frac{T_n}{2^n P_n} \leqslant 1 + \lim_{n\to\infty} \Big(\frac{Q_n}{P_n} - 1\Big). \tag{14}$$

Finally, taking into account (6), we conclude that the inequalities (12) and (14) are equalities. This proves the theorem. □

Corollary 2. *The sequences $\{T_n\}_{n=1}^{\infty}$ and $\{2^n Q_n\}_{n=1}^{\infty}$ are asymptotically equal, i.e.*

$$\frac{T_n}{2^n Q_n} \to 1 \quad \textit{for} \quad n \to \infty. \tag{15}$$

P r o o f. The assertion immediately follows from (6) and (11). □

4. On periodicity of the last figures of the sequence P_n

In the end we introduce a short remark on periodicity of the last figures of the sequence P_n. Z. I. Borevich proved in 1982 a wonderful result on the sequence P_n. He proved the following assertion (see [2]).

Theorem 3 (Borevich). *Let m be an arbitrary positive integer. Then there exists an index n_0 from which the sequence $\{P_n \bmod m\}_{n=1}^{\infty}$ is periodical. Specifically, when $m = p_1 \dots p_k$, where $p_1, \dots, p_k$ are different primes, then $\{P_n \bmod m\}_{n=1}^{\infty}$ is periodical and the length of its period is equal to the least common multiple of the numbers $p_1 - 1, \dots, p_k - 1$.*

From the above-mentioned theorem another interesting consequence follows. The following fact was not explicitly emphasized in Borevich's work.

Corollary 3. *For every non-negative integer k the following holds: The last figure of the number P_{4k+1} is 1, P_{4k+2} is 3, P_{4k+3} is 9 and P_{4k+4} is 9. Also the sequence of the last figures of P_n is periodical, the length of this period is 4 and the members of this sequence are $1, 3, 9, 9, \ldots$ (see Table). This immediately implies that the values P_n are odd for every positive integer n.*

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