A NOTE ON SEPARATE CONTINUITY AND CONNECTIVITY PROPERTIES

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(Received October 3, 1995)

Abstract. Separately continuous functions are shown to have certain properties related to connectedness.

Keywords: separate continuity, cluster sets, O-connectedness, closed graph, local w^* continuity

MSC 1991: 54C10, 26A15

I. Introduction

The following elementary example shows that separately continuous functions are not connected functions:
Define $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that

$$f((x,y))=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0).\end{cases}$$

Now let $E=\{(x,y)\colon x\geqslant 0, y\geqslant 0 \text{ and } \frac{1}{3}x\leqslant y\leqslant 3x\}$. Then the image of E is not connected. In this paper, we show that, for separately continuous functions, if the connected set is also open, then its image is a connected set in the range space. This condition, which we call "O-connectedness," is strictly weaker than connectedness, as shown by the following example:

$$f(x)=\begin{cases}0, & x<0\\ 1, & x=0\\ \sin\frac{1}{x}, & x>0\end{cases}$$

In Theorem 2 and Corollary 1, we show that, with suitable restrictions on the domain and range spaces, O-connected functions (including separately continuous functions) have connected cluster sets. Theorem 4 and Corollary 2 show that the closed graph property, combined with O-connectedness, yields continuity. Corollary 3 presents a similar result for separately continuous functions.

Throughout this paper a function f from a space X into a space Y will be denoted by $f\colon X \to Y$. We say that a function $f\colon X \to Y$ is O-connected if the image of every connected open set in X is a connected set in Y.

II. Separate continuity and O-connectedness

The following lemma is similar to Theorem 3.5 of [2]:

Lemma. *Let $f\colon X \times Y \to \mathbb{R}$ be a real-valued separately continuous function, where X and Y are topological spaces. Let $A \subset X$ and $B \subset Y$ be connected sets in the topologies on X and Y respectively. Then $f(A \times B)$ is a connected set in $\mathbb{R}$.*

Proof. Let $E = \{f(x,y)\colon x \in A \text{ and } y \in B\}$. If the set E consists of a single point, we are done. Let z_1 and z_2 be any two points in E such that $z_1 \neq z_2$. There exist points (x_1,y_1) and (x_2,y_2) in $A\times B$ such that $f(x_1,y_1) = z_1$ and $f(x_2,y_2) = z_2$. Since f is continuous in each variable separately, if $x_1 = x_2$ or $y_1 = y_2$, then every value between z_1 and z_2 is in E. If $x_1 \neq x_2$ and $y_1 \neq y_2$, consider the point (x_2,y_1) in $A\times B$. Again, since f is separately continuous, every value between $f(x_1,y_1) = z_1$ and $f(x_2,y_1) = z_3$ is in E. Similarly, every value between z_3 and z_2 is in E. That is, E contains every value between z_1 and z_2. Since z_1 and z_2 were chosen arbitrarily, the set E must be an interval in $\mathbb{R}$. $\square$

Before presenting the next result, we recall that if O is an open cover of a connected set S in a space X, then any two points a and b of S can be connected by a simple chain consisting of elements of O. (See, for example, Theorem 26.15 of [4], the proof of which is readily adapted to the subspace topology.)

Theorem 1. *Let $f\colon X\times Y \to \mathbb{R}$ be a real-valued separately continuous function, where X and Y are locally connected spaces. Then f is O-connected.*

Proof. Let G be a connected open subset of $X \times Y$. Then G is the union of a collection of basis elements of the form $U \times V$, where each U and each V is open and connected. Since these basis elements form an open cover of the connected set G, any two points (x_1,y_1) and (x_2,y_2) in G can be joined by a finite collection $[U_1 \times V_1, U_2 \times V_2, \ldots, U_n \times V_n]$ of such basis elements, such that $(x_1,y_1) \in (U_1 \times V_1)$

and $(x_2, y_2) \in (U_n \times V_n)$ and any two successive sets $(U_i \times V_i)$ and $(U_{i+1} \times V_{i+1})$ have at least one common point. Thus, if $f(G)$ is not a singleton, by mimicking the argument in the proof of the Lemma above, we can show that, for any two points z_1 and z_2 in $f(G)$, every value between z_1 and z_2 is in $f(G)$. Hence, $f(G)$ is connected in $\mathbb{R}$. $\square$

III. Cluster sets, O-connectedness and separate continuity

For a function $f\colon X \to Y$, where X and Y are first countable spaces, we say that the cluster set of f at $x \in X$, denoted by $C(f;x)$, is the set of all y in Y such that there exists a sequence (x_n) in X converging to x and $\big(f(x_n)\big)$ converges to y. It is easy to show that the set $C(f;x)$ is always closed. Also, $C(f;x)$ is never empty, since $f(x)$ is always an element of $C(f;x)$. In [2] W. Pervin and N. Levine showed that for a connected function $f\colon X \to Y$, where X is first countable and locally connected, and Y is first countable and compact Hausdorff, the cluster set $C(f;x)$ is connected for every x in X. Only slight modifications of the proof of Pervin and Levine are needed to prove the next result. For the convenience of the reader, we set forth the entire proof.

Theorem 2. *Let X be a locally connected and first countable space, and let Y be compact Hausdorff and first countable. Suppose that $f\colon X \to Y$ is O-connected. Then for any x in X, $C(f;x)$ is a connected subset of Y.*

Proof. Assume that $C(f;x)$ is disconnected for some x in X. Then let $C(f;x) = A\big|B$ be a separation. Since $C(f;x)$ is closed, then A and B are closed subsets of Y. But Y is compact and Hausdorff and therefore normal. Thus, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Then $C(f;x) \subset U \cup V$. The claim now is there exists an open set G containing x such that $f(G) \subset U \cup V$. Assume that for every open set G containing x there exists a point x' in G such that $f(x') \in Y \setminus (U \cup V)$. As we shall see, this will lead to a contradiction. Since X is first countable, we can construct a sequence (x'_n) in X such that (x'_n) converges to x. Consider the sequence $\big(f(x'_n)\big)$ in Y. Since $Y \setminus (U \cup V)$ is a closed subset of the compact space Y, it is also compact. Thus, $\big(f(x'_n)\big)$ has a convergent subsequence converging to some y' in $Y \setminus (U \cup V)$. But y' is in $C(f;x)$, and this contradicts the fact that $C(f;x) \subset U \cup V$. Therefore, there is some open set G containing x such that $f(G) \subset U \cup V$. Since X is locally connected, there exists a connected open set H in G containing x such that $f(H) \subset U \cup V$. Since f is O-connected, $f(H)$ is connected in Y, and thus $f(H)$ lies entirely in U or V. Then either A or B must be empty, because the other can have no points of $C(f;x)$ in it; i.e., H contains the tail of every sequence (x_n) converging to x. Hence, $C(f;x)$ is connected. $\square$

Corollary 1. *Let $f\colon \mathbb{R}\times\mathbb{R}\to I$ be a separately continuous function from the real plane into a closed interval I. Then for any point (x,y) in the domain of f, the cluster set of f at (x,y) is connected.*

Proof. Apply Theorem 1 and Theorem 2. $\square$

Remark 1. In Corollary 1 the cluster set is degenerate at points of joint continuity. We also remark that the converse of Corollary 1 is not true, as illustrated by the following function of the form $f\colon \mathbb{R}\times\mathbb{R}\to[-1,1]$:

$$f((x,y))=\begin{cases}\sin\left((x^2+y^2)^{-1}\right), & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Now by application of Theorem 1 and Corollary 1 above, we obtain the following:

Theorem 3. *Let $f\colon \mathbb{R}\times\mathbb{R}\to I$ be a separately continuous function from the real plane into a closed interval I. Let (x',y') be any point in $\mathbb{R}\times\mathbb{R}$. Then in any connected open set containing (x',y'), f takes on every value in $C\big(f;(x',y')\big)$ [except possibly the end points if $C\big(f;(x',y')\big)$ is an interval].*

Proof. If $C\big(f;(x',y')\big)=\big\{f(x',y')\big\}$, we are done. If $C\big(f;(x',y')\big)$ is a closed interval $[a,b]$, then any open set containing (x',y') contains the tail of a sequence (x_n,y_n) such that the sequence $f(x_n,y_n)$ converges to a. A similar sequence converges to b. Now apply Theorem 1. $\square$

IV. Closed graph, O-connectedness and separate continuity

We say that a function $f\colon X\to Y$ is locally w^* continuous if there exists an open basis B for the topology on Y such that $f^{-1}[\mathrm{Fr}(V)]$ is closed in X for any $V\in B$, where Fr() denotes the frontier operator [1]. Local w^* continuity is a generalization of the closed graph property for functions of the form $f\colon X\to Y$, where Y is locally compact and Hausdorff [1]. The next theorem and its corollary generalize the well-known result that a connected function with a closed graph, is continuous.

Theorem 4. *Let X be a locally connected space and let $f\colon X\to Y$ be locally w^* continuous. If f is O-connected, then f is continuous.*

Proof. Let $x\in X$ and let $W\subset Y$ be an open set containing $f(x)$. By local w^* continuity, there exists a basic open set $V\subset Y$ such that $f(x)\in V\subset W$ and

$f^{-1}[\mathrm{Fr}(V)]$ is closed in X. Then the complement of $f^{-1}[\mathrm{Fr}(V)]$, which we shall call G, is open and contains x. Since X is locally connected, there exists an open connected set U such that $x \in U \subset G$. Claim: $f(U) \subset V \subset W$. Assume there exists $x' \in U$ such that $f(x') \notin V$. Now $Y \setminus \mathrm{Fr}(V)$ is a disconnected subspace of Y. Since $f(U)$ is connected, $f(U)$ is contained in V or in $Y \setminus \mathrm{Cl}(V)$. But this is impossible. $\square$

Corollary 2. *Let $f\colon X \to Y$ be a function, where X is locally connected and Y is locally compact and Hausdorff. Suppose that f has the closed graph property. Then if f is O-connected, f is continuous.*

P r o o f. The function f is locally w^* continuous. Now apply Theorem 4. $\square$

Corollary 3. *Let $f\colon X \times Y \to \mathbb{R}$ be a separately continuous real-valued function, where X and Y are locally connected spaces. If f is locally w^* continuous, then f is continuous.*

R e m a r k 2. For a more general result, see [1].

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