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IDEAL BANACH CATEGORY THEOREMS AND FUNCTIONS

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Abstract. Based on some earlier findings on Banach Category Theorem for some "nice" σ-ideals by J. Kaniewski, D. Rose and myself I introduce the h operator (h stands for "heavy points") to refine and generalize kernel constructions of A. H. Stone. Having obtained in this way a generalized Kuratowski's decomposition theorem I prove some characterizations of the domains of functions having "many" points of h-continuity. Results of this type lead, in the case of the σ-ideal of meager sets, to important statements of Abstract Analysis such as Blumberg or Namioka-type theorems.

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1. Introduction

Given a topological space (X,τ), let $\mathcal{T}\subseteq\mathcal{P}(X)$ be an ideal of subsets of X. For any subset $A\subseteq X$, let $A^*(\tau,\mathcal{T})$ or simply A^* if τ and $\mathcal{T}$ are understood, be the adherence of A modulo $\mathcal{T}$. In particular, $A^*=\{x\in X\colon x\in U\in\tau$ implies $U\cap A\notin\mathcal{T}\}$. Observe that A^* is a closed subset of cl$A$. For convenience $A^0(\tau,\mathcal{T})$ or simply A^0, if τ and $\mathcal{T}$ are understood, denote the set $A\setminus A^*$. In the terminology of A. M. Stone, $A\cap A^*(\tau,\mathcal{T})$ is the kernel of the subspace $(A,\tau|A)$ (relative to the ideal $\mathcal{T}|A=\mathcal{T}\cap\mathcal{P}(A)$).

The following three conditions have been intensively studied in [Kaniewski, Piotrowski and Rose, 5].

B_1: Let $D\subset X$. Suppose that for every $\emptyset\neq U_{\rm open}$ there is a nonempty open $V\subset U$ such that $V\cap D$ is an I-set in X. Then D is an I-set in X.

B_2: Let $D \subset X$. The set of all points x of D for which there is an open nbd $U \ni x$ such that $(U \cap D) \in I$, is an I-set. Actually, B_2 may be formulated as follows:

$$\forall D \subset X, \text{ the set } K_I(D) = D \cap \bigcup\{U_{\text{open}} \subset X : (U \cap D) \in I\} \in I \tag{1}$$

B_3: In a topological space X, the union of any family of open I-sets is an I-set in X, or equivalently $K_I(X) \in I$.

2. Kuratowski's Decomposition Theorem and the h operator

As was already noticed in Introduction, the set A* is a closed subset of cl A, the closure of A in X. So, if $A = X$, we derive that the set X^*, the kernel of X, is closed in X. Now, since X^0, the co-kernel of X, is defined as the complement of X^* in X, we conclude that X^0 is an open subset of X.

Summarizing, we have

Generalized Kuratowski's Decomposition Theorem—first version. *Let (X, τ) be a topological space and let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a σ-ideal of subsets of X satisfying B_2. Then X can be uniquely decomposed into a closed subspace A^*, possibly empty, which is the adherence of A modulo $\mathcal{I}$, and an open subspace A^0, possibly empty, where $A^0 \in \mathcal{I}$.*

The original decomposition theorem, due to K. Kuratowski, formulated for the σ-ideal $\mathcal{M}(\tau)$ of meager sets, can be easily obtained from the following version of the Banach Category Theorem [Blumberg, 1].

($\star$) If X is a metric space, $D \subseteq X$ and $S \subseteq D$ is the set of points in D which are meager relative to D, then S is meager in X.

It is easy to see that ($\star$) above is a special case of B_2, see Introduction. Also, it can be easily checked that B_2 *suffices* to derive *this version* of the decomposition theorem.

We shall now exhibit an example from which the reader will see the need of further refinement of the notion of the $(-)^*$ operator.

Example 2.1. Let (X, τ) be the closed upper half-plane with the Euclidean topology and $\mathcal{I} = \mathcal{M}(\tau)$. Further, let

$$\begin{aligned} A_+ &= \{(x, y) : (x \in \mathbb{Q} \wedge x > 0) \wedge (y \in \mathbb{Q} \wedge y > 0)\}, \\ A_- &= \{(x, y) : x < 0 \wedge y > 0 \wedge [(x \in \mathbb{R} \setminus \mathbb{Q}) \vee (y \in \mathbb{R} \setminus \mathbb{Q})]\}. \end{aligned} \tag{2}$$

Define $A = A_+ \cup A_-$.

Observe that

$$A^0 = \{(x, y) : x > 0 \wedge y \geqslant 0\} \text{ and } A^* = \{(x, y) : x \leqslant 0 \wedge y \geqslant 0\}.$$

The fact that A^* (resp. A^0) is a closed (resp. open) subset of the closed upper half-plane illustrates only the general situation, where A^* (resp. A^0) is a closed (resp. open) subset of cl$A$, the closure of A in X.

What about the *open* subspace $H = \{(x, y) \colon x < 0 \wedge y \geqslant 0\}$ made of the "true," deep-inside non-meager points? Observe that while the points of H *do* have the property $(\mathcal{P})$ below, *none* of the points of the "border"—the non-negative part of the y-axis, has the property $(\mathcal{P})$.

First, we need the following definition: Let X be a space and let $A \subseteq X$. A point $x \in X$ is said to be *non-meager relative to* A if every open neighborhood U of x contains a subset B of A which is non-meager in X.

And now the promised property:

$(\mathcal{P})$ Let X be a space and let $A \subseteq X$. Let $x \in X$ be arbitrary. Then there is an open neighborhood V of x such that every point $y \in V$ is non-meager relative to A.

We shall now formulate a general case of a "good" property of all points of H.

As in Section 1, (X, τ) denotes a topological space, $\mathcal{I} \subseteq \mathcal{P}(X)$ is a σ-ideal of subsets of X. Let $A \subseteq X$; associate with A the set A^h, the *heavy part of* A defined by

$A^h = \{x \in X \colon \exists U \in \tau$ such that $x \in U$ and $U \subseteq A^*\}$.

The reader will notice

Fact 2.1. A^h is an open (possibly empty) subspace.

Using quite elementary properties of the relative topology we deduce that A^h is the *maximal* open subset contained in A^*, that is :

Proposition 2.1. $A^h = \mathrm{Int} A^*$.

Let us turn to another version of the decomposition theorem.

Observe that $A^* \setminus A^h = A^* \setminus$ Int A^* is nowhere dense in X. So, if we assume that nowhere dense sets are in the σ-ideal $\mathcal{I}$, then we can "shift" $A^* \setminus A^h$ to A^0(!)

In view of Theorem 1 of Part 1, an ideal $\mathcal{I}$ satisfies B_1 if and only if $\mathcal{N}(\tau) \subseteq \mathcal{I}$ and $\mathcal{I}$ satisfies B_2.

We are now ready for

Generalized Kuratowski's Decomposition Theorem—second version.
Let (X, τ) be a topological space and let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a σ-ideal of subsets of X satisfying B_1. Then X can be uniquely decomposed into an open subspace A^h, a possibly empty, the heavy part of A, and the closed subspace $A^0 \cup (A^* \setminus A^h)$ which is an element of $\mathcal{I}$.

Proof. Consider the first version of Generalized Kuratowski's Decomposition Theorem. By earlier remarks, the set A^h is open, and it is the maximal set with this property. So, $A^* \setminus A^h$ is nowhere dense.

Now, by Theorem 1 of Part 1—the equivalence of B_1—the set $A^* \setminus A^h \in \mathcal{I}$. Thus $A^0 \cup (a^* A^h) \in \mathcal{I}$, the subspace $A^0 \cup (A^* \setminus A^h)$ being closed in X, as the complement of A^h. $\square$

R e m a r k 2.1. Observe that the second version of the decomposition theorem requires *the stronger condition* B_1 rather than B_2.

3. Functions on $(X, \tau, \mathcal{I})$

Let (X, τ) be a space and let $\mathcal{I}$ be a σ-ideal of subsets of X.

Throughout this section we will need the following definitions.

Given a set $A \subseteq X$. Every element $x \in A^h$, the heavy part of A, will be called a *heavy point relative to* A.

In other words, $x \in X$ is a heavy point relative to A if and only if there is an open neighborhood U of x such that $U \subseteq A^*$.

Let $f\colon X \to Y$ be a function. We say that f *is h-continuous at* x_0, if for every open set G containing $f(x_0)$, the set $f^{-1}(G)$ has x as a heavy point relative to X.

E x a m p l e 3.1. The "salt & pepper" function, $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$, if x is rational and $f(x) = 1$, if x is irrational, is h-continuous at every irrational $\mathcal{I}$ being the σ-ideal of countable sets.

This type of "almost continuity" is very strong. For a fairly large class of spaces and σ-ideals (or just ideals) on them, if f is h-continuous at *every* point of the domain space, then f is continuous—see [Kaniewski and Piotrowski, 5] for appropriate generalizations.

If both the domain and the range of f is the set of reals, then h-continuity of f at x_0 can be characterized by: $\exists S\colon X \setminus S \in \mathcal{I}$ and $\lim\limits_{x \in S, x \to x_0} f(x) = f(x_0)$.

The following result is found in [Thomson, 10], see [Kaniewski and Piotrowski, 5] for generalizations.

Proposition 3.1. ([Thomson, 10], Thm 34.1, p. 78) *Let $\mathcal{I}$ be an ideal of sets in $\mathbb{R}$ which does not contain an open nonempty set. If a function $f\colon \mathbb{R} \to \mathbb{R}$ is h-continuous at every point of f, then f is continuous.*

In what follows a *network* $\mathcal{N}$ for a space X is a collection of subsets N of X such that whenever $x \in U$ with U open, then there exists $N \in \mathcal{N}$ with $x \in N \subset U$.

So, a network is like a base, but its elements need not be open. In some cases a network can be "fattened up" to a base for a space, e.g., a compact space with a countable network has a countable base [Engelking, 4].

Given a space Y having a network $\mathcal{N}$, let $f\colon X \to Y$ be a function.

Define $C_h(f, \mathcal{N})$ (or $C_n(f, \mathcal{N})$) to be the set of all points $x \in X$ such that for every $N \in \mathcal{N}$ we have:

$$f(x) \in N \text{ implies } x \in [f^{-1}(N)]^h \ \ (\text{or } x \in \operatorname{Int} \operatorname{cl} f^{-1}(N), \text{ respectively}). \tag{3}$$

Since for every $S \subseteq X$ we have $S^* \subseteq$ cl S we have

$$C_h(f,\mathcal{N}) \subseteq C_n(f,\mathcal{N}). \tag{4}$$

When $\mathcal{N}$ is a base for τ then $C_h(f,\mathcal{N})$ (or $C_n(f,\mathcal{N})$) stands for the set $C_h(f)$ (or $C_n(f)$) of points of *h-continuity* (or *near continuity*) of f.

For any network $\mathcal{N}$ the following relations are true:

$$C_h(f,\mathcal{N}) \subset C_h(f) \text{ and } C_n(f,\mathcal{N}) \subseteq C_n(f). \tag{5}$$

Theorem 3.1. *Let $\mathcal{I}$ be a σ-ideal satisfying B_1 and let $\mathcal{N}$ be countable. Then $C_n(f,\mathcal{N})$ is "almost everywhere" in X, i.e.,*

$$X \setminus C_h(f,\mathcal{N}) \in \mathcal{I}. \tag{6}$$

P r o o f. By the definition of $C_h(f,\mathcal{N})$ we have

$$X \setminus C_h(f,\mathcal{N}) = \bigcup_{N\in\mathcal{N}} \{f^{-1}(N) \setminus [f^{-1}(N)]^h\}.$$

By the Decomposition Theorem—second version, $f^{-1}(N) \setminus [f^{-1}(N)]^h \in \mathcal{I}, N \in \mathcal{N}$. So, the countable union of elements $f^{-1}(N) \setminus [f^{-1}(N)]^h$ of the σ-ideal is in the σ-ideal, so $X \setminus C_h(f,\mathcal{N}) \in \mathcal{I}$. □

The same assertion for $C_n(f,\mathcal{N})$ does not require the strength of the Banach Category Theorem, since for any $S \subseteq X, S\setminus$ Int cl S is nowhere dense. Clearly, $C(f) \subseteq C_n(f)$, where $C(f)$ stands for the set of points of continuity of f.

Proposition 3.2. *Assume $X = X^*$. Then $C(f) \subseteq C_h(f)$.*

P r o o f. The proof of this claim is routine and as such is left to the reader. The fact that $X = X^*$ is necessary in Proposition 3.2 easily follows from Example 3.2. Let $\mathcal{M}(\tau)$ be the σ-ideal of meager sets in $\mathbb{Q}$ (the rationals) and let $f\colon \mathbb{Q} \to \mathbb{R}$ be any constant function, i.e., $f(x) = c$. Clearly, $C(f) = \mathbb{Q}$, whereas $C_h(f) = \emptyset$. □

Before we elaborate on the consequences of Theorem 3.1 and its importance in General Topology and Real Analysis, let us consider a simple condition which implies the converse of the statement made in Proposition 3.1.

Proposition 3.3. *Let (Y,γ) be a regular space with a topology γ and a network $\mathcal{N}$. If for every $V \in \gamma$ and $y \in V$ there is $N \in \mathcal{N}$ such that $y \in N \subseteq V$ and $[f^{-1}(N)]^* \subset f^{-1}(\text{cl } V)$, then $C_h(f,\mathcal{N}) \subset C(f)$.*

P r o o f. Let $x \in C_h(f,\mathcal{N})$ and $y = f(x) \in V_0 \in \gamma$. Let $V \in \gamma$ be such that $y \in V \subseteq \text{cl } V \subseteq V_0$. Further, let N be as in the assumption of our Proposition. Then $x \in [f^{-1}(N)]^h \subseteq f^{-1}(\text{cl } V) \subseteq f^{-1}(V_0)$, which shows that $x \in C(f)$. □

The following result may be also viewed as a converse to Theorem 3.1.

Theorem 3.2. *Let Y contain a countably infinite discrete subset N. Then, if every function $f\colon X \to Y$ has a dense set of h-continuity, then $X = X^*$.*

P r o o f. Suppose $X \subseteq X^*$, that is, there is an open nonempty set U s.t. $U \in \mathcal{I}$, i.e., $U = \bigcup\limits_{i=1}^{\infty} F_i$, $F_i \in \mathcal{I}$, $i = 1, 2, 3, \ldots$. Let $N = \{n_0, n_1, n_2, \ldots\}$. Define $f\colon X \to Y$ as follows:

$$f(x) = \begin{cases} n_i, & \text{if } \ x \in F_i \\ n_0, & \text{if } \ x \in X \setminus U. \end{cases} \tag{7}$$

We claim that f does not have a dense set D of h-continuity, more specifically f does *not* have a point of h-continuity in U.

If D is dense in X, then there is an $x \in U$ such that $x \in D \cap U$, U being open in X. Then $x \in F_i$ for some i. So, $f(x) \in \{n_i\}$. The set $\{n_i\}$ is open in Y, so let us consider $f^{-1}\{n_i\}$. Observe that $f^{-1}\{n_i\} = F_i$; recall F_i's belong to the σ-ideal $\mathcal{I}$. This indicates that f is not nowhere h-continuous in U. □

The following characterization of spaces that are equal to their kernels easily follows from Theorems 3.1 and 3.2, namely:

Theorem 3.3. *Let $\mathcal{I}$ be a σ-ideal of subsets of X satisfying B_1, and let Y be an infinite, second countable, Hausdorff space. Then $X = X^*$ if and only if every function $f\colon X \to Y$ has a dense set of points of h-continuity.*

P r o o f. The range space Y has a countable infinite discrete subset N as an infinite Hausdorff space; being second countable it has a countable network. So, the assumptions of Theorem 3.1 and 3.2 are met. The conclusion of Theorem 3.3 follows easily as a corollary from the above two theorems. □

Recall that $f\colon X \to Y$ is called *categorically almost continuous* at x_0 if f is h-continuous at x_0 and $\mathcal{M}(\tau)$ is the σ-ideal of meager sets in X.

Corollary 3.1. *A space X is Baire if and only if every function $f\colon X \to \mathbb{N}$ has a dense set of points of categorical almost continuity. Here $\mathbb{N}$ stands for the set of natural numbers with the Euclidean topology.*

4. Concerning Theorem 3.1

A special case of Theorem 3.1, namely the one for the σ-ideal of meager sets is known in literature since 1922 [Blumberg, 1]; we shall refer to this special case as the Lemma on the existence of a residual set of categorical almost continuity points.

Its original statement, see [Bradford and Goffman, 2], asserts that if X is *any* topological space and Y is second countable then *every* function $f\colon X \to Y$ has a residual set of points of categorical almost continuity.

In other words, if the domain space X is Baire and the range space is second countable, then *any* function has a "thick," in fact a dense G_δ set of points of almost continuity.

We now exhibit Example 4.1 showing that the assumption that Y has a countable network (or a weaker assumption that Y is second countable) in Theorem 3.1 cannot be weakened.

Example 4.1. Let $\mathcal{E}$ and $\mathcal{D}$ denote the Euclidean and the discrete topology, respectively. Consider the identity function $F\colon (\mathbb{R},\mathcal{E}) \to (\mathbb{R},\mathcal{D})$ defined by $f(x) = x$. With the σ-ideal $\mathcal{M}(\mathcal{E})$, we see that $C_h(f,\mathcal{N})$ is empty.

We shall now exhibit Example 4.2 proving that the requirement of Theorem 3.1 that Y is second countable can *not* be relaxed to one such that Y has *both* an open-hereditarily countable pseudo-base and is hereditarily Lindelöf, see [Piotrowski, 8] for further results.

Example 4.2. Let $\mathcal{E}$ and $\mathcal{S}$ denote the Euclidean and Sorgenfrey topology, respectively. Consider the identity function $f\colon (\mathbb{R},\mathcal{E}) \to (\mathbb{R},\mathcal{S})$ given by $f(x) = x$. Again, with $\mathcal{M}(\mathcal{E})$ being the σ-ideal of meager sets in the domain space we see that $C_h(f,\mathcal{N}) = \emptyset$.

The Lemma on the existence of a residual set of categorical almost continuity points—not surprisingly—led in the past to two important results of Abstract Analysis:

Blumberg Theorem. (See [Bradford and Goffman, 2]; see also [White, 11].) *Let X be a metric Baire space and let $f\colon X \to \mathbb{R}$ be a function. Then there is a dense subset D of X such that $f|D$ is continuous on D (in the relative topology), and*

Separate and Joint Continuity Theorem. ([Ke], where one needs to prove an analogue of the lemma for multivalued functions ($\equiv$ relations) first.) *Let X be a space, let Y be second countable and let Z be a regular, second countable space. If $f\colon X \times Y \to Z$ is separately continuous, i.e., is continuous in one variable, while the other is fixed, then there is a residual subset A of X such that f is (jointly) continuous at every point of the set $A \times Y$.*

Remark 4.1. There have been studies [Brown, 3] or more recently [Recław, 9], of Blumberg theorem vis-à-vis various σ-ideals in the domain space.

An interested reader is especially urged to consult a work of B. S. Thomson [10] who proves

Theorem 4.2. ([10], Theorem 34.2, p. 79.) *Let $\mathcal{I}$ be a σ-ideal of sets that contains no interval. Then, if $f\colon \mathbb{R} \to \mathbb{R}$ is h-continuous except at the points of a set N_0 that belongs to $\mathcal{I}$, then there is a set M, whose complement is in $\mathcal{I}$ and f is continuous relative to M at each point of M.*

The proof clearly uses the Lindelöfness of the domain space. It would be interesting to get a generalization of the just quoted result of [Thomson, 10] to general topological spaces.

In conclusion, one can now apply Theorem 3.1 to obtain appropriate analogues of Blumberg's Theorem or Kenderov's theorem on separate and joint continuity.

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