BOOLEAN CLUSTER MODELS: MEAN CLUSTER DILATIONS AND SPHERICAL CONTACT DISTANCES

Jan Rataj and Ivan Saxl, Praha[1]

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Abstract. Boolean cluster point processes with various cluster distributions are examined by means of their spherical contact distribution function. Special attention is paid to clusters with strong independence properties (Poisson clusters) and regular clusters.

Keywords: Poisson process, cluster process, Boolean model, Matérn model, spherical contact distribution function

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1. Introduction

Given a stationary random closed set (RACS) X, the spherical contact distribution function (or equivalently, the distribution function of contact distances) is the conditional probability that a ball of given radius centred at the origin o hits X, conditional on o not belonging to X. Together with the distribution of other random distances, it creates an important alternative to the commonly used second order characteristics for the description of the spatial arrangement of a RACS. If the volume fraction $p(X)$ of X is zero (e.g. point, fibre and surface sets), the probability $\Pr\{o \notin X\} = 1$ a.s. and the spherical contact distribution function is equal to the Choquet (capacity) functional $T_X(B_r)$ for a family of balls B_r of diameter $0 \leqslant r < \infty$ [10].

The main task of this paper is to investigate the capability of the spherical contact distribution function to reveal relatively small differences in the local arrangement of spatial patterns with a particular attention to local regularities. As a model structure, the Neyman-Scott processes (i.e. Boolean models with point clusters as primary

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grains) will be considered. The types of clusters examined range from deterministic arrangement (vertices of simplices and cubes) and random N-tuples of points in a ball (binomial clusters) to mixtures of clusters random in arrangement as well as in the number of points (Matérn model [9]). The general results are illustrated by several numerical examples showing the behaviour of models depending on the cluster parameters and on the dimension of the embedding space.

As is well known, the key term in the calculation of the spherical contact distribution function of a Boolean model is the mean volume of dilation of the primary grain. This quantity is interesting even in other applications, namely in image analysis, description of grain growth from germs, spatial tessellations etc. As the results for non-convex grains are rather scarce [4], [7], the conclusions obtained in the present paper can be useful also in a more general context.

2. Boolean cluster model

The setting of this note will be the d-dimensional Euclidean space $\mathbb{R}^d$ with the scalar product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $\|\cdot\|$. The closed ball of radius r and centre a will be denoted by $B_r(a)$; if $a = o$ we write simply B_r. $\mathbb{S}^{d-1}$ is the unit sphere. The sign $\oplus$ stands for the Minkowski addition of subsets of $\mathbb{R}^d$. Further, ν_k is the k-dimensional Hausdorff measure in $\mathbb{R}^d$, $0 \leqslant k \leqslant d$ (in particular, ν_d is the Lebesgue measure) and $\kappa_d = \nu_d(B_1)$, $O_{d-1} = \nu_{d-1}(\mathbb{S}^{d-1})$.

Let $\mathcal{Z}$ be the space of all finite subsets of $\mathbb{R}^d$ with the induced myopic topology from the space of compact subsets (this topology is metrizable by the Hausdorff metric). A random element $Z \in \mathcal{Z}$ will be called a *random cluster*. Its main characteristics are $\operatorname{card} Z$ (number of points, called the daughters) and its size, e.g. the diameter $\operatorname{diam} Z$ or the volume $\nu_d(\operatorname{conv} Z)$ of its convex hull. Given the distribution of Z, the expected values will be denoted by $N = \mathbb{E}(\operatorname{card} Z)$ and $V_Z = \mathbb{E}\big(\nu_d(\operatorname{conv} Z)\big)$. The minimum interdaughter distance is denoted by $\sigma_Z = \min\{\|y-z\|\colon y \neq z, y, z \in Z\}$.

A *Boolean cluster model* is the union set of a stationary Poisson point process on $\mathcal{Z}$ and can be represented in the form

$$X = \bigcup_{i=1}^{\infty}(x_i + Z_i),$$

where $\{x_i\colon i = 1, 2, \ldots\}$ is the realization of a stationary Poisson point process in $\mathbb{R}^d$ (germ process) and Z_i are independent copies of a random cluster Z (cf. [5], §8.3). The Boolean cluster model is a random closed set in the usual sense (cf. [10]) and its distribution is described by the intensity $\lambda > 0$ of the germ process and the distribution of the cluster Z. The spherical contact distribution function of X has the form

$$F_X(r) = \Pr\{B_r \cap X \neq \emptyset\} = 1 - \exp\big(-\lambda \mathrm{E}\nu_d(Z \oplus B_r)\big),\ r > 0 \tag{1}$$

(see [10], §3.1). Note that for an arbitrary random cluster Z, the Boolean cluster models with dispersed clusters $X_R = \bigcup_i (x_i + RZ_i)$ converge weakly as $R \to \infty$ to the stationary Poisson point process of intensity $N\lambda$ (cf. [5], §§9.1, 9.2).

3. Mean cluster dilation

The formula (1) indicates that the calculation of the mean dilation volume of the cluster Z,

$$\psi_Z(r) = \mathrm{E}\nu_d(Z \oplus B_r),$$

is crucial for obtaining the spherical contact distribution function. In this section, we shall give formulae for particular random clusters. Since general formulae would be too complicated and hardly applicable in practice, we concentrate on two particular types of clusters, namely deterministic clusters with some strong regularity and random clusters with independent elements.

3.1. Non-random clusters

Let $Z = \{z_1, \dots, z_N\}$ be any finite deterministic subset of $\mathbb{R}^d$. We have

$$\psi_Z(r) = \nu_d\Big(\bigcup_{i=1}^N B_r(z_i)\Big) = \sum_{i=1}^N \nu_d\big(B_r(z_i) \cap W_i\big), \tag{2}$$

where

$$W_i = \{x \in \mathbb{R}^d\colon \|x - z_i\| \leqslant \|x - z_j\| \text{ for all } j\}$$

is the "influence zone" of z_i. For $r \geqslant 0$, $b_1, \dots, b_N \in \mathbb{R}$ and $u_1, \dots, u_N \in \mathbb{S}^{d-1}$, denote

$$K_N^d(r; u_1, b_1, \dots, u_N, b_N) = \nu_d\big(B_r \cap \{x\colon \langle x, u_1\rangle \leqslant b_1, \dots, \langle x, u_N\rangle \leqslant b_N\}\big).$$

In this notation we have e.g.

$$\nu_d\big(B_r(z_1) \cap W_1\big) = K_{N-1}^d\big(r; u_2, b_2, \dots, u_N, b_N\big)$$

with $u_k = (z_k - z_1)/\|z_k - z_1\|$ and $b_k = \|z_k - z_1\|/2$ (analogously for $B_r(z_i) \cap W_i$).

Simplicial cluster. If Z is formed by the vertices of an N-simplex, we have $N \leqslant d + 1$ and the vectors u_i are linearly independent. We shall find a recurrence relation for this case. The section of the halfspace $\{\langle\cdot,u_k\rangle \leqslant b_k\}$ by the hyperplane $L_t = \{\langle\cdot,u_N\rangle = t\}$ $(t \in \mathbb{R})$ is the half-hyperplane

$$\{\langle\cdot,u_{k|N}\rangle \leqslant b'_k(t)\},$$

where $u_{k|N}$ is the normalized orthogonal projection of u_k onto L_t and

$$b'_k(t) = \frac{1 - t\langle u_k,u_N\rangle}{\sqrt{1-\langle u_k,u_N\rangle^2}}b_k.$$

Using the Fubini theorem, we get the relation $(1 \leqslant j \leqslant i)$

$$\begin{aligned}(3)\qquad & K^i_j(r;u_1,b_1,\,\dots,u_j,b_j)\\ &= \int_{-r}^{(b_j)^*_r} K^{i-1}_{j-1}\Big(\sqrt{r^2-t^2};u_{1|j},b'_1(t),\dots,u_{j-1|j},b'_{j-1}(t)\Big)\,\mathrm{d}t,\end{aligned}$$

where $(b_j)^*_r = \max\{-r,\min\{r,b_j\}\}$. In the case $j = 0$ we clearly have

$$K^d_0(r) = \kappa_d r^d.$$

The formula (3) can thus be used iteratively to obtain the dilation volume of a general simplex. If the simplex is regular, then all summands in (2) coincide and we have

$$(4)\qquad \psi_Z(r) = NK^d_{N-1}\Big(r;\frac{z_2-z_1}{\|z_2-z_1\|},\frac{\|z_2-z_1\|}{2},\dots,\frac{z_N-z_1}{\|z_N-z_1\|},\frac{\|z_N-z_1\|}{2}\Big).$$

Explicit formulae for dimensions 2 and 3 can be found in Appendix.

Rectangular cluster. Let Z be formed by the vertices of an m-dimensional rectangular parallelepiped of edge lengths $a_1,\dots,a_m$ in $\mathbb{R}^d$. Since rotations do not affect the dilation volume, we can assume that the edge of length a_i is parallel to the i-th canonical basis vector e_i of $\mathbb{R}^d$. Using the symmetry of Z w.r.t. hyperplanes $\{\langle\cdot,e_i\rangle = \frac{1}{2}a_i\}$, $1 \leqslant i \leqslant m$, we get

$$\begin{aligned}\psi_Z(r) &= 2^m\nu_d\big(B_r \cap \{x_1 \leqslant \tfrac{1}{2}a_1,\dots,x_m \leqslant \tfrac{1}{2}a_m\}\big)\\ (5)\qquad &= K^d_m\big(r;e_1,\tfrac{1}{2}a_1,\dots,e_m,\tfrac{1}{2}a_m\big)\end{aligned}$$

(cf. [7]). If, in particular, $a_1 = \dots = a_m = a$ (Z is a cubic cluster), then the last formula simplifies to

$$(6)\qquad \psi_Z(r) = 2^m k^d_m(r,\tfrac{1}{2}a),$$

with

$$k_m^d(r,b) = K_m^d(r; e_1, b, \ldots, e_m, b).$$

The recurrence relation (3) reads in this case

$$k_j^i(r,b) = \int_{-r}^{b_r^*} k_{j-1}^{i-1}\big(\sqrt{r^2 - t^2}, b\big)\,\mathrm{d}t. \tag{7}$$

Explicit formulae for the lower dimensions are again given in Appendix.

In the following, we shall consider random clusters with strong independence properties.

3.2. Poisson clusters

In general, a Poisson cluster Z is the support set of a (non-stationary) Poisson point process in $\mathbb{R}^d$ with finite intensity measure Λ of expectation

$$N = \mathbb{E}(\Lambda(\mathbb{R}^d))$$

(mean number of cluster points). Such a random cluster is described by the property that for any Borel subset A of $\mathbb{R}^d$, $\mathrm{card}(A \cap Z)$ is a Poissonian random variable of mean $\Lambda(A)$. The mean dilation volume is in this case

$$\begin{aligned} \psi_Z(r) &= \mathrm{E} \int 1_{Z \oplus B_r}(x)\,\mathrm{d}x \\ &= \int \Big(1 - \Pr\big\{Z \cap B_r(x) = \emptyset\big\}\Big)\,\mathrm{d}x \\ &= \int \Big(1 - \mathrm{e}^{-\Lambda(B_r(x))}\Big)\,\mathrm{d}x. \end{aligned} \tag{8}$$

We will consider two particular cases of Poisson clusters.

Poisson globular cluster. Here Λ is the $N/(\kappa_d R^d)$-multiple of the Lebesgue measure restricted to the ball B_R. The resulting Boolean cluster model is known as the *Mat rn model* (see [9], [10]). We have

$$\Lambda\big(B_r(x)\big) = \frac{N}{\kappa_d R^d}\nu_d\big(B_R \cap B_r(x)\big) \tag{9}$$

and, using the function k_1^d introduced in the preceding section, we can express

$$\Lambda\big(B_r(x)\big) = \frac{N}{\kappa_d R^d}\big(k_1^d(R, -p) + k_1^d(r, -q)\big), \tag{10}$$

where

$$p = \frac{\|x\|^2 + R^2 - r^2}{2\|x\|}, \; q = \frac{\|x\|^2 + r^2 - R^2}{2\|x\|}.$$

Poisson spherical cluster. In this model, Λ is the restriction of ν_{d-1} to the d-sphere ∂B_R, multiplied by $N/(O_{d-1}R^{d-1})$. Then,

$$\begin{aligned} \Lambda\big(B_r(x)\big) &= \frac{N}{O_{d-1}R^{d-1}}\nu_{d-1}\big(\partial B_R \cap B_r(x)\big) \\ &= \frac{NO_{d-2}}{O_{d-1}} \int\limits_{\{p/R \leqslant \sin\theta \leqslant 1\}} \cos^{d-2}\theta \, \mathrm{d}\theta. \end{aligned} \tag{11}$$

In particular, we get

$$\Lambda\big(B_r(x)\big) = \begin{cases} \frac{N}{2}\Big(\frac{\pi}{2} - \arcsin\frac{p}{R}\Big) & \text{for } d = 2, \\ \frac{N}{\pi}(1 - \frac{p}{R}) & \text{for } d = 3. \end{cases}$$

3.3. Binomial clusters

Given a natural number N and a probability distribution μ on $\mathbb{R}^d$, the binomial cluster Z is the collection of N independent random points of distribution μ. In fact, the binomial cluster is a Poisson cluster with a fixed total number of points. We have

$$\begin{aligned} \psi_Z(r) &= \int \Big(1 - \Pr\big\{Z \cap B_r(x) = \emptyset\big\}\Big)\,\mathrm{d}x \\ &= \int \Big(1 - \big(1 - \mu\big(B_r(x)\big)\big)^N\Big)\,\mathrm{d}x. \end{aligned} \tag{12}$$

We introduce again two types of binomial clusters.

Binomial globular cluster. $\mu = \Lambda$ from Eq. (10) with $N = 1$.

Binomial spherical cluster. $\mu = \Lambda$ from Eq. (11) with $N = 1$.

Five types of clusters in $\mathbb{R}^2$, $\mathbb{R}^3$ described above will be now compared numerically at selected equal values of the mean number of daughters N and equal cluster size parameter $R \in (0, 2.5/\lambda^{1/d}]$. R is the radius of the minimum circumscribed ball in the case of regular clusters and plays a similar role also in the case of Poisson (binomial) globular and spherical clusters (cf. Section 3.2). The selected upper bound for R corresponds roughly to five times the mean nearest neighbour distance of the parent point process $\Gamma(1 + 1/d)/(\lambda\kappa_d)^{1/d}$, so that the clusters can overlap extensively. In particular, the following clusters are considered: singletons ($N = 1$), point pairs ($N = 2$), regular clusters with $N = 2^d$ (the clusters formed by the vertices of a d-cube inscribed into a d-ball of radius R) in $\mathbb{R}^2$, $\mathbb{R}^3$, and regular simplicial clusters with $N = 3$ (the regular cluster formed by the vertices of an equilateral triangle inscribed into the circle of radius R) in $\mathbb{R}^2$. The various cases are denoted by the following indices marking special cases of Z: BS, BG denote the binomial spherical (circular) and globular clusters, resp., the indices PS, PG the corresponding Poisson clusters, resp.—hence PG is the Matérn cluster; RG denotes the regular clusters. Further, P is used for the Poisson point process of intensity λ (parent process) and D for the Poisson point process of intensity $N\lambda$ ("Poisson point process of daughters"). The unit intensity $\lambda = 1$ of the parent process is assumed in all calculations. The mean cluster cardinality N will be sometimes added in parentheses (e.g., $BS(3)$ is the binomial spherical random cluster with $N = 3$).

Before examining the mean volumes of dilations and distributions of spherical contacts, the mean cluster sizes $V_Z = \mathbb{E}\big(\nu_d(\operatorname{conv} Z)\big)$ will be compared.

4.1. Cluster sizes

The expected volume for binomial clusters, namely for N independent randomly distributed points either in the interior of a unit ball – $V_{BG(N)}$ – or on its boundary – $V_{BS(N)}$ – have been calculated by Miles [6], Buchta [3] and Affentranger [1] (in their notation, $V^d_{N,0}$, $V^d_{0,N}$ stand for $V_{BG(N)}$, $V_{BS(N)}$, respectively). The expected volumes of the convex hulls of the corresponding Poisson clusters are

$$V_{PG(N)} = \sum_{i=d+1}^{\infty} V_{BG(i)} \Pr\{n = i\}, \quad V_{PS(N)} = \sum_{i=d+1}^{\infty} V_{BS(i)} \Pr\{n = i\}, \tag{13}$$

where n is a random Poissonian variable of mean N, i.e. $\Pr\{n = i\} = \mathrm{e}^{-N} N^i/i!$. The corresponding estimates of the expected values of V for all clusters considered are given in Table 1 (some higher values of N not considered here are included in order to show that the limit values π, 2π and, in particular, $4\pi/3$, 4π are approached rather slowly). The comparison of the values is partly obscured by the presence of binomial clusters of zero volumes $V_{BS(N)}$, $V_{BG(N)}$ ($N = 1, 2$ for $d = 2$ and $N = 1, 2, 3$ for $d = 3$). Consequently, the Poisson clusters are greater than the binomial ones in all these cases as they always include also clusters of non-zero Lebesgue measure. Beginning with $N = 4$ (6 in $\mathbb{R}^3$), the binomial clusters are always greater than the Poisson ones. It can be simply shown that the spherical dilations of regular clusters considered are always the greatest ones even when $V(BG) = 0$.

	N	$V_{PG(N)}$	$V_{BG(N)}$	$V_{PS(N)}$	$V_{BS(N)}$	$V_{RG(N)}$
	1	0.024	0	0.049	0	0
	2	0.123	0(0.905)	0.250	0(1.273)	0(2.000)
$d = 2$	3	0.274	0.232	0.554	0.478	1.299*
	4	0.446	0.464	0.889	0.955	2.000*
	5	0.617	0.666	1.210	1.350	2.378*
	10	1.259	1.317	2.252	2.376	2.939*
	1	0.001	0	0.003	0	0
	2	0.014	0(1.029)	0.031	0(1.333)	0(2.000)
$d = 3$	3	0.045	0(0.367)	0.102	0(0.628)	–
	4	0.096	0.053	0.215	0.120	0.513**
	8	0.397	0.403	0.856	0.889	1.540†
	10	0.555	0.573	1.173	1.230	–

* regular polygon, ** regular tetrahedron, † cube

Table 1: Volumes of convex hulls of binomial, Poisson and regular clusters with $R = 1$ for selected mean numbers of daughters N. The $(N-1)$-dimensional measures of $\operatorname{conv} Z$ for binomial clusters with $N = 1, 2$ and 3 are given in parentheses.

4.2. Mean volumes of dilation

The convex hull of a point cluster can be roughly considered as a convex germ of the dilation $\psi_Z(r) \approx \psi_{\operatorname{conv} Z}(r)$ for sufficiently high r (i.e. exceeding one half of the diameter of Z). Consequently, on the basis of Table 1, the following sequence of inequalities can be expected for clusters of the same parameter N and sufficiently high r:

$$\begin{aligned} \psi_D(r) \equiv N\psi_P(r) &\geqslant \psi_{RG}(r) \geqslant \psi_{BS}(r) \\ &\geqslant \Psi_{PS}(r) \geqslant \psi_{BG}(r) \geqslant \psi_{PG}(r) \geqslant \psi_P(r). \end{aligned} \tag{14}$$

Here $\psi_P(r) = \kappa_d r^d$ and the remaining functions are calculated using the formulae (2), (5), (8) and (12). Figure 1 demonstrates this behaviour; the seemingly smooth passage from $\psi_D(r)$ to $\psi_P(r)$ is disturbed only in the case of regular clusters which follow the ψ_D-curve up to one half of the minimum interdaughter distance $r = \sigma_{RG}/2$ and only then start to grow more slowly ($\sigma_{RG} = 2R$ in the case of pairs, $2R/\sqrt{d}$

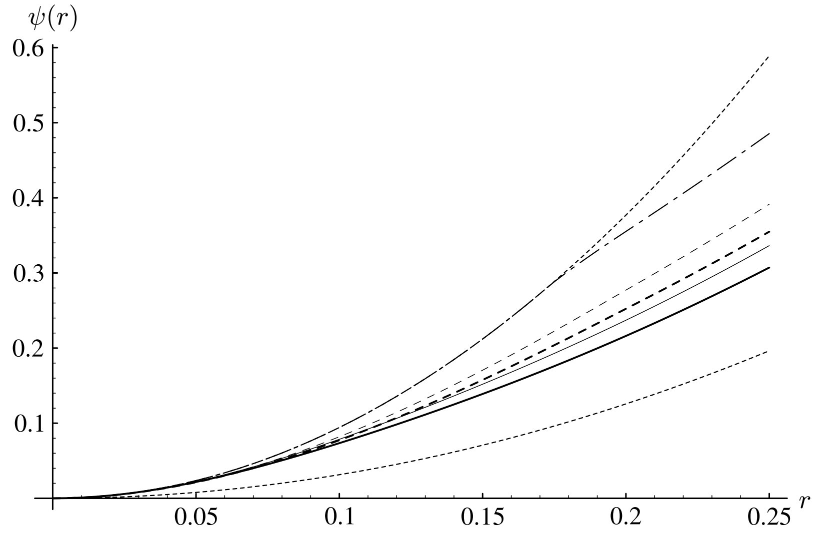

Figure 1. The mean spherical dilation $\psi_Z(r)$ for planar Boolean cluster models with clusters of $N = 3$ points in mean and size $R = 0.2$ (the regular cluster is formed by the vertices of the equilateral triangle). The upper (D) and lower (P) dotted envelopes correspond to the daughter and parent Poisson processes, dash-dotted line to regular clusters, dashed lines to circular clusters and full lines to globular clusters; thick lines describe Poisson clusters, thin lines the binomial ones.

in the case of 2^d-clusters and $R\sqrt{2(1+d^{-1})}$ for regular simplices). The detailed behaviour of $\psi_Z(r)$ at small values of r is seen best by considering the derivative $g_Z(r) = \mathrm{d}\psi_Z(r)/\,\mathrm{d}(r^d)$, which for the cases P and D takes on the constant values κ_d and $N\kappa_d$, respectively – Figure 2. Note that this derivative equals, up to a constant factor, the hazard rate introduced recently by Baddeley and Gill [2] into distance methods by analogy with the survival analysis.

The differences between the shapes of $\psi(r)$ for the three basic cases, namely regular, globular and spherical clusters, as well as the range of substantial changes in the passage from $\psi_D(r)$ to $\psi_P(r)$ are clearly revealed in Figure 2. At the chosen value of R, the inflexion points of $g_Z(r)$ correspond to the maxima or inflexion points of the density function $f_Z(r) = \mathrm{d}F_Z(r)/\,\mathrm{d}r$ of the corresponding Boolean cluster models visible in Figure 3 at $r = \sigma_{RG}/2$ or slightly below it. These features are much less perceptible at other values of R. At small N (especially at $N = 1$) or at high R, the inequality (14) does not hold any more because of the effects considered below in connection with the spherical contact distribution function and its moments. A more detailed discussion concerning the behaviour of $\psi_Z(r)$ and $g_Z(r)$ will be given in [8].

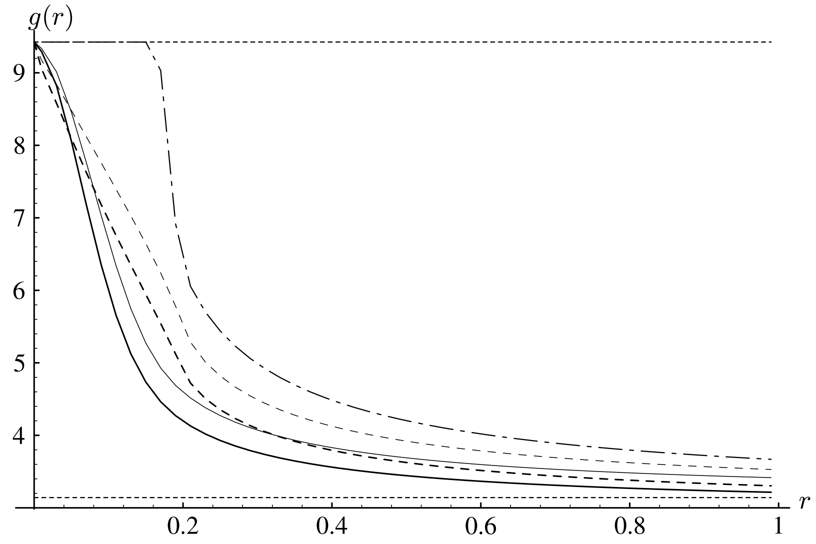

Figure 2. The derivative $g_Z(r) = \mathrm{d}\psi_Z(r)/\,\mathrm{d}(r^d)$ for the same case as in Figure 1.

4.3. Spherical contact distribution function and its moments

We consider now the spherical contact distribution function (1) of the Boolean cluster model corresponding to the given random cluster Z and the Poisson parent process of unit intensity. The distribution function will be denoted by $F_Z(r)$ and its probability density function by $f_Z(r)$.

An example of the shape and order of $F_Z(r)$ and $f_Z(r)$ is given in Figure 3 (the sequence of inequalities (14) implies an analogous sequence for $F_Z(r)$ everywhere and for $f_Z(r)$ in a certain limited range of r). The basic frame of the picture are the envelopes $F_{D(N)}(r)$, $F_P(r)$. The equality $F_{D(N)}(r) = F_{RG(N)}(r)$ holds always for $r \leqslant \sigma_{RG}/2$ and is the most typical feature of regular clusters. Further, the inequalities

$$F_{D(N)}(r) \geqslant F_{RG(N)}(r) \geqslant F_{BS(N)}(r) \geqslant F_{BG(N)}(r) \geqslant F_P(r)$$

also nearly always hold (the exception is the behaviour of BS clusters of great size, see below). The distribution functions of regular and binomial clusters approach $F_P(r)$ at $R \to 0$, they shift without crossing toward $F_{D(N)}(r)$ with growing R and approach $F_{D(N)}(r)$ at $R \to \infty$. These rules also explain the corresponding values of the general moments

$$\mu_Z^{(k)} = \int_0^\infty r^k \,\mathrm{d}F_Z(r).$$

The first moments ($k = 1$) are given in Table 2. Its inspection shows that the values of moments intermediate between those of the parent ($\mu_P^{(1)} = \kappa_d^{-1/d}\Gamma(1 + d^{-1})$) and

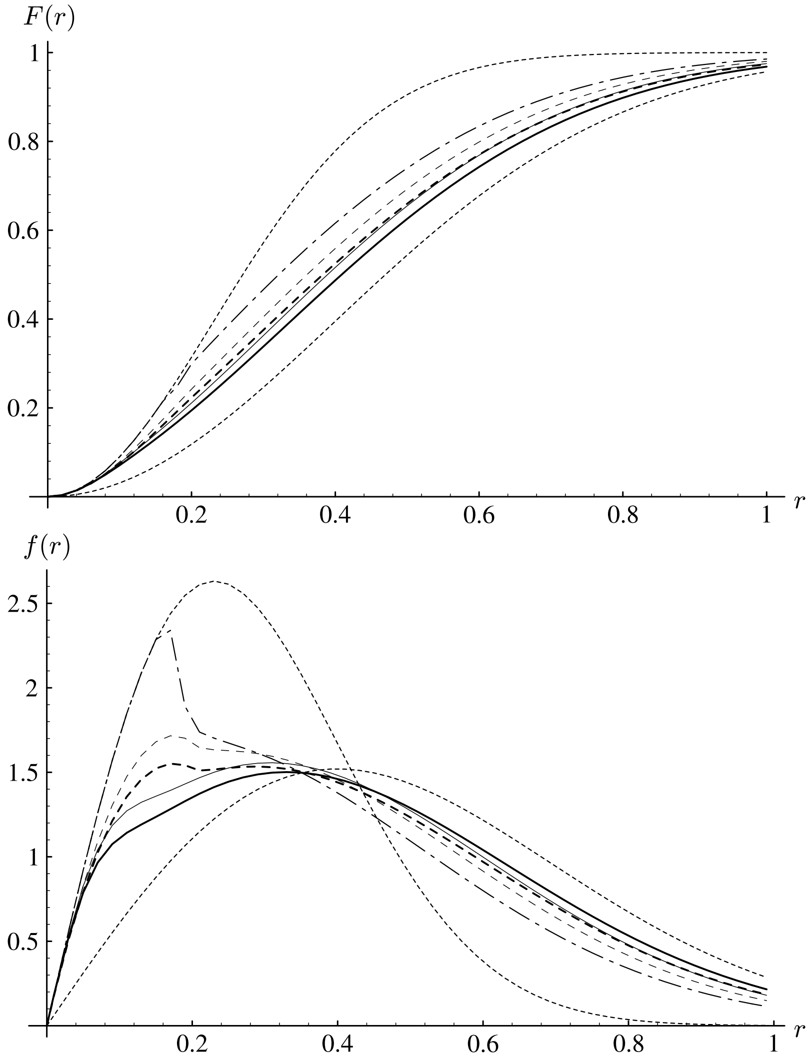

Figure 3. The spherical contact distribution function $F_Z(r)$ and the corresponding p.d.f. $f_Z(r)$ for the same case as in Figure 1.

the daughter ($\mu^{(1)}_{D(N)} = (N\kappa_d)^{-1/d}\Gamma(1+d^{-1})$) Poisson point processes correspond to the cluster sizes $0.1 \leqslant R \leqslant 0.5$. The values of $\mu^{(2)}_Z$ behave similarly.

The behaviour of Poisson clusters is more complex and its understanding requires a more detailed consideration. The probability of a void cluster $\Pr\{n = 0\} = \mathrm{e}^{-N}$ is appreciable for small N. Therefore, when $R \to 0$, the Poisson cluster processes approach the Poisson process of "multiple" points of reduced intensity $1-\mathrm{e}^{-N}$, which is as low as 0.632, 0.856 and 0.950 at $N = 1, 2, 3$, respectively. Consequently, the

Singletons $N = 1$

R				P	PG	BG	PS	BS	RG	D
.00001		**.627124**	*.627124*							
.0001		**.6271**	*.6271*							
.01		**.6257**	*.6256*							
.1		**.6133**	*.6079*							
.2		**.6002**	*.5904*							
.5		**.5674**	*.5517*	.5		.5		.5	.5	.5
1.0		**.5344**	*.5239*							
1.5		**.5194**	*.5153*							
2.0		**.5119**	*.5112*							
2.5	*.5089*	**.5080**								
5.0	*.5014*	**.5003**								

Pairs $N = 2$ in the plane

R			P		PG		BG	PS	BS	RG	D
.01	**.5348**	*.5337*					.4970		.4959	.4936	
.1	**.5112**	*.5013*					.4728		.4627	.4441	
.5			.5		**.4313**	*.4095*	.4015		.3850	.3589	.3536
1.0					**.3860**	*.3777*	.3706		.3658	.3536	
1.5					**.3696**	*.3688*	.3616		.3613	.3536	
2.5				*.3625*	**.3596**	.3580	.3566			.3536	

Pairs $N = 2$ in the space

R			P		PG		BG	PS	BS	RG	D
.01	**.5789**	*.5782*					.5514		.5506	.5490	
.1	**.5578**			*.5512*			.5298		.5233	.5098	
.5			.5540		**.4872**	*.4722*	.4685		.4579	.4423	.4397
1.0					**.4533**	*.4480*	.4467		.4439	.4397	
1.5					**.4446**	*.4433*	.4422		.4415	.4397	
2.5				*.4410*	**.4409**	.44032	.44029			.4937	

Table 2. The first moment $\mu_Z^{(1)}$ for various Boolean cluster models of different parameters $0 < R < 2.5$ ($\lambda = 1$). The columns of the table are arranged in the sequence P, PG, BG, PS, BS, RG, D, following the inequality (14). The values of moments in rows are arranged in the decreasing setup from left to right. Consequently, if some moment value does not conform with the prescribed order of columns, it is put into a frame and shifted (the path is usually shown and marked by an arrow) to the correct place in the row. The values travelling most of all are denoted by different fonts: $\mu_{PG}^{(1)}$ are **bold**, $\mu_{PS}^{(1)}$ are *italics*.

Regular triads $N = 3$ in the plane

R			P			PG		BG	PS	BS	RG	D
.0001	**.5128**	*.5128*						.5		.5	.5	
.01	**.5090**	*.5076*						.4956		.4939	.4917	
.1						**.4760**	*.4628*	.4590		.4439	.4241	
.5			.5			**.3708**		.3526	*.3465*	.3303	.2962	.2887
1.0						**.3189**	*.3130*	.3095		.3051	.2887	
1.5					*.3041*	**.3028**	.2991	.2980			.2887	
2.5				*.2976*	.2947	**.2938**		.2921			.2887	

Regular quadruples (square vertices) $N = 4$ in the plane

R			P			PG		BG	PS	BS	RG	D
.0001	**.5045**	*.5045*						.5		.5	.5	
0.01						**.4999**	*.4982*	.4947		.4927	.4910	
0.1						**.4595**		.4505	*.4439*	.4328	.4155	
0.5			.5			**.3346**		.3219	*.3093*	.2974	.2607	.25
1.0						**.2785**	*.2747*	.2718		.2687	.2500	
1.5					*.2657*	**.2628**	.2618	.2596			.2501	
2.5				*.2590*	.2568	**.2546**		.2534			.2500	

Regular octuples (cube vertices) $N = 8$ in the space

R		P			PG		BG	PS	BS	RG	D
.0001	**.55393**					*.55391*	.55387		.55385	.55384	
.01					**.5480**		.5477	*.5465*	.5462	.5453	
.1					**.4964**		.4944	*.4825*	.4799	.4708	
.5		.5540			**.3401**		.3360	*.3169*	.3132	.2888	.277
1.0					**.2888**		.2874	*.2856*	.2845	.2770	
1.5					**.2807**	*.2806*	.2802		.2802	.2770	
2.5			*.2783*	.2781	**.2778**		.2777			.2770	

(The wandering of Poisson clusters at low values of R is caused by the presence of "void clusters" and its range decreases with growing N. The movement of spherical clusters at high values of R is caused by their slower convergence to the daughter Poisson process and is more important at higher N.)

inequalities

$$F_{PG(N)}(r) \leqslant F_{PS(N)}(r) \leqslant F_P(r), \quad \mu_P^{(k)} \leqslant \mu_{PS(N)}^{(k)} \leqslant \mu_{PG(N)}^{(k)}$$

hold in all examined cases for sufficiently small R and the reversed order of distribution functions and moments is the more pronounced the smaller is N – compare Table 2. In fact, in the case of clusters with $N = 1$, the moments of Poisson cluster processes are higher than $\mu_P^{(k)}$ for arbitrary R and, for $N = 2$, the above inequalities hold even at $R = 0.1$ in the planar case.

The second effect is connected with great size clusters and is less pronounced. It seems that the convergence of spherical clusters to the daughter process D is slightly slower than that of globular clusters; therefore especially the PS clusters differ considerably from the process P. The suspicion that this effect might be caused by numerical inaccuracies has not been confirmed. On the other hand, the random arrangement of points tied firmly to the sphere is not entirely dissolved unless R is very high and some manifestation of it can be reasonably expected.

In order to summarize the results of Table 2, we can evaluate the relative increase due to clustering of given type Z in the moments of the spherical distances, namely (neglecting the effect of void clusters occurring at $R \to 0$ for Poisson clusters and important at small N only)

$$\Delta_Z^{(k)} = \frac{\mu_Z^{(k)}}{\mu_D^{(k)}} - 1 \leqslant N^{k/d} - 1 = \Delta_P^{(k)}. \tag{15}$$

Hence, the effect of clustering decreases with growing dimension if N is constant (compare the results for pairs in $\mathbb{R}^2$ and $\mathbb{R}^3$) and is independent of dimension for 2^d-clusters. Considering the effect of R, the cluster processes are mutually indistinguishable at very small R and the equality in (15) approximately applies. At medium values $(0.1 \leqslant R \leqslant 0.5)$, the possibility to resolve various processes is greater and can be roughly evaluated by the difference $\Delta_{PG}^{(1)} - \Delta_{RG}^{(1)}$. It equals roughly $0.5\Delta_P^{(1)}$ for pairs and lies between $0.1\Delta_P^{(1)}$ and $0.3\Delta_P^{(1)}$ for 2^d-clusters in $\mathbb{R}^2$ and $\mathbb{R}^3$. At the values of $R \geqslant 1$, the above difference quickly decreases, in particular for $d > 2$.

5. Appendix

We shall give explicit formulae for the dilation volume of particular regular clusters in dimensions 2 and 3.

Regular simplex. Let u, v be two unit vectors in $\mathbb{R}^2$ of mutual angle $\pi/3$. If Z is formed by the vertices of an equilateral triangle of edge length a, then by (4),

$$\psi_Z(r) = 3K_2^2\Big(r; u, \frac{a}{2}, v, \frac{a}{2}\Big) = \frac{3a^2}{4}K_2^2\Big(\frac{2r}{a}; u, 1, v, 1\Big).$$

For its calculation we shall use simple geometrical considerations, rather than the recurrence relation (3). The set

$$\{x \in \mathbb{R}^2 \colon \|x\| \leqslant r, \langle x, u\rangle \leqslant 1, \langle x, v\rangle \leqslant 1\}$$

can be partitioned into a finite number of sectors and rectangular triangles and its area is

$$K_2^2(r; u, 1, v, 1) = \begin{cases} \pi r^2, & r \leqslant 1, \\ (\pi - 2\arccos\frac{1}{r})r^2 + 2\sqrt{r^2-1}, & 1 \leqslant r \leqslant \frac{2}{\sqrt{3}}, \\ (\frac{5\pi}{6} - \arccos\frac{1}{r})r^2 + \sqrt{r^2-1} + \frac{1}{\sqrt{3}}, & r \geqslant \frac{2}{\sqrt{3}}. \end{cases} \tag{16}$$

For the three-dimensional case we obtain by integration

$$K_2^3(r; u, 1, v, 1) = \begin{cases} \frac{4\pi}{3} r^3, & r \leqslant 1, \\ 2\pi(r^2 - \frac{1}{3}), & 1 \leqslant r \leqslant \frac{2}{\sqrt{3}}, \\ \frac{2r^3}{3} \arccos\frac{r}{2\sqrt{r^2-1}} + \frac{2\sqrt{3}-2}{9}\sqrt{r^2 - \frac{4}{3}} & \\ \qquad + \big(\pi - \arccos\frac{1}{\sqrt{3}\sqrt{r^2-1}}\big)(r^2 - \frac{1}{3}), & r \geqslant \frac{2}{\sqrt{3}}. \end{cases} \tag{17}$$

Cubic cluster. Using the formula (7), the following expressions for the function $k_j^i(r, 1)$ can be obtained.

$$k_1^1(r, 1) = \begin{cases} 2r, r \leqslant 1 \\ r + 1, r \geqslant 1 \end{cases} \tag{18}$$

$$k_1^2(r, 1) = \begin{cases} \pi r^2, r \leqslant 1 \\ (\pi - \arccos\frac{1}{r})r^2 + \sqrt{r^2-1}, r \geqslant 1 \end{cases} \tag{19}$$

$$k_2^2(r, 1) = \begin{cases} \pi r^2, r \leqslant 1 \\ (\pi - 2\arccos\frac{1}{r})r^2 + 2\sqrt{r^2-1}, 1 \leqslant r \leqslant \sqrt{2} \\ (\frac{3\pi}{4} - \arccos\frac{1}{r})r^2 + \sqrt{r^2-1} + 1, r \geqslant \sqrt{2} \end{cases} \tag{20}$$

$$k_1^3(r, 1) = \begin{cases} \frac{4\pi}{3} r^3, r \leqslant 1 \\ \pi\big(\frac{2r^3}{3} + r^2 - \frac{1}{3}\big), r \geqslant 1 \end{cases} \tag{21}$$

$$k_2^3(r, 1) = \begin{cases} \frac{4\pi}{3} r^3, r \leqslant 1 \\ 2\pi\big(r^2 - \frac{1}{3}\big), 1 \leqslant r \leqslant \sqrt{2} \\ \frac{2r^3}{3} \arccos\frac{1}{r^2-1} + \frac{2}{3}\sqrt{r^2-2} \\ \qquad + (r^2 - \frac{1}{3})\big(\pi + 2\arcsin\frac{1}{\sqrt{r^2-1}}\big), r \geqslant \sqrt{2} \end{cases} \tag{22}$$

$$
k_3^3(r,1) = \begin{cases} \frac{4\pi}{3} r^3, r \leqslant 1 \\ \pi\big(-\frac{2r^3}{3} + 3r^2 - 1\big), 1 \leqslant r \leqslant \sqrt{2} \\ \frac{4r^3}{3} \arcsin \frac{r(r^2-3)}{\sqrt{2}(r^2-1)^{3/2}} + 2\sqrt{r^2-2} \\ \qquad +(6r^2-2) \arcsin \frac{1}{\sqrt{r^2-1}}), \sqrt{2} \leqslant r \leqslant \sqrt{3} \\ \frac{4r^3}{3} \arcsin \frac{r(r^2-3)}{\sqrt{2}(r^2-1)^{3/2}} + \sqrt{r^2-2} + 1 \\ \qquad +(3r^2-1)\big(\frac{\pi}{4} + \arcsin \frac{1}{\sqrt{r^2-1}}\big), r \geqslant \sqrt{3}. \end{cases} \tag{23}
$$

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Author's address: *Jan Rataj*, Mathematical Institute of the Charles University, Sokolovská 83, 180 00 Praha 8, Czech Republic; *Ivan Saxl*, Mathematical Institute of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic.