REES IDEAL ALGEBRAS

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Abstract. We describe algebras and varieties for which every ideal is a kernel of a one-block congruence.

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The concept of Rees congruences was introduced for semigroups by D. Rees [3]. R. F. Tichy [5] generalized this concept to universal algebras. The author with J. Duda described Rees algebras in [1] and, moreover, gave a characterization of varieties all of whose members are Rees algebras. Some particular results for lattices can be found in [2] and [4]. Our aim is to study the Rees congruences induced in algebras by ideals in the sense of A. Ursini [6]. We will describe such ideals and characterize varieties of algebras having Rees ideal congruences.

1. Preliminaries

For an algebra $\mathcal{A} = (A, F)$ we denote by $Con\ \mathcal{A}$ the lattice of congruences of $\mathcal{A}$. By ω_A we denote the least congruence on $\mathcal{A}$, i.e. ω_A is the identity relation alias the diagonal of A. Further, we denote by ι_A the greatest congruence on $\mathcal{A}$, i.e. $\iota_A = A \times A$. We call $\Theta \in Con\ A$ a *one-block congruence* if the partition of A induced by Θ contains at most one non singleton congruence class. Trivially, ω_A and ι_A are one-block congruences.

Lemma 1. *Let $\Theta, \Phi \in Con\ \mathcal{A}$ be one-block congruences. Then Θ, Φ are 3-permutable, i.e. $\Theta \vee \Phi = \Theta \circ \Phi \circ \Theta = \Phi \circ \Theta \circ \Phi$.*

The proof is elementary. □

R e m a r k 1. It is obvious that the join of two one-block congruences need not be a one-block congruence.

Definition 1. Let B be a subalgebra of an algebra $\mathcal{A} = (A, F)$. B is called a *Rees subalgebra* whenever $B^2 \cup \omega_A \in Con\ \mathcal{A}$. Any congruence of the form $B^2 \cup \omega_A$ for some subalgebra B of $\mathcal{A}$ is called a *Rees congruence.* An algebra $\mathcal{A}$ is a *Rees algebra* if its every subalgebra is a Rees one.

Hence, every Rees congruence is a one-block congruence and therefore, every two Rees congruences on an algebra $\mathcal{A}$ are 3-permutable.

The concept of an ideal was generalized by A. Ursini [6] for algebras with 0. In what follows, let $\mathcal{C}$ be a class of algebras of a fixed similarity type τ. For $\mathcal{A} \in \mathcal{C}$, the set of all fundamental operations of $\mathcal{A}$ will be denoted by F. We require that all algebras of $\mathcal{C}$ have a constant 0 which is either a nullary operation of F or at least equationally defined. For $\mathcal{A} \in \mathcal{C}$, this constant will be denoted by 0_A.

An $(n+m)$-ary term $p(x_1, \dots, x_n, y_1, \dots, y_m)$ of type τ is called an *ideal term* in $y_1, \dots, y_m$ if

$$p(x_1, \dots, x_n, 0, \dots, 0) = 0$$

is an identity in $\mathcal{C}$. For $\mathcal{A} = (A, F) \in \mathcal{C}$, a non-void subset I of A is called an *ideal* of $\mathcal{A}$ if for every ideal term $p(x_1, \dots, x_n, y_1, \dots, y_m)$ in $y_1, \dots, y_m$ and all elements $a_1, \dots, a_n$ of A and $b_1, \dots, b_m$ of I we have

$$p(a_1, \dots, a_n, b_1, \dots, b_m) \in I;$$

is such a case, we say that I is *closed under the ideal term p.* In other words, a non-void subset of A is an ideal of $\mathcal{A}$ if it is closed under every ideal term.

It is worth mentioning that for rings and for lattices with 0 this concept coincides with common concept of an ideal in these algebras. For groups, it coincides with the concept of the normal subgroup.

For an algebra $\mathcal{A} \in \mathcal{C}$, we denote by Id $\mathcal{A}$ the set of all ideals of $\mathcal{A}$. Evidently, $\{0_A\}$ and the whole algebra $\mathcal{A}$ are ideals of $\mathcal{A}$. It is easy to show that Id $\mathcal{A}$ is a complete lattice with respect to set inclusion where meet coincides with set intersection.

Further, denote by $\mathcal{IT}(\mathcal{A})$ the set of all ideal terms of $\mathcal{A} \in \mathcal{C}$. It can be shown that $\mathcal{IT}(\mathcal{A})$ is a clone and, moreover, either $\mathcal{IT}(\mathcal{A})$ consists only of 0_A and all the projections or $\mathcal{IT}(\mathcal{A})$ is infinite. We say that $\mathcal{A}$ has a *finite basis of ideal terms* if there exists a finite subset of $\mathcal{IT}(\mathcal{A})$ generating the clone $\mathcal{IT}(\mathcal{A})$. It is well-known that groups, rings or lattices with 0 have finite bases of ideal terms.

For any $\Theta \in Con\ \mathcal{A}$, the congruence kernel $[0]_\Theta$ is an ideal of $\mathcal{A}$. On the other hand, there can exist ideals of $\mathcal{A}$ which are not congruence kernels.

An algebra $\mathcal{A} = (A, F)$ is said to have a *finite type* if F is a finite set.

Definition 2 Let $\mathcal{C}$ be a class of algebras with 0. An ideal I of an algebra $\mathcal{A} \in \mathcal{C}$ is called a *Rees ideal* if $I^2 \cup \omega_A \in Con\ \mathcal{A}$; any congruence of this form is called a *Rees ideal congruence* (*induced by* I). An algebra $\mathcal{A}$ is a *Rees ideal algebra* if every ideal of $\mathcal{A}$ is a Rees ideal. A class $\mathcal{C}$ is a *Rees ideal class* if each $\mathcal{A} \in \mathcal{C}$ is a Rees ideal algebra.

Evidently, for any $\mathcal{A} \in \mathcal{C}$, $\{0_A\}$ and A are Rees ideals of $\mathcal{A}$ and ω_A, ι_A are Rees ideal congruences.

Rees congruences were intensively studied on lattices, see [2], [4]. These results are summarized by J. Duda (see [2], Theorem 3):

Proposition. *Let $\mathcal{C}$ be a class of lattices with 0. Then $\mathcal{C}$ is a Rees ideal class if and only if $\mathcal{C}$ is a class of chains.*

Example 1. Consider the commutative groupoid $\mathcal{G} = (\{0, a, b, c\}, \cdot)$ given as follows:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	b	a	a
b	0	a	a	b
c	0	a	b	c

Evidently, the subset $\{0, a, b\}$ is a congruence kernel, thus $\{0\}$, $\{0, a, b\}$, $\{0, a, b, c\}$ are ideals of $\mathcal{G}$. It is an easy exercise to check that $\mathcal{G}$ has no other ideals. Evidently, each of these ideals is a Rees one, i.e. $\mathcal{G}$ is a Rees ideal algebra.

For an algebra $\mathcal{A}$, denote by $Con_R\mathcal{A}$ the set of all Rees ideal congruences of $\mathcal{A}$.

We are able to characterize Rees ideal algebras by two-generated ideals as follows:

Lemma 2. *Let $\mathcal{A}$ be an algebra with* 0. *The following conditions are equivalent:*

(1) *$\mathcal{A}$ is a Rees ideal algebra;*

(2) *every ideal of $\mathcal{A}$ generated by two elements is a Rees ideal;*

(3) *for every unary polynomial p over $\mathcal{A}$ and for any elements a, b, of $\mathcal{A}$ we have either*

(i) *$p(a) = p(b)$, or*

(ii) *there exist ideal terms $q(x_1, \ldots, x_n, y_1, y_2)$, $r(x_1, \ldots, x_n, y_1, y_2)$ in y_1, y_2 such that $p(a) = q(c_1, \ldots, c_n, a, b)$, $p(b) = r(c_1, \ldots, c_n, a, b)$ for some elements $c_1, \ldots, c_n$ of $\mathcal{A}$.*

Proof. (1) $\Rightarrow$ (2) is trivial. Prove (2) $\Rightarrow$ (3): Let a, b be elements of $\mathcal{A}$ and p a unary polynomial over $\mathcal{A}$. Consider an ideal I of $\mathcal{A}$ generated by the set $\{a, b\}$. By (2), I is a Rees ideal, i.e. $\Theta_I = I^2 \cup \omega_A \in Con\ \mathcal{A}$. Moreover, $a, b \in I$ implies

$\langle a,b\rangle \in \Theta_I$. Hence also $\langle p(a),p(b)\rangle \in \Theta_I$, i.e. either $p(a)=p(b)$ or $p(a),p(b)\in I$, i.e. there exist ideal terms q,r as desired in (3), see [6] for some details.

(3) $\Rightarrow$ (1): Let I be an ideal of $\mathcal{A}$. Evidently, $\Theta_I = I^2 \cup \omega_A$ is an equivalence on $\mathcal{A}$. To prove $\Theta_I \in Con\ \mathcal{A}$ we need only to prove the substitution property of Θ_I. Since Θ_I is reflexive and transitive, it remains only to show the substitution property with respect to unary polynomials over $\mathcal{A}$. Let $\langle a,b\rangle \in \Theta_I$ and let p be a unary polynomial *over* $\mathcal{A}$. By (3), either $p(a)=p(b)$ or $p(a),p(b)\in I$, i.e. $\langle p(a),p(b)\rangle \in \Theta_I = I^2\cup\omega_A$, which completes the proof. $\square$

Lemma 3. *Every homomorphic image of a Rees ideal algebra is a Rees ideal algebra.*

P r o o f. Let $\mathcal{A}$ be a Rees ideal algebra and let $\mathcal{B}=h(\mathcal{A})$ for some homomorphism h of $\mathcal{A}$. Let I be an ideal of $\mathcal{B}$. Let $J=h^{-1}(I)$. It is a routine to show that J is an ideal of $\mathcal{A}$, i.e. $J^2\cup\omega_A \in Con\ \mathcal{A}$. Since $I^2\cup\omega_B$ is an equivalence on $\mathcal{B}$, it remains only to prove the substitution property of $I^2\cup\omega_B$ with respect to unary polynomials over $\mathcal{B}$. Let p be a unary polynomial over $\mathcal{B}$. Then $p(x)=t(x,b_1,\dots,b_n)$ for some term function t over $\mathcal{B}$ and elements $b_1,\dots,b_n$ of $\mathcal{B}$. Suppose $\langle a,b\rangle\in I^2\cup\omega_B$. The case $a=b$ is trivial. Let $a\neq b$. Then $a,b\in I$, i.e. there are $a',b'\in A$ with $h(a')=a$, $h(b')=b$. Hence $a',b'\in J$, thus $\langle a',b'\rangle \in J^2\cup\omega_A$ and, by the assumption, also $\langle t(a',c_1,\dots,c_n),t(b',c_1,\dots,c_n)\rangle \in J^2\cup\omega_A$ for $c_i\in h^{-1}(b_i)$, $i=1,\dots n$. Since h is a homomorphism, we conclude $p(a)=p(b)$ or $p(a),p(b)\in I$. $\square$

R e m a r k 2. A class $\mathcal{C}$ of Rees ideal algebras of the same type need not be closed under direct products as one may check using Proposition 2. Moreover, $\mathcal{C}$ need not be closed under subalgebras as the following example shows.

E x a m p l e 2. Let $\mathcal{A}=(A,\cdot)$, where $A=\{0,a,b,c,d\}$ and the binary operation $\cdot$ is defined as follows:

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	d
b	0	a	a	c	d
c	0	a	b	c	d
d	0	a	b	c	d

Evidently, $\{0\}$ and A are the only ideals of $\mathcal{A}$, i.e. $\mathcal{A}$ is a Rees ideal algebra. Further, $\mathcal{B}=(\{0,a,b,c\},\cdot)$ is a subalgebra of $\mathcal{A}$ having an ideal $I=\{0,a\}$. However, $I^2\cup\omega_B\notin Con\ \mathcal{B}$, i.e. $\mathcal{B}$ is not a Rees ideal algebra.

3. Rees ideal varieties

Varieties of Rees algebras were characterized in [1]. It was proved that $\mathcal{V}$ is a variety of Rees algebras if and only if $\mathcal{V}$ is at most unary. We are going to establish a characterization of varieties of Rees ideal algebras showing that these varieties have not restricted their similarity types.

Theorem. *For a variety $\mathcal{V}$ with* 0*, the following conditions are equivalent:*

(1) *$\mathcal{V}$ is a Rees ideal variety;*

(2) *for any integer $n \geqslant 1$ and any n-ary term t and each $i \in \{1, \dots, n\}$ either t does not depend on the i-th variable or*

$$t(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0$$

is an identity of $\mathcal{V}$.

Proof. (1) $\Rightarrow$ (2) Let t be an n-ary term of $\mathcal{V}$ and $\mathcal{A} = F_{\mathcal{V}}(x_1, \dots, x_n, y)$ a free algebra of $\mathcal{V}$. By (3) of Lemma 2, either

$$(*) \qquad t(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = t(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n)$$

or there exists an ideal term q in the last two variables such that

$$v(x_i) = t(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = q(a_1, \dots, a_k, x_i, y)$$

for some $a_1, \dots, a_k \in A$. Since $v(x_i)$ does not depend on y, this implies also

$$v(x_i) = q(a_1, \dots, a_k, x_i, x_i).$$

In the case of $(*)$, t does not depend on the *i-th* variable. The latter case gives

$$t(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = q(a_1, \dots, a_k, 0, 0) = 0.$$

(2) $\Rightarrow$ (1): Let $\mathcal{A} \in \mathcal{V}$ and let I be an ideal of $\mathcal{A}$. Set $\Theta_I = I^2 \cup \omega_A$. Since Θ_I is an equivalence on A, it remains to show the substitution property with respect to unary polynomials over $\mathcal{A}$. Suppose $\langle a, b\rangle \in \Theta_I$ and p is a unary polynomial over $\mathcal{A}$. If $a = b$ then $p(a) = p(b)$. If $a \neq b$ then $a, b \in I$. By (2), p is either constant, i.e. $p(a) = p(b)$, or $p(x) = t(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$ for some n-ary term function t over $\mathcal{A}$ and for some elements $a_1, \dots, a_n$ of $\mathcal{A}$. By (2), t is an ideal term in the i-th variable, so also $p(0) = 0$. Hence $a, b \in I$ implies $p(a), p(b) \in I$. In all cases we conclude $\langle p(a), p(b)\rangle \in \Theta_I$ proving $I^2 \cup \omega_A = \Theta_I \in Con\ \mathcal{A}$. $\square$

E x a m p l e 3. (a) The variety of all $\wedge$-semilattices with 0 is a Rees ideal variety. (b) More generally, any variety of groupoids with 0 satisfying the identities $x \cdot 0 = 0 = 0 \cdot x$ is a Rees ideal variety. (c) Every variety of at most unary algebras with 0 satisfying $f(0) = 0$ for any unary fundamental operation f is a Rees ideal variety.

Corollary. *Let $\mathcal{V}$ be a Rees ideal variety of a finite similarity type. Then $\mathcal{V}$ has a finite basis of ideal terms.*

P r o o f. By Theorem, every n-ary term either is an ideal term in the i-th variable or it does not depend on the i-th variable. Hence, for $\mathcal{A} = (A, F) \in \mathcal{V}$ and $\emptyset \neq I \subseteq A$, I is an ideal of $\mathcal{A}$ if and only if I is closed under every ideal term which is of the form $f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$, where $f \in F$ and f depends on the *i-th* variable. Since F is finite and every $f \in F$ is finitary, we conclude the assertion. $\square$

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