SPECTRAL PROPERTIES OF FOURTH ORDER DIFFERENTIAL OPERATORS

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Abstract. Necessary and sufficient conditions for discreteness and boundedness below of the spectrum of the singular differential operator $\ell(y) \equiv \frac{1}{w(t)}(r(t)y'')''$, $t \in [a,\infty)$ are established. These conditions are based on a recently proved relationship between spectral properties of ℓ and oscillation of a certain associated second order differential equation.

Keywords: singular differential operator, property BD, oscillation criteria, principal solution

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1. Introduction

In this paper we deal with spectral properties of the one-term differential operator

$$\ell(y) \equiv \frac{1}{w(t)}(r(t)y'')'', \tag{1.1}$$

where $t \in I := [a,\infty)$, $r^{-1}, w \in L_{\mathrm{loc}}(I)$, $r, w > 0$ on I are real valued functions, for some $a \in \mathbb{R}$. In particular, we look for necessary and sufficient conditions which guarantee that the spectrum of any self-adjoint extension of the minimal differential operator generated by ℓ in the weighted Hilbert space $L^2_w(I)$ (with the inner product $(y,z)_w = \int_I w(t)y(t)z(t)\,\mathrm{d}t$) is discrete and bounded below.

Let us first recall briefly the basic facts of spectral theory of singular differential operators (a comprehensive treatment of this topic may be found e.g. in Glazman [8], Naimark [12], Weidmann [16]). Consider the formally self-adjoint, even order,

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differential expression

$$\tilde{\ell}(y) \equiv \frac{1}{w(t)} \sum_{k=0}^{n} (-1)^k \left(p_k(t) y^{(k)}\right)^{(k)}, \tag{1.2}$$

where $t \in I$, $w, p_0, \dots, p_{n-1}, p_n^{-1} \in L_{\mathrm{loc}}(I)$, $w, p_n > 0$ on I are real-valued functions. Denote the quasi-derivatives

$$\begin{aligned} y^{[j]} &= y^{(j)},\ j = 0, 1, \dots, n-1, \\ y^{[n]} &= p_n y^{(n)}, \\ y^{[n+j]} &= p_{n-j} y^{(n-j)} - (y^{[n+j-1]})',\ j = 1, \dots, n, \end{aligned}$$

and let

$$\mathcal{D}(L) = \{y \in L^2_w(I) \colon y^{[k]} \in AC(I), k = 0, \dots, 2n-1, y^{[2n]} \in L^2_w(I)\}.$$

The differential operator $L \colon \mathcal{D}(L) \subseteq L^2_w(I) \to L^2_w(I)$ given by $L(y) = \tilde{\ell}(y)$, $y \in \mathcal{D}(L)$ and its adjoint $L_0 := L^*$ are called respectively the *maximal* and the *minimal operator* defined by $\tilde{\ell}$. Any self-adjoint extension K of L_0 (which exists, since the functions p_k are real, i.e., L_0 has the same deficiency indices γ_+, γ_- for which $n \leqslant \gamma_+ = \gamma_- \leqslant 2n$ holds) satisfies $L_0 \subseteq K \subseteq L$ and all self-adjoint extensions of L_0 have the same essential spectrum (= continuous spectrum + cluster points of discrete spectrum).

One of the most important problems in the theory of singular differential operators is to find conditions which guarantee that the essential spectrum of any self-adjoint extension K of L_0 is empty, i.e. K has a spectrum which is discrete and bounded below—the so-called property BD according to the terminology introduced in Hinton, Lewis [10]. Property BD means, roughly speaking, that the singular operator behaves like a regular one since it is known that the spectrum of regular operators consists only of eigenvalues of finite multiplicities with the only possible cluster point at ∞.

Spectral properties of singular differential operators of the form (1.1) are closely related to the oscillation theory of self-adjoint, even order differential equations. Two points $t_1, t_2 \in I$ are said to be *conjugate* relative to the equation

$$\sum_{k=0}^{n} (-1)^k \left(p_k(t) y^{(k)}\right)^{(k)} = 0, \tag{1.3}$$

if there exists a nontrivial solution y of this equation for which $y^{(i)}(t_1) = 0 = y^{(i)}(t_2)$, $i = 0, \dots, n-1$. Equation (1.3) is said to be *nonoscillatory* if there exists $c \in I$ such that the interval (c, ∞) contains no pair of points conjugate relative to (1.3), in the opposite case (1.3) is said to be *oscillatory at* ∞. The relationship between oscillation theory of (1.3) and spectral properties of singular operators generated by (1.2) is given by the next fundamental theorem.

Theorem A. (Glazman [8]) *The following are equivalent.*

(i) *The operator* $\tilde{\ell}$ *possesses property BD.*

(ii) *The equation* $\tilde{\ell}(y) = \lambda y$ *is nonoscillatory for every* $\lambda \in \mathbb{R}$.

(iii) *For every* $\lambda \in \mathbb{R}$ *there exists* $N \in \mathbb{R}$ *such that*

$$\int_N^\infty \left[\sum_{k=0}^n p_k(t)(y^{(k)})^2\right] \mathrm{d}t \geqslant \lambda \int_N^\infty w(t)y^2(t)\,\mathrm{d}t$$

for any $y \in W^{n,2}(N,\infty)$ *with compact support in* $[N,\infty)$.

By this theorem, nonoscillation criteria for (1.3) are sufficient conditions for $\tilde{\ell}$ to have property BD while oscillation criteria for (1.3) are necessary conditions for $\tilde{\ell}$ not to have property BD.

The Tkachenko-Lewis [11] classical result of spectral theory of one term differential operators of the form

$$\bar{\ell}(y) \equiv \frac{(-1)^n}{w(t)} \Bigl(r(t)y^{(n)}\Bigr)^{(n)} \tag{1.4}$$

with the weight function $w(t) \equiv 1$ states that $\bar{\ell}$ has property BD if and only if

$$\lim_{t\to\infty} t^{2n-1} \int_t^\infty \frac{1}{r(s)}\,\mathrm{d}s = 0. \tag{1.5}$$

Later, see Ahlbrandt, Hinton, Lewis [1], Došlý [4], Fiedler [6], it was shown that conditions similar to (1.5) are also necessary and sufficient for $\bar{\ell}$ to possess property BD with the weight functions $w(t) = t^\alpha$ or $w(t) = \mathrm{e}^{\beta t}$, $\alpha, \beta \in \mathbb{R}$. However, for the general weight function w, no condition in terms of w and r is known which is simultaneously necessary and sufficient for property BD of $\bar{\ell}$.

In this paper we fill this gap, in a certain way, for fourth order operator (1.1). We present a quite general condition for the weight function w under which (1.5)-like condition is necessary and sufficient for property BD of (1.1).

2. Transformation and oscillation of self-adjoint equations

Self-adjoint equations (1.3) are closely related to linear Hamiltonian systems (further LHS)

$$x' = Ax + B(t)u, \quad u' = C(t)x - A^T u, \tag{2.1}$$

where

$$\begin{aligned} B(t) &= \operatorname{diag}\{0, \dots, 0, p_n^{-1}(t)\}, \\ C(t) &= \operatorname{diag}\{p_0(t), \dots, p_{n-1}(t)\}, \\ A = A_{i,j} &= \begin{cases} 1, & \text{for } j = i+1, \quad i = 1, \dots, n-1, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned} \tag{2.2}$$

Let y be a solution of (1.3) and set $x = (y^{[0]}, \dots, y^{[n-1]})^T$, $u = (y^{[2n-1]}, \dots, y^{[n]})^T$. Then (x, u) is a solution of LHS (2.1). In this case we say that the solution (x, u) is generated by y. Simultaneously with (2.1) we will consider its matrix analogy

$$X' = AX + B(t)U, \quad U' = C(t)X - A^T U, \tag{2.3}$$

where X, U are $n \times n$ matrices. A solution (X, U) of (2.3) is said to be *isotropic* if $X^T(t)U(t) - U^T(t)X(t) \equiv 0$. An alternative terminology is prepared—Hartman [9], self-conjoined—Reid [13], self-conjugate—Sternberg [14]; our terminology is due to Coppel [3]. An isotropic solution (X, U) of (2.3) is said to be *principal at* ∞ if X is nonsingular on an interval $[c, \infty) \subseteq I$ and

$$\lim_{t\to\infty} \left(\int_c^t X^{-1}(s) B(s) X^{T-1}(s) \, \mathrm{d}s \right)^{-1} = 0.$$

Let $(\tilde{X}, \tilde{U})$ be a solution of (2.3) which is linearly independent of (X, U) (i.e. (X, U), $(\tilde{X}, \tilde{U})$ form the base of the solution space of (2.3)), then $(\tilde{X}, \tilde{U})$ is said to be *nonprincipal at* ∞. A system $y_1, \dots, y_n$ of solutions of (1.3) is said to form a *principal (nonprincipal) system at* ∞ if the solution (X, U) of the corresponding LHS (2.3) whose columns are generated by $y_1, \dots, y_n$ is principal (nonprincipal) at ∞. For example, concerning the equation $y^{(2n)} = 0$, the functions $y_i = t^{i-1}$, $i = 1, \dots, n$, form the principal system of solutions while $\tilde{y}_i = t^{n+i-1}$, $i = 1, \dots, n$, form the nonprincipal one. A principal (nonprincipal) system of solutions at ∞ exists whenever (1.3) is nonoscillatory.

One of the main tools we use in the sequel is the transformation of the one-term differential operators described in the next statement.

Lemma 2.1. (Ahlbrandt, Hinton, Lewis [2], Došlý [4]) *Let h be a positive function such that the quasi-derivatives $h^{[j]}$, $j = 0, \dots, 2n-1$, are absolutely continuous on I. Then the transformation $y = h(t)z$ gives*

$$(-1)^n h(t)\left(r(t)y^{(n)}\right)^{(n)} = \sum_{k=0}^{n} (-1)^k \left(R_k(t)z^{(k)}\right)^{(k)},$$

where $R_0 = (-1)^n h\left(rh^{(n)}\right)^{(n)}$, $R_n = h^2 r$ and also the "middle" functions R_k, $k = 1, \dots, n-1$, may be computed explicitly. In particular, if $n = 2$, we have

$$R_1 = -2h(rh')' - 2hrh'' + 2rh'^2.$$

Observe that if g and h are solutions of

$$\left(r(t)y^{(n)}\right)^{(n)} = 0, \tag{2.4}$$

then $v = z' = (g/h)'$ is a solution of the $(2n-2)$ order equation

$$\sum_{k=1}^{n} (-1)^{k-1} \left(R_k(t)v^{(k-1)}\right)^{(k-1)} = \kappa, \tag{2.5}$$

where κ is a real constant. In our investigation it will be important to determine when $\kappa = 0$. To answer this question, we need the following transformation of linear Hamiltonian systems.

Lemma 2.2. (Ahlbrandt, Hinton, Lewis [2]) *Let $H, K \in C^1(I)$ be $n \times n$ matrices such that H is nonsingular and $H^T K \equiv K^T H$ on I. Then the transformation*

$$x = H(t)\tilde{x}, \quad u = K(t)\tilde{x} + H^{T-1}(t)\tilde{u} \tag{2.6}$$

transforms (2.1) *into a linear Hamiltonian system*

$$\tilde{x}' = \tilde{A}(t)\tilde{x} + \tilde{B}(t)\tilde{u}, \quad \tilde{u}' = \tilde{C}(t)\tilde{x} - \tilde{A}^T(t)\tilde{u}, \tag{2.7}$$

where

$$\tilde{A} = H^{-1}(-H' + AH + BK), \quad \tilde{B} = H^{-1}BH^{T-1},$$
$$\tilde{C} = K^T(H' - AH - BK) - H^T(K' - CH + A^T K).$$

In particular, if (2.1) *corresponds to the self-adjoint equation* (1.3), *i.e. the matrices A, B, C are given by* (2.2) *and*

$$H = H_{i,j} = \begin{cases} \binom{i-1}{j-1} h^{(i-j)}, & i \geqslant j, \\ 0, & i < j, \end{cases}$$

then there exists an $n \times n$ matrix K such that (2.6) *transforms* (2.1) *into a linear Hamiltonian system corresponding to the equation*

$$\sum_{k=0}^{n}(-1)^k \left(R_k(t)y^{(k)}\right)^{(k)} = 0, \tag{2.8}$$

i.e. $\tilde{A} = A$, $\tilde{B} = \operatorname{diag}\{0,\dots,0,R_n^{-1}\}$ *and* $\tilde{C} = \operatorname{diag}\{R_0,\dots,R_{n-1}\}$.

Now, we are able to prove the next important auxiliary statement.

Lemma 2.3. *Let g, h be solutions of* (1.3) *and let*

$$\{h,g\} = \sum_{j=1}^{n}\left(h^{[j-1]}g^{[2n-j]} - h^{[2n-j]}g^{[j-1]}\right)$$

denotes Lagrange's bracket of g and h. Then $\kappa = \{h,g\}$.

Proof. Let H, K be the matrices which transform the linear Hamiltonian system corresponding to (1.3) into the LHS corresponding to equation (2.8). Denote by (x,u) and $(\hat{x},\hat{u})$ the solutions of (2.1) generated by g and h, respectively. Transformation (2.6) transforms (x,u) into a solution $(\tilde{x},\tilde{u})$ and $(\hat{x},\hat{u})$ into $(e_1,0)$, where $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$. Then

$$\tilde{u}_1 = z^{[2n-1]} = \sum_{k=1}^{n}(-1)^{k-1}\left(R_k(t)z^{(k)}\right)^{(k-1)}$$

and, on the other hand,

$$\begin{aligned}\{g,h\} &= x^T\hat{u} - u^T\hat{x} = \tilde{x}^T H^T(Ke_1 + H^{T-1}.0) - (\tilde{x}^T K^T + \tilde{u}^T H^{-1})He_1\\ &= -\tilde{u}^T e_1 = -\tilde{u}_1.\end{aligned}$$

□

Combining Lemma 2.1 and Lemma 2.3 we get

Lemma 2.4. *Let g, h be solutions of the equation*

$$(r(t)y'')'' = 0, \tag{2.9}$$

such that $h > 0$, $f := (g/h)' \neq 0$ for large t, $\{g,h\} = 0$ and let R_1 be given in Lemma (2.1). *Then the transformation $v = f(t)z$ gives*

$$f\left[(rh^2v')' - R_1 v\right] = \left[r\,\frac{W^2(h,g)}{h^2}\,z'\right]', \tag{2.10}$$

where $W(h,g)$ is the wronskian of h and g.

P r o o f. According to Lemmas 2.1, 2.3 equation (2.5) takes the form

$$(rh^2v')' - R_1 v = 0. \tag{2.11}$$

By Lemma 2.3 the function f solves equation (2.11) and applying Lemma 2.1 again with $v = f(t)z$ we get the result. □

The next statement summarizes some oscillation and nonoscillation criteria for the second order differential equation

$$(r(t)y')' + q(t)y = 0, \tag{2.12}$$

where $r, q > 0$ for large t are real-valued functions. The proofs of these criteria may be found e.g. in the monograph Swanson [15].

Lemma 2.5. (i) *If $\int^\infty r^{-1}(t)\,\mathrm{d}t = \infty = \int^\infty q(t)\,\mathrm{d}t$, then (2.12) is oscillatory.*

(ii) *If $\int^\infty r^{-1}(t)\,\mathrm{d}t = \infty$, $\int^\infty q(t)\,\mathrm{d}t < \infty$ or $\int^\infty r^{-1}(t)\,\mathrm{d}t < \infty$, $\int^\infty q(t)\,\mathrm{d}t = \infty$, then (2.12) is oscillatory provided*

$$\limsup_{t\to\infty} \int^t r^{-1}(s)\,\mathrm{d}s \int_t^\infty q(s)\,\mathrm{d}s > 1$$

or

$$\limsup_{t\to\infty} \int^t q(s)\,\mathrm{d}s \int_t^\infty r^{-1}(s)\,\mathrm{d}s > 1$$

and nonoscillatory provided

$$\liminf_{t\to\infty} \int^t r^{-1}(s)\,\mathrm{d}s \int_t^\infty q(s)\,\mathrm{d}s < \tfrac{1}{4}$$

or

$$\liminf_{t\to\infty} \int^t q(s)\,\mathrm{d}s \int_t^\infty r^{-1}(s)\,\mathrm{d}s < \tfrac{1}{4},$$

respectively.

(iii) *If $\int^\infty r^{-1}(t)\,\mathrm{d}t < \infty$, $\int^\infty q(t)\,\mathrm{d}t < \infty$ then (2.12) is nonoscillatory.*

In particular cases which we will study in this paper it is sometimes difficult to verify convergence or divergence of one of the integrals $\int^\infty r^{-1}(t)\,\mathrm{d}t$, $\int^\infty q(t)\,\mathrm{d}t$. However, if $\int^\infty r^{-1}(t)\,\mathrm{d}t < \infty$, the transformation

$$y = \left(\int_t^\infty r^{-1}(s)\,\mathrm{d}s\right) z \tag{2.13}$$

transforms (2.12) into the equation

$$\left[r(t)\left(\int_t^\infty r^{-1}(s)\,\mathrm{d}s\right)^2 z'\right]' + \left(\int_t^\infty r^{-1}(s)\,\mathrm{d}s\right)^2 q(t)z = 0$$

and

$$\int^\infty \frac{1}{r(t)\left(\int_t^\infty r^{-1}(s)\,\mathrm{d}s\right)^2}\,\mathrm{d}t = \lim_{t\to\infty}\frac{1}{\int_t^\infty r^{-1}(s)\,\mathrm{d}s} = \infty.$$

3. Property BD of fourth order operators

In this section we present the main results of the paper—necessary and sufficient conditions for the fourth order operator (1.1) to possess property BD. We use the following general statement proved in Došlý [4]. We say that a system $y_1,\dots,y_{2n}$ of solutions of the equation

$$\left(\frac{1}{w(t)}y^{(n)}\right)^{(n)} = 0 \tag{3.1}$$

forms an *ordered* system at ∞ if $y_j > 0$, $j = 1,\dots,2n$, for large t and

$$\frac{y_j}{y_{j+1}} \to 0, \quad j = 1,\dots,2n-1,$$

for $t \to \infty$.

Let $y_1,\dots,y_n,\tilde y_1,\dots,\tilde y_n$ be an ordered system of solutions of equation (3.1). Then the functions $y_1,\dots,y_n$ form a principal system of solutions of (3.1) and the functions $\tilde y_1,\dots,\tilde y_n$ form a nonprincipal one (see Coppel [3, Chap. III]). Denote by (X,U) and $(\tilde X,\tilde U)$ the solutions of the associated linear Hamiltonian system generated respectively by $y_1,\dots,y_n$ and $\tilde y_1,\dots,\tilde y_n$ and let $K = K_{i,j} = X^T\tilde U - U^T\tilde X$.

Theorem B. (Došlý [4]) *Suppose that there exists an index $i \in \{1,\dots,n\}$ and $\lambda_0 > 0$ such that the equation*

$$\sum_{k=1}^{n}(-1)^{k-1}\left(R_k(t)u^{(k-1)}\right)^{(k-1)} = \lambda\,\frac{K_{i,l}W_{l,i}^2(t)}{W'(t)W_{l,i}(t) - W(t)W_{l,i}'(t)}\,u \tag{3.2}$$

is nonoscillatory for $\lambda < \lambda_0$, where

$$l := \min\{j \in \{1,\dots,n\} \mid K_{i,j} \neq 0\},$$

the functions $R_k(t)$, $k=1,\dots,n$, are given in Lemma 2.1 with $h=y_i$, $r=\frac{1}{w}$ and $W:=W(\tilde y_1,\dots,\tilde y_n)$, $W_{l,i}:=W(\tilde y_1,\dots,\tilde y_{l-1},y_i,\tilde y_{l+1},\dots,\tilde y_n)$ are the wronskians of the functions in brackets. Then the operator $\bar\ell$ given by (1.4) has property BD if and only if

$$\lim_{t\to\infty}\frac{W(t)}{K_{i,l}W_{l,i}(t)}\int_t^\infty\frac{y_i^2(s)}{r(s)}\,\mathrm{d}s=0. \tag{3.3}$$

The above theorem contains—for general n—a rather unpleasant assumption concerning nonoscillation of (3.2). However, if $n=2$ then (3.2) is a second order equation and its oscillation theory is deeply developed; some of its results which we will use in the sequel were given in Lemma 2.5.

We start with the explicit description of principal and nonprincipal systems of solutions of the equation

$$\left(\frac{1}{w(t)}y''\right)''=0$$

which satisfy the assumptions of Theorem 3. Here we use the results of Došlý, Komenda [5] where an explicit description of these systems of solutions is given for general n.

Notation. To simplify expressions and formulas we use an abbreviated notation for integrals—without the integration variable t, s, $\mathrm{d}t$, $\mathrm{d}s$ and without the lower intergration limit. For example, under the symbol $\int^t(t-s)w$ we understand the function $t\mapsto\int_c^t(t-s)w(s)\,\mathrm{d}s$, or the symbol $\int^t w\int^s\tau w$ means the function $t\mapsto\int_c^t w(s)\left(\int_c^s\tau w(\tau)d\tau\right)\,\mathrm{d}s$ for some $c\in\mathbb{R}$.

We distinguish the following cases:

(A) $\int^\infty w=+\infty$. Then

$$y_1=1,\ y_2=t,\ \tilde y_1=\int^t(t-s)w,\ \tilde y_2=\int^t(t-s)sw.$$

(B) $\int^\infty w<+\infty$, $\int^\infty tw=+\infty$. Then

$$y_1=1,\ y_2=t\int_t^\infty w+\int^t sw,\ \tilde y_1=t,\ \tilde y_2=\int^t(t-s)sw.$$

(C) $\int^\infty tw<+\infty$, $\int^\infty t^2w=+\infty$. Then

$$y_1=\int_t^\infty(s-t)w,\ y_2=1,\ \tilde y_1=t\int_t^\infty sw+\int^t s^2w,\ \tilde y_2=t.$$

(D) $\int^{\infty} t^2 w < +\infty$. Then

$$y_1 = \int_t^{\infty} (s-t)w,\ y_2 = \int_t^{\infty} (s-t)sw,\ \tilde{y}_1 = 1,\ \tilde{y}_2 = t.$$

One may easily verify that in all the above cases the solutions $y_1, y_2, \tilde{y}_1, \tilde{y}_2$ form an ordered system of solutions of (3.1). The matrix $K = K_{i,j} = X^T\tilde{U} - U^T\tilde{X} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Now let us turn our attention to the explicit form of equation (3.2) (for $n=2$) in all the particular cases **(A)**–**(D)**. Denote

$$Y^2(t) := \frac{w(t)K_{i,l}W_{l,i}^2(t)}{W'(t)W_{l,i}(t) - W(t)W_{l,i}'(t)} = \frac{w(t)}{[W(t)/(K_{i,l}W_{l,i}(t))]'}. \tag{3.4}$$

It is not hard to see that the expression on the right-hand side is indeed eventually positive. Intuitively, $W(t)$ is the wronskian of a nonprincipal system of solutions, whereas in $W_{l,i}(t)$ the l-th nonprincipal component is replaced by a definitely smaller i-th principal one. Thus the ratio $W/(K_{i,l}W_{l,i})$ is a monotone increasing function.

Using (3.4) we may write equation (3.2) in the form

$$-\left(\frac{1}{w(t)}h^2(t)u'\right)' + R_1(t)u = \lambda\,\frac{Y^2(t)}{w(t)}\,u \tag{3.5}$$

and the transformation $u = (g/h)'z$, where g, h are the same as in Lemma 2.4, transforms (3.5) into the equation

$$\left(\frac{W^2(h,g)}{wh^2}\,z'\right)' + \lambda\,\frac{Y^2\,W^2(h,g)}{wh^4}\,z = 0. \tag{3.6}$$

Proposition 3.1. *Let h and g be as above. If*

$$\int^{\infty} \frac{wh^2}{W^2(h,g)} = +\infty \quad \text{and} \quad \lim_{t\to\infty} \frac{Y\,W^2(h,g)}{wh^3} \int^{t} \frac{wh^2}{W^2(h,g)} =: L < +\infty \tag{3.7}$$

or

$$\int^{\infty} \frac{wh^2}{W^2(h,g)} < +\infty \quad \text{and} \quad \lim_{t\to\infty} \frac{Y\,W^2(h,g)}{wh^3} \int_t^{\infty} \frac{wh^2}{W^2(h,g)} =: L < +\infty \tag{3.8}$$

then the equation (3.6) is nonoscillatory for $\lambda < \lambda_0 := \frac{1}{4L^2}$.

Proof. For (3.7): from Lemma 2.5(ii)—the nonoscillation part—we have that (3.6) is nonoscillatory provided

$$\lim_{t\to\infty} \int^{t} \frac{wh^2}{W^2(h,g)} \int_t^{\infty} \lambda\,\frac{Y^2\,W^2(h,g)}{wh^4} < \tfrac{1}{4}.$$

Using l'Hospital's rule the last limit takes the form

$$\lim_{t\to\infty}\frac{\lambda\displaystyle\int_t^\infty\frac{Y^2\,W^2(h,g)}{wh^4}}{\frac{1}{\int^t\frac{wh^2}{W^2(h,g)}}}=\lim_{t\to\infty}\frac{-\lambda\,\frac{Y^2\,W^2(h,g)}{wh^4}}{-\frac{\frac{wh^2}{W^2(h,g)}}{\left[\int^t\frac{wh^2}{W^2(h,g)}\right]^2}}=\lambda\left[\lim_{t\to\infty}\frac{Y\,W^2(h,g)}{wh^3}\int^t\frac{wh^2}{W^2(h,g)}\right]^2$$
$$=\lambda\,L^2<\lambda_0\,L^2=\tfrac{1}{4}.$$

For (3.8): we transform (3.6) by (2.13) and then proceed in a similar way as above. Note that the limit in (3.8) is the same as in (3.7) except of the interval of integration.

□

In the particular cases **(A)**–**(D)** the limit in (3.7) or in (3.8) takes the following form:

Case (A1). $h=y_1=1$ (i.e. $i=1$ and $l=2$), $g=y_2=t$ (then $R_1\equiv 0$ by Lemma 2.1 and the transformation from Lemma 2.4 is actually not needed). Equation (3.6) reads

$$\left(\frac{1}{w}z'\right)'+\lambda\,\frac{\left(\int^t w\right)^2}{w\left[\int^t(t-s)w\right]^2}\,z=0 \tag{3.9}$$

and

$$\frac{W}{K_{i,l}W_{l,i}}=\frac{\int^t s^2w\int^t w-\left(\int^t sw\right)^2}{\int^t w}. \tag{3.10}$$

Case (A2). $h=y_2=t$ (i.e. $i=2$ and $l=1$), $g=y_1=1$. Equation (3.6) reads

$$\left(\frac{1}{t^2w}z'\right)'+\lambda\,\frac{\left(\int^t s^2w\right)^2}{wt^4\left[\int^t(t-s)w\right]^2}\,z=0 \tag{3.11}$$

and

$$\frac{W}{K_{i,l}W_{l,i}}=\frac{\int^t s^2w\int^t w-\left(\int^t sw\right)^2}{\int^t s^2w}. \tag{3.12}$$

Using Proposition 1 and Theorem 3 we get

Criterion A. *Let* $\int^\infty w=\infty$.

(i) Case (A1). *If*

$$\lim_{t\to\infty}\frac{\left(\int^t w\right)^2}{w\int^t (t-s)w} =: M < \infty \tag{3.13}$$

then the operator ℓ *given by* (1.1) *has property BD if and only if*

$$\lim_{t\to\infty}\frac{\int^t s^2w\int^t w-\left(\int^t sw\right)^2}{\int^t w}\int_t^\infty \frac{1}{r}=0. \tag{3.14}$$

(ii) Case (A2). *If*

$$\lim_{t\to\infty}\frac{\left(\int^t s^2w\right)^2}{t^3w\int^t (t-s)sw} =: L < \infty \tag{3.15}$$

then the operator ℓ *given by* (1.1) *has property BD if and only if*

$$\lim_{t\to\infty}\frac{\int^t s^2w\int^t w-\left(\int^t sw\right)^2}{\int^t s^2w}\int_t^\infty \frac{s^2}{r}=0. \tag{3.16}$$

Proof. (i) Since $Y=\frac{\int^t w}{\int^t (t-s)w}$ and $W(h,g)=1$, Proposition 1 states that equation (3.9) is nonoscillatory for $\lambda<\frac{1}{4M^2}$. Relation (3.3) from Theorem 3 reads as (3.14) in this particular case.

(ii) Similarly as in (A1) we have $Y=\frac{\int^t s^2w}{\int^t (t-s)sw}$, $W(h,g)=-1$ and the statement follows from Proposition 1 and Theorem 3. □

Before continuing with cases **(B)**–**(D)**, we give two examples which illustrate the application of Criterion A.

Example 3.1. (i) Let $w(t)=\frac{1}{t}$. Then the values of the constants from Criterion 1 are $M=\infty$, $L=\frac{1}{2}$. Hence part (ii) of this criterion (part (i) does not apply to this case) states that ℓ has property BD if and only if

$$\lim_{t\to\infty}\ln t\int_t^\infty \frac{s^2}{r(s)}\,\mathrm{d}s=0.$$

This criterion complies with a general criterion for $2n$-order differential operators with weight functions of the form $t^{-\alpha}$, $\alpha\in\{1,3,\dots,2n-1\}$, given in Došlý [4].

(ii) Let $w(t) = \frac{1}{t \ln t}$. Then the constants M, L in Criterion A are $M = \infty$, $L = \frac{1}{2}$, to compute these limits we have used the fact that

$$\lim_{t\to\infty} \frac{\int^t \frac{s}{\ln s}}{\frac{t^2}{2\ln t}} = 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{\int^t \frac{1}{\ln s}}{\frac{t}{\ln t}} = 1.$$

This means that part (i) of Criterion A does not apply to this case whereas part (ii) gives the following necessary and sufficient condition for property BD of ℓ with this weight function:

$$\lim_{t\to\infty} \ln\ln t \int_t^\infty \frac{s^2}{r(s)}\, \mathrm{d}s = 0.$$

In cases **(B)–(D)** we may proceed in the same way as in (A). We give the corresponding necessary and sufficient conditions for property BD without proofs. The values of Y and $W/(K_{i,l}W_{l,i})$ appearing in these criteria may be evaluated by a direct computation.

Criterion B. *Let* $\int^\infty w < \infty$, $\int^\infty tw = \infty$.
(i) Case (B1). $i = 1$: $h = y_1 = 1$, $g = \tilde{y}_1 = t$, $Y = \frac{1}{t}$, $W(h,g) = 1$. *If*

$$\lim_{t\to\infty} \frac{1}{tw} \int_t^\infty w < \infty$$

then the operator ℓ *given by* (1.1) *has property BD if and only if*

$$\lim_{t\to\infty} \int^t s^2 w \int_t^\infty \frac{1}{r} = 0.$$

(ii) Case (B2). $i = 2$: $h = y_2 = t\int_t^\infty w + \int^t sw$, $g = y_1 = 1$, $W(h,g) = \int_t^\infty w$, $Y = \frac{\int^t s^2 w \int_t^\infty w + \left(\int^t sw\right)^2}{\int^t (t-s)sw}$. *If*

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty w\right)^2 \left[\int^t s^2 w \int_t^\infty w + \left(\int^t sw\right)^2\right]}{w\int^t (t-s)sw \left(t\int_t^\infty w + \int^t sw\right)^3} \int^t \frac{w}{\int_s^\infty w} \left(s\int_s^\infty w + \int^s \tau w\right)^2 < \infty$$

then the operator ℓ *given by* (1.1) *has property BD if and only if*

$$\lim_{t\to\infty} \frac{\int^t s^2 w}{\int^t s^2 w \int_t^\infty w + \left(\int^t sw\right)^2} \int_t^\infty \frac{\left[s\int_s^\infty w + \int^s \tau w\right]^2}{r} = 0.$$

Criterion C. *Let* $\int^\infty tw < \infty$, $\int^\infty t^2 w = \infty$.

(i) Case (C1). $i = 1$: $h = y_1 = \int_t^\infty (s-t)w$, $g = y_2 = 1$, $W(h,g) = \int_t^\infty w$, $Y = \frac{\int^t s^2 w \int_t^\infty w + \left(\int_t^\infty sw\right)^2}{t\int_t^\infty sw + \int^t s^2 w}$. *Denote*

$$I_{C1} := \int^\infty \frac{w}{\left(\int_t^\infty w\right)^2}\left[\int_t^\infty (s-t)w\right]^2.$$

If $I_{C1} = \infty$ *and*

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty w\right)^2 \left[\int^t s^2 w \int_t^\infty w + \left(\int_t^\infty sw\right)^2\right]}{w\left[\int_t^\infty (s-t)w\right]^3 \left(t\int_t^\infty sw + \int^t s^2 w\right)} \int^t \frac{w}{\left(\int_s^\infty w\right)^2}\left[\int_s^\infty (\tau - s)w\right]^2 < \infty$$

or if $I_{C1} < \infty$ *and*

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty w\right)^2 \left[\int^t s^2 w \int_t^\infty w + \left(\int_t^\infty sw\right)^2\right]}{w\left[\int_t^\infty (s-t)w\right]^3 \left(t\int_t^\infty sw + \int^t s^2 w\right)} \int_t^\infty \frac{w}{\left(\int_s^\infty w\right)^2}\left[\int_s^\infty (\tau - s)w\right]^2 < \infty$$

then the operator ℓ given by (1.1) *has property BD if and only if*

$$\lim_{t\to\infty} \frac{\int^t s^2 w}{\int^t s^2 w \int_t^\infty w + \left(\int_t^\infty sw\right)^2} \int_t^\infty \frac{\left[\int_s^\infty (\tau - s)w\right]^2}{r} = 0.$$

(ii) Case (C2). $i = 2$: $h = y_2 = 1$, $g = \tilde{y}_2 = t$, $Y = \frac{1}{t}$, $W(h,g) = 1$. *If*

$$\lim_{t\to\infty} \frac{1}{tw}\int_t^\infty w < \infty$$

then the operator ℓ given by (1.1) *has property BD if and only if*

$$\lim_{t\to\infty} \int^t s^2 w \int_t^\infty \frac{1}{r} = 0.$$

Criterion D. *Let* $\int^\infty t^2 w < \infty$.

(i) Case (D1). $i = 1$: $h = y_1 = \int_t^\infty (s-t)w$, $g = \tilde{y}_1 = 1$, $W(h,g) = \int_t^\infty w$, $Y = \int_t^\infty w$. *Denote*

$$I_{D1} := \int^\infty \frac{w}{\left(\int_t^\infty w\right)^2}\left[\int_t^\infty (s-t)w\right]^2.$$

If $I_{D1} = \infty$ *and*

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty w\right)^3}{w\left[\int_t^\infty (s-t)w\right]^3} \int^t \frac{w}{\left(\int_s^\infty w\right)^2}\left[\int_s^\infty (\tau - s)w\right]^2 < \infty$$

or if $I_{D1} < \infty$ *and*

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty w\right)^3}{w\left[\int_t^\infty (s-t)w\right]^3} \int_t^\infty \frac{w}{\left(\int_s^\infty w\right)^2} \left[\int_s^\infty (\tau - s)w\right]^2 < \infty$$

then the operator ℓ *given by* (1.1) *has property BD if and only if*

$$\lim_{t\to\infty} \frac{1}{\int_t^\infty w} \int_t^\infty \frac{\left[\int_s^\infty (\tau-s)w\right]^2}{r} = 0.$$

(ii) Case (D2). $i = 2$: $h = y_2 = \int_t^\infty (s-t)sw$, $g = \tilde{y}_2 = t$, $W(h,g) = \int_t^\infty s^2 w$, $Y = \frac{\int_t^\infty s^2 w}{t}$. *Denote*

$$I_{D2} := \int^\infty \frac{w}{\left(\int_t^\infty s^2 w\right)^2} \left[\int_t^\infty (s-t)sw\right]^2.$$

If $I_{D2} = \infty$ *and*

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty s^2 w\right)^3}{tw\left[\int_t^\infty (s-t)sw\right]^3} \int^t \frac{w}{\left(\int_s^\infty \tau^2 w\right)^2} \left[\int_s^\infty (\tau - s)\tau w\right]^2 < \infty$$

or if $I_{D2} < \infty$ *and*

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty s^2 w\right)^3}{tw\left[\int_t^\infty (s-t)sw\right]^3} \int_t^\infty \frac{w}{\left(\int_s^\infty \tau^2 w\right)^2} \left[\int_s^\infty (\tau - s)\tau w\right]^2 < \infty$$

then the operator ℓ *given by* (1.1) *has property BD if and only if*

$$\lim_{t\to\infty} \frac{1}{\int_t^\infty s^2 w} \int_t^\infty \frac{\left[\int_s^\infty (\tau-s)\tau w\right]^2}{r} = 0.$$

Remark 3.2. (i) By a direct computation one may verify that criteria **(A)**–**(D)** apply to weight functions of the form $w(t) = t^\alpha \ln^\beta t$ or $w(t) = t^\alpha e^{\gamma t}$, $\alpha, \beta, \gamma \in \mathbb{R}$.

(ii) To remove the function $R_1(t)$ from equation (3.5) we need to use any solution g of (3.1) for which $\{g, h\} = 0$. In the above criteria we have used the solution g which gives formally the simplest condition for property BD of the operator ℓ. Suppose that we use another function, say $\tilde{g}$, satisfying our assumptions, and denote by $\tilde{L}$ the constant from Proposition 1 where g is replaced by $\tilde{g}$. If $L = \infty$ then equation (3.5) is oscillatory for all $\lambda \in \mathbb{R}$. On the other hand, if $\tilde{L} < \infty$ then (3.5) is nonoscillatory for $\lambda < 1/(4\tilde{L}^2)$, which is a contradiction. So either $L = \infty = \tilde{L}$ or both L and $\tilde{L}$ are finite. This means that the choice of the function g has no effect on the oscillation behavior of (3.5). In cases (A1), (A2), (B1), (C2) it even occurs that the corresponding criteria for g and $\tilde{g}$ are totally the same.

(iii) (3.14) or (3.16)-like conditions can be found in Fiedler [7] as necessary but yet not sufficient conditions for property BD of the operator (1.1), since only cases $\int^\infty t^2 w = \infty$ or $\int^\infty t^2 w < \infty$ are treated.

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