ON SOME PROPERTIES OF THE CLASS $\mathcal{P}(B, b, \alpha)$

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Abstract. Let $\mathcal{P}$ denote the well known class of functions of the form $p(z) = 1 + q_1 z + \dots$ holomorphic in the unit disc $\mathbf{D}$ and fulfilling the condition $\operatorname{Re} p(z) > 0$ in $\mathbf{D}$. Let $0 \leqslant b < 1$, $b < B$, $0 < \alpha < 1$ be fixed real numbers. $\mathcal{P}(B, b, \alpha)$ denotes the class of functions $p \in \mathcal{P}$ such that there exists a measurable subset $\mathbf{F}$ of the unit circle $\mathbf{T}$, of Lebesgue measure $2\pi\alpha$, such that the function p fulfils $\operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta}) \geqslant B$ a.e. on $\mathbf{F}$ and $\operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta}) \geqslant b$ a.e. on $\mathbf{T} \setminus \mathbf{F}$. In this paper further properties of the class $\mathcal{P}(B, b, \alpha)$ are examined. In particular, the investigations included in it constitute a direct continuation of papers [6]–[8] and concern mainly the form of the closed convex hull of the class $\mathcal{P}(B, b, \alpha)$ as well as the estimates of the functional $\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\lambda} p(z)\}$, $0 \neq z \in \mathbf{D}$, $\lambda \in \langle -\pi, \pi)$, $p \in \mathcal{P}(B, b, \alpha)$. This article belongs to the series of papers ([1]–[8]) where different classes of functions defined by conditions on the circle $\mathbf{T}$ were studied.

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1. Introduction and general remarks

As usual, we denote by $\mathbf{C}$ the complex plane, by $\mathbf{D} = \{z;\ |z| < 1\}$ the unit disc and by $\mathbf{T} = \{z;\ |z| = 1\}$ the unit circle. Let $S(M)$, $M > 1$, denote the class of functions of the form

$$f(z) = z + a_2 z^2 + \dots + a_n z^n + \dots \tag{1.1}$$

holomorphic, univalent and such that $|f(z)| < M$ in $\mathbf{D}$ and let $S = S(+\infty)$. Denote by $\mathcal{P}_\beta$, $0 \leqslant \beta < 1$, the class of functions of the form

$$p(z) = 1 + q_1 z + \dots + q_n z^n + \dots \tag{1.2}$$

holomorphic in $\mathbf{D}$ with $\operatorname{Re} p(z) > \beta$ for $z \in \mathbf{D}$, and let $\mathcal{P} = \mathcal{P}_0$.

As is well known, in the geometrical theory of functions one has studied, among other things, properties of selected classes of functions of the form (1.1) or (1.2). These classes were usually defined by imposing suitable geometrical conditions (e.g. the convexity of the domain $f(\mathbf{D})$) or analytic ones (e.g. $\operatorname{Re} f'(z) > 0$) in the disc $\mathbf{D}$. One can also find papers (for example [10]) in which functions (1.1) of the class being defined were subjected to a condition they should satisfy on the circle $\mathbf{T}$ or for $z \in \mathbf{D}$ with modulus sufficiently close to 1.

The article belongs to the series of papers [1]–[8]. In these papers, some classes of functions holomorphic in $\mathbf{D}$ (of form (1.1) or (1.2)) defined by two different conditions on the unit circle $\mathbf{T}$ were studied.

In the paper [1] the authors investigated the class $S(M, m; \alpha)$, $0 < m \leqslant M < \infty$, $0 \leqslant \alpha \leqslant 1$, of bounded functions of form (1.1) such that there exists an open arc $I_\alpha = I_\alpha(f) \subset \mathbf{T}$ of length $2\pi\alpha$ such that, for each $z_1 \in I_\alpha$,

$$\overline{\lim}_{\mathbf{D} \ni z \to z_1} |f(z)| \leqslant M$$

and for every $z_2 \in \mathbf{T} \setminus \overline{I_\alpha}$

$$\overline{\lim}_{\mathbf{D} \ni z \to z_2} |f(z)| \leqslant m.$$

Of course, $S(M, m; 1) = S(M)$, $M > 1$, and $S(M, m; 0) = S(m)$, $m > 1$.

In the paper [2], the class $\widetilde{\mathcal{P}}(B, b; \alpha) \subset \mathcal{P}$, $0 \leqslant b < 1$, $b < B$, $0 < \alpha < 1$, of functions of the form (1.2) fulfilling the conditions

$$\begin{aligned} &\liminf_{\mathbf{D} \ni z \to z_1} \operatorname{Re} p(z) \geqslant B \text{ for each } z_1 \in I_\alpha \\ &\liminf_{\mathbf{D} \ni z \to z_2} \operatorname{Re} p(z) \geqslant b \text{ for each } z_2 \in \mathbf{T} \setminus \overline{I_\alpha} \end{aligned} \tag{1.3}$$

was introduced, $I_\alpha = I_\alpha(p)$ still being an open arc of length $2\pi\alpha$ of the circle $\mathbf{T}$. Evidently, $\widetilde{\mathcal{P}}(B, b; 1) = \mathcal{P}_B$, $b \leqslant B < 1$ and $\widetilde{\mathcal{P}}(B, b; 0) = \mathcal{P}_b$, $0 \leqslant b < 1$.

The idea of using open arcs of $\mathbf{T}$ of lengths $2\pi\alpha$ and $2\pi(1-\alpha)$ in the above definitions has certain analogies in the papers by P. T. Mocanu (see [9] and [1]).

In the subsequent articles ([3]–[8]), various subclasses of the family $\mathcal{P}$ were again considered, with the difference that in place of arcs various subsets $\mathbf{F}$ of the circle $\mathbf{T}$ appeared. It is also known that if $p \in \mathcal{P}$, then $\operatorname{Re} p(z)$ has nontangential limits $\operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta})$ a.e. on $\langle -\pi, \pi)$. So, instead of (1.3) the conditions

$$\operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta}) \geqslant B \text{ a.e. on } \mathbf{F} \text{ and } \operatorname{Re} p(\mathrm{e}^{\mathrm{i}\theta}) \geqslant b \text{ a.e. on } \mathbf{T} \setminus \mathbf{F} \tag{1.4}$$

were adopted.

In particular, in the preprint [3] $\mathbf{F}$ is a given closed subset of $\mathbf{T}$ of Lebesgue measure $2\pi\alpha$ and the class $\breve{\mathcal{P}}(B, b, \alpha; \mathbf{F})$ of functions (1.2) satisfying conditions (1.4) on this set $\mathbf{F}$ is considered.

In [4] and [5] $\mathbf{F}$ is still a given closed subset of $\mathbf{T}$ of Lebesgue measure $2\pi\alpha$ but the authors consider the class $\mathcal{P}(B,b,\alpha;\mathbf{F})$ of functions (1.2) satisfying conditions (1.4) on the set $\mathbf{F}$ or on the set $\mathbf{F}_\tau = \{\xi \in \mathbf{T};\ \mathrm{e}^{-\mathrm{i}\tau}\xi \in \mathbf{F}\}$ for some $\tau = \tau(p) \in \langle -\pi, \pi)$. In reports [4], [5] they also consider the class

$$\mathcal{P}(B,b,\alpha) = \bigcup_{\mathbf{F}} \mathcal{P}(B,b,\alpha;\mathbf{F})$$

where $\mathbf{F} \subset \mathbf{T}$ satisfies the conditions mentioned above.

Since the rotations $(I_\alpha)_\tau$ of the arcs I_α are admissible already in the class $\widetilde{\mathcal{P}}(B,b,\alpha)$, whereas $\overline{I_\alpha}$ are closed sets, therefore the replacement of the arcs I_α by arbitrary closed sets of measure $2\pi\alpha$ seemed natural. What is more, the fulfilment of suitable conditions at each point of the set was abandoned and replaced by the fulfilment of them almost everywhere.

It turned out ([6]) that all main results from [4], [5] are preserved when, in the above-mentioned definitions of the classes, in place of $\mathbf{F}$ we take a given measurable subset of the unit circle $\mathbf{T}$ of Lebesgue measure $2\pi\alpha$. The analogous assumption about the sets $\mathbf{F}$ is used also in paper [8]. The following definition was adopted there.

Definition 1.1. Let $0 \leqslant b < 1$, $b < B$, $0 < \alpha < 1$ be fixed real numbers. By $\mathcal{P}(B,b,\alpha)$ we denote the class of functions $p \in \mathcal{P}$ such that there exists a measurable set $\mathbf{F} = \mathbf{F}(p)$, $\mathbf{F} \subset \mathbf{T}$, of Lebesgue measure $m(\mathbf{F}) = 2\pi\alpha$, such that the function p fulfils (1.4) a.e. on $\mathbf{F}$ and $\mathbf{T} \setminus \mathbf{F}$.

In [8] (Ths. 4, 5) it was shown that a) $\mathcal{P}(B,b,\alpha)$ is not convex, b) $\mathcal{P}(B,b,\alpha)$ is not compact, i.e. not closed, in the topology given by the uniform convergence on compact subsets of $\mathbf{D}$. It was shown there that the functions $p_n \in \mathcal{P}(B,b,\alpha)$ which realize the maximum modulus of the n-th coefficient in the class $\mathcal{P}(B,b,\alpha)$ ([6], Th. 8) converge to a function $p_0 \in \mathcal{P}$ not belonging to $\mathcal{P}(B,b,\alpha)$. So, the following three natural questions arise:

(a) What is the closure of $\mathcal{P}(B,b,\alpha)$?

(b) What is the closed convex hull of $\mathcal{P}(B,b,\alpha)$?

(c) Which are the compact subsets of $\mathcal{P}(B,b,\alpha)$?

The replacement of the closed sets $\mathbf{F}$ by measurable sets, mentioned before, not only allowed to generalize the results known earlier, but it is just essential on account of questions (a), (b) and (c) to be considered in the second section of this article.

In the third section we will obtain an estimate of the functional $\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\lambda}p(z)\}$, $0 \neq z \in \mathbf{D}$, $\lambda \in \langle -\pi, \pi)$, defined in the class $\mathcal{P}(B,b,\alpha)$. The theorem proved there generalizes the corresponding results from the paper [8] (Th.3).

Questions (a), (b), (c) and formulations of the main theorems were given in [7].

2. The closed convex hull of $\mathcal{P}(B,b,\alpha)$

In this section we give answers to questions (a), (b) and (c).

We denote by $\lambda(A)$ the normalized Lebesgue measure on $\mathbf{T}$, i.e. $\lambda(\mathbf{T})=1$. Denote by χ_A the characteristic function of the set A. Here we treat the function $f(\mathrm{e}^{\mathrm{i}t})$: $\mathbf{T}\to\mathbf{C}$ as the function $f(t)\colon\langle-\pi,\pi)\to\mathbf{C}$, too.

(a) We will need the following lemmas.

Lemma 2.1. *Let $\alpha\in(0,1)$ be a given real number. Let c, μ be arbitrary real numbers fulfilling $0<c\leqslant 1$, $0<\mu\leqslant 1$, $c\mu=\alpha$. Let $I_\mu\subset\mathbf{T}$ be an arbitrary arc on $\mathbf{T}$, $\lambda(I_\mu)=\mu$. Then there exists a sequence $\{\mathbf{F}_n\}_{n=1}^\infty$ of measurable subsets of I_μ such that*

$$\lambda(\mathbf{F}_n)=\alpha,\quad n=1,2,\ldots,\tag{2.1}$$

and

(2.2) *the sequence of measures $\chi_{\mathbf{F}_n}\,\mathrm{d}\lambda$ converges weakly to the measure $c\chi_{I_\mu}\,\mathrm{d}\lambda$.*

Proof. Since $c\in(0,1\rangle$, $c\mu=\alpha$, we have $\mu\geqslant\alpha$. In the trivial case $\mu=\alpha$ put $\mathbf{F}_n=I_\mu$, $n=1,2,\ldots$, thus (2.1) and (2.2) are fulfilled. So, let $\mu>\alpha$. Without loss of generality, suppose that the midpoint of I_μ is the point $z=1$, write $l=2n-1$ and define

$$\mathbf{F}_n=\bigcup_{k=-n+1}^{n-1}\mathbf{F}_n^{(k)}$$

where

$$\mathbf{F}_n^{(k)}=\Big\{z\in\mathbf{T};\ z=\mathrm{e}^{\frac{2k\pi\mu}{l}\mathrm{i}}\mathrm{e}^{\varrho\mathrm{i}},\ -\frac{\pi\alpha}{l}<\varrho<\frac{\pi\alpha}{l}\Big\}$$
$$\text{for } k=-n+1,\ldots,-1,0,1,\ldots,n-1.$$

Clearly, the intervals $\mathbf{F}_n^{(k)}$ are mutually disjoint, $\mathbf{F}_n\subset I_\mu$ and $\lambda(\mathbf{F}_n)=\alpha$.

Let f be a given function continuous on $\mathbf{T}$. We have to show that

$$\lim_{n\to\infty}\int_{\mathbf{T}}f(t)\chi_{\mathbf{F}_n}(t)\,\mathrm{d}\lambda(t)=\int_{\mathbf{T}}cf(t)\chi_{I_\mu}(t)\,\mathrm{d}\lambda(t)=:\int_{\mathbf{T}}cf\chi_{I_\mu}\,\mathrm{d}\lambda.$$

First,

$$\int_{\mathbf{T}} f(t)\chi_{\mathbf{F}_n}(t)\,\mathrm{d}\lambda(t) = \sum_{k=-n+1}^{n-1} \int_{\mathbf{F}_n^{(k)}} f(t)\,\mathrm{d}\lambda(t) \tag{2.3}$$

$$= \sum_{k=-n+1}^{n-1} \int_{(2k\pi\mu-\alpha\pi)/l}^{(2k\pi\mu+\alpha\pi)/l} f(t)\frac{\mathrm{d}t}{2\pi} = \sum_{k=-n+1}^{n-1} \int_{\sigma_n^{(k)}-\alpha\pi/l}^{\sigma_n^{(k)}+\alpha\pi/l} f(t)\frac{\mathrm{d}t}{2\pi}$$

where $\mathrm{e}^{\mathrm{i}\sigma_n^{(k)}} = \exp(2k\pi\mu\mathrm{i}/l)$ is the midpoint of the arc $\mathbf{F}_n^{(k)}$, $k = -n+1, \ldots, -1, 0, 1, \ldots, n-1$.

On the other hand, by the definition of the Riemann integral and taking into occount that $c = \alpha/\mu$, we have

$$\int_{\mathbf{T}} cf(t)\chi_{I_\mu}(t)\,\mathrm{d}\lambda(t) = \frac{\alpha}{\mu}\lim_{n\to\infty} \sum_{k=-n+1}^{n-1} \Big[f(\mathrm{e}^{\mathrm{i}\sigma_n^{(k)}})\frac{\mu}{l}\Big]$$

$$= \lim_{n\to\infty}\Bigg[\frac{\alpha}{l}\sum_{k=-n+1}^{n-1} f(\mathrm{e}^{\mathrm{i}\sigma_n^{(k)}})\Bigg].$$

Now, let $\varepsilon > 0$ be given. There exists n_1 such that, for $n \geqslant n_1$,

$$\Big|\int_{\mathbf{T}} cf(t)\chi_{I_\mu}(t)\,\mathrm{d}\lambda(t) - \frac{\alpha}{l}\sum_{k=-n+1}^{n-1} f(\mathrm{e}^{\mathrm{i}\sigma_n^{(k)}})\Big| < \frac{\varepsilon}{2}. \tag{2.4}$$

Since f is continuous on the compact $\mathbf{T}$, there exists $\delta > 0$ such that, for $|t_1 - t_2| < \delta$, $t_1, t_2 \in \langle -\pi, \pi)$,

$$|f(t_1) - f(t_2)| < \frac{\varepsilon}{2\alpha}. \tag{2.5}$$

From (2.5) we obtain, for $n \geqslant n_2 = [\pi\alpha/\delta] + 1$,

$$\Bigg|\sum_{k=-n+1}^{n-1}\Big[f(\mathrm{e}^{\mathrm{i}\sigma_n^{(k)}})\frac{\alpha}{l}\Big] - \sum_{k=-n+1}^{n-1}\int_{\sigma_n^{(k)}-\pi\alpha/l}^{\sigma_n^{(k)}+\pi\alpha/l} f(t)\frac{\mathrm{d}t}{2\pi}\Bigg|$$

$$= \Bigg|\sum_{k=-n+1}^{n-1}\int_{\sigma_n^{(k)}-\pi\alpha/l}^{\sigma_n^{(k)}+\pi\alpha/l}\Big[f(\mathrm{e}^{\mathrm{i}\sigma_n^{(k)}}) - f(t)\Big]\frac{\mathrm{d}t}{2\pi}\Bigg| < \frac{\varepsilon}{2\alpha}\alpha = \frac{\varepsilon}{2}.$$

From this inequality and (2.3), (2.4) it follows that, for $n \geqslant \max(n_1, n_2)$,

$$\Big|\int_{\mathbf{T}} cf(t)\chi_{I_\mu}(t)\,\mathrm{d}\lambda(t) - \int_{\mathbf{T}} f(t)\chi_{\mathbf{F}_n}(t)\,\mathrm{d}\lambda(t)\Big| < \varepsilon$$

and Lemma 2.1 is proved. $\square$

Lemma 2.2. *Let $\alpha \in (0,1)$ be a given real number. Let $\sigma = \sum\limits_{k=1}^{n} c_k\chi_{I_k}$ be a step function on $\mathbf{T}$ such that*

(i) *$I_k \subset \mathbf{T}$, $k = 1, 2, \ldots, n$, are mutually disjoint arcs with $\lambda(I_k) = \mu_k > 0$,*

(ii) *$c_k \in (0, 1\rangle$,*

(iii) *$\sum\limits_{k=1}^{n} c_k\mu_k = \alpha$.*

Then there exists a sequence $\{\mathbf{F}_m\}_{m=1}^{\infty}$ of measurable subsets of $\bigcup\limits_{k=1}^{n} I_k \subset \mathbf{T}$ such that

$$\lambda(\mathbf{F}_m) = \alpha, \quad m = 1, 2, \ldots, \tag{2.6}$$

and

(2.7) *the sequence of measures $\chi_{\mathbf{F}_m}\,\mathrm{d}\lambda$ converges weakly to the measure $\sigma\,\mathrm{d}\lambda$.*

P r o o f. Denote $\alpha_k = c_k\mu_k$, $k = 1, \ldots, n$. Because of (i), (ii), (iii), we have $\alpha_k \in (0, \alpha\rangle$. By (ii), $\mu_k \geqslant \alpha_k$. Putting $I_\mu = I_k$, $k = 1, \ldots, n$, in Lemma 2.1, we see that, for each $k = 1, \ldots, n$, there exists a sequence $\{\mathbf{F}_m^{(k)}\}_{m=1}^{\infty}$ of measurable subsets $\mathbf{F}_m^{(k)} \subset I_k$ such that $\lambda(\mathbf{F}_m^{(k)}) = \alpha_k$, $m = 1, 2, \ldots$, and the measures $\chi_{\mathbf{F}_m}\,\mathrm{d}\lambda$ converge weakly to the measure $c_k\chi_{I_k}\,\mathrm{d}\lambda$. Putting $\mathbf{F}_m = \sum\limits_{k=1}^{n} \mathbf{F}_m^{(k)}$, we see that $\mathbf{F}_m$ is measurable, $\mathbf{F}_m \subset \bigcup\limits_{k=1}^{n} I_k$ and

$$\lambda(\mathbf{F}_m) = \sum_{k=1}^{n} \lambda(\mathbf{F}_m^{(k)}) = \sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} c_k\mu_k = \alpha,$$

so (2.6) is fulfilled.

On the other hand, the measures $\chi_{\mathbf{F}_m}\,\mathrm{d}\lambda = \sum\limits_{k=1}^{n} \chi_{\mathbf{F}_m^{(k)}}\,\mathrm{d}\lambda$ converge weakly to the measure $\big(\sum\limits_{k=1}^{n} c_k\chi_{I_k}\big)\,\mathrm{d}\lambda = \sigma\,\mathrm{d}\lambda$. The proof is complete. $\square$

Let $\alpha \in (0,1)$. Denote by $\mathcal{S}_\alpha$ the set of all step functions $\sigma = \sum\limits_{k=1}^{n} c_k \chi_{I_k}$ fulfilling conditions (i), (ii), (iii) from Lemma 2.2. Denote

$$\mathcal{L}_\alpha = \Big\{ f \in L^1(\mathbf{T});\ f(t) \in \langle 0,1\rangle \text{ a.e. } \mathrm{d}\lambda, \int_{\mathbf{T}} f(t)\,\mathrm{d}\lambda(t) = \alpha \Big\}.$$

Lemma 2.3. *$\mathcal{S}_\alpha$ is dense in $\mathcal{L}_\alpha$ in the metric of $L^1(\mathbf{T})$.*

Proof. Let $\varepsilon > 0$ and an arbitrary function $f \in \mathcal{L}_\alpha$ be given. We have to show that there exists a function $\sigma \in \mathcal{S}_\alpha$ such that $\int_{\mathbf{T}} |f - \sigma|\,\mathrm{d}\lambda < \varepsilon$.

Since the set of all continuous functions on $\mathbf{T}$ is dense in $L^1(\mathbf{T})$, there exists a continuous function g on $\mathbf{T}$, $g(t) \not\equiv 0$ on $\mathbf{T}$, such that

$$\int_{\mathbf{T}} |f - g|\,\mathrm{d}\lambda < \tfrac{1}{4}\varepsilon. \tag{2.8}$$

Put $h_1 = \min(1, g)$. Then $h_1(t) \not\equiv 0$ on $\mathbf{T}$ and

$$\int_{\mathbf{T}} |f - h_1|\,\mathrm{d}\lambda = \int_{\mathbf{T}_1} |f - h_1|\,\mathrm{d}\lambda + \int_{\mathbf{T}_2} |f - h_1|\,\mathrm{d}\lambda$$

where $\mathbf{T}_1 = \{t;\ g(t) \geqslant 1\}$, $\mathbf{T}_2 = \{t;\ g(t) < 1\}$. So, by the fact that $0 \leqslant f \leqslant 1$ and by (2.8), we have

$$\int_{\mathbf{T}} |f - h_1|\,\mathrm{d}\lambda = \int_{\mathbf{T}_1} (1 - f)\,\mathrm{d}\lambda + \int_{\mathbf{T}_2} |f - g|\,\mathrm{d}\lambda \leqslant \int_{\mathbf{T}_1} (g - f)\,\mathrm{d}\lambda + \int_{\mathbf{T}_2} |f - g|\,\mathrm{d}\lambda,$$

therefore

$$\int_{\mathbf{T}} |f - h_1|\,\mathrm{d}\lambda \leqslant \int_{\mathbf{T}} |f - g|\,\mathrm{d}\lambda < \tfrac{1}{4}\varepsilon. \tag{2.9}$$

Next we put $h = \max(0, h_1)$. Then h is a continuous function and we obtain

$$\begin{aligned}\int_{\mathbf{T}} |f - h|\,\mathrm{d}\lambda &= \int_{h_1 \leqslant 0} |f - h|\,\mathrm{d}\lambda + \int_{0 < h_1 \leqslant 1} |f - h|\,\mathrm{d}\lambda \\ &= \int_{h_1 \leqslant 0} (f - 0)\,\mathrm{d}\lambda + \int_{0 < h_1 \leqslant 1} |f - h_1|\,\mathrm{d}\lambda \\ &\leqslant \int_{h_1 \leqslant 0} (f - h_1)\,\mathrm{d}\lambda + \int_{0 < h_1 < 1} |f - h_1|\,\mathrm{d}\lambda,\end{aligned}$$

so, by (2.9), we have

$$\int\limits_{\mathbf{T}} |f-h| \,\mathrm{d}\lambda \leqslant \int\limits_{\mathbf{T}} |f-h_1| \,\mathrm{d}\lambda < \tfrac{1}{4}\varepsilon \tag{2.10}$$

and

$$0 \leqslant h \leqslant 1, \quad h(t) \not\equiv 0 \quad t \in \mathbf{T}. \tag{2.11}$$

Because of the uniform continuity of h on $\mathbf{T}$, there exists $\delta > 0$ such that

$$|h(t_1) - h(t_2)| < \tfrac{1}{4}\varepsilon \text{ for } |t_1 - t_2| < \delta, \quad t_1, t_2 \in \mathbf{T}. \tag{2.12}$$

Divide $\mathbf{T}$ into $n > 1/\delta$ mutually disjoint congruent arcs I_l, $\lambda(I_l) = \frac{1}{n}$, $l = 1, \ldots, n$, choose an arbitrary point $a_l \in I_l$ and put

$$s(t) = h(a_l) \ \text{ for } t \in I_l, \ l = 1, \ldots, n.$$

Of course, this choice may be made so that there exists l_0 such that

$$h(a_{l_0}) \neq 0. \tag{2.13}$$

From (2.12) we have $|h(t) - s(t)| < \frac{1}{4}\varepsilon$ and so

$$\int\limits_{\mathbf{T}} |h-s| \,\mathrm{d}\lambda < \tfrac{1}{4}\varepsilon. \tag{2.14}$$

Summing up, we have constructed a step function s fulfilling (i) and, by virtue of (2.11), (2.13), also (ii) from Lemma 2.2. Moreover, by (2.10) and (2.14),

$$\int\limits_{\mathbf{T}} |f-s| \,\mathrm{d}\lambda \leqslant \int\limits_{\mathbf{T}} |f-h| \,\mathrm{d}\lambda + \int\limits_{\mathbf{T}} |h-s| \,\mathrm{d}\lambda < \tfrac{1}{2}\varepsilon. \tag{2.15}$$

It remains to take account of condition (iii). In virtue of $\int_{\mathbf{T}} f \,\mathrm{d}\lambda = \alpha$, from (2.15) we get

$$\alpha - \tfrac{1}{2}\varepsilon < \int\limits_{\mathbf{T}} s \,\mathrm{d}\lambda < \alpha + \tfrac{1}{2}\varepsilon.$$

If $\int_{\mathbf{T}} s \,\mathrm{d}\lambda = \alpha$, the assertion is obvious. So, let first

$$\alpha < \int\limits_{\mathbf{T}} s \,\mathrm{d}\lambda < \alpha + \tfrac{1}{2}\varepsilon.$$

We have to modify s to a step function σ fulfilling (i), (ii) and, moreover,

$$\int_{\mathbf{T}} \sigma \,\mathrm{d}\lambda = \alpha, \quad \int_{\mathbf{T}} |f-\sigma| \,\mathrm{d}\lambda < \varepsilon.$$

So, for $c \in \langle 0,1\rangle$, let us consider the function

$$\varphi(c) = \int_{\mathbf{T}} \min(s,c) \,\mathrm{d}\lambda.$$

From the definition of $\min(s(t),c)$ we have

$$|\varphi(c) - \varphi(c+\Delta c)| < |\Delta c|,$$

so, φ is continuous on $\langle 0,1\rangle$. As $\min(s(t),0) \equiv 0$ and $\varphi(1) = \int_{\mathbf{T}} s \,\mathrm{d}\lambda$, we have $\varphi(0) = 0$ and $\varphi(1) > \alpha$. By the Bolzano theorem, there exists $c_0 \in (0,1)$ such that $\varphi(c_0) = \alpha$ and $0 < \int_{\mathbf{T}} (s - \min(s,c_0)) \,\mathrm{d}\lambda < \alpha + \frac{1}{2}\varepsilon - \alpha = \frac{1}{2}\varepsilon$.

Hence, putting $\sigma = \min(s,c_0)$, we have $\sigma \in \mathcal{S}_\alpha$ and

$$\int_{\mathbf{T}} |f-\sigma| \,\mathrm{d}\lambda \leqslant \int_{\mathbf{T}} |f-s| \,\mathrm{d}\lambda + \int_{\mathbf{T}} |s-\sigma| \,\mathrm{d}\lambda < \varepsilon.$$

If $\alpha - \frac{1}{2}\varepsilon < \int_{\mathbf{T}} s \,\mathrm{d}\lambda < \alpha$, we consider the function

$$\psi(c) = \int_{\mathbf{T}} \max(s,c) \,\mathrm{d}\lambda$$

for $c \in \langle 0,1\rangle$ and obtain quite similarly the desired result. Lemma 2.3 is proved. $\square$

Theorem 2.1. *The closure of $\mathcal{P}(B,b,\alpha)$ is the set of all functions $p \in \mathcal{P}$ which can be represented in the form*

$$p(z) = b + \frac{B-b}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z} \,\mathrm{d}t + (1-\eta) q(z), \tag{2.16}$$
$$z \in \mathbf{D}, \quad \eta = B\alpha + b(1-\alpha),$$

where f is an arbitrary measurable function on $\langle -\pi,\pi)$ with the following properties:

$$0 \leqslant f(t) \leqslant 1 \ \text{ a.e. on } \langle -\pi,\pi), \tag{2.17}$$

$$\tag{2.18} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \alpha,$$

and q is an arbitrary function in $\mathcal{P}$.

Proof. Of course, functions (2.16) are holomorphic in $\mathbf{D}$, $p(0) = 1$ and $\operatorname{Re} p(z) > b$ in $\mathbf{D}$, therefore $p \in \mathcal{P}$.

Let us recall the following well known fact. The set of positive measures μ of total mass 1, endowed with the topology given by the weak convergence of measures, is homeomorphic to the set $\mathcal{P}$, endowed with the topology given by the uniform convergence on compact sets in $\mathbf{D}$.

Let now p be a given function of the form (2.16). In virtue of Lemma 2.3, the function f can be approximated by functions $\sigma_n \in \mathcal{S}_\alpha$ fulfilling conditions (i), (ii) and (iii).

So, the functions

$$\frac{1}{2\pi\alpha}\int_{-\pi}^{\pi} \sigma_n(t)\frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}\,\mathrm{d}t$$

converge uniformly on compact sets in $\mathbf{D}$ to the function

$$\frac{1}{2\pi\alpha}\int_{-\pi}^{\pi} f(t)\frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}\,\mathrm{d}t.$$

Hence the functions $p_n \in \mathcal{P}(B, b, \alpha)$ of the form (2.16), where $f = \sigma_n$, converge uniformly on compact sets in $\mathbf{D}$ to the function p, so the assertion of Theorem 2.1 is proved. □

(b) An immediate consequence of Theorem 2.1 is the following

Corollary 2.1. *The closed convex hull of $\mathcal{P}(B, b, \alpha)$ is the same as in Theorem* 2.1.

It is essential here that the set of all functions f fulfilling conditions (2.17) and (2.18) is convex, and that the class $\mathcal{P}$ is convex.

(c) Let $A \subset \mathcal{P}(B, b, \alpha)$. Denote by A_0 the set of the characteristic functions of all sets $\mathbf{F} \subset \mathbf{T}$, $\lambda(\mathbf{F}) = \alpha$, on which (1.4) holds for some $p \in A$.

Theorem 2.2. *The subset $A \subset \mathcal{P}(B, b, \alpha)$ is compact if and only if the set A_0 is closed in $L^1(\mathbf{T})$.*

This theorem can be obtained by a thorough analysis of the proof of compactness of the class $\mathcal{P}(B,b,\alpha;\mathbf{F})$ (see Th. 3 in [6]). Detailed considerations will be carried out in the next paper.

3. Estimation of functional $\operatorname{Re}\{e^{i\lambda}p(z)\}$

A. In the paper [8] (Th.3), sharp estimates from below and from above of the functional $\operatorname{Re} p(z)$, $0 \neq z \in \mathbf{D}$, $p \in \mathcal{P}(B,b,\alpha)$, were established. At present, we will take up an analogous task, but for the functional $\operatorname{Re}\{e^{i\lambda}p(z)\}$ where λ is an arbitrary fixed parameter from the interval $\langle -\pi, \pi)$. We will also formulate some corollaries concerning the set of values of the functional $G(p) = p(z)$, $p \in \mathcal{P}(B,b,\alpha)$.

Let us first recall (see [3] Th. 4 and Th. 5, [6] Corol.2) that the extreme points in the class $\breve{\mathcal{P}}(B,b,\alpha;\mathbf{F})$ are of form

$$p(z;\gamma,\mathbf{F}) = b + \frac{B-b}{2\pi}\int\limits_{\mathbf{F}} \frac{e^{it}+z}{e^{it}-z}\,dt + (1-\eta)\frac{e^{i\gamma}+z}{e^{i\gamma}-z}, \quad \gamma \in \mathbf{R},\ z \in \mathbf{D}. \tag{3.1}$$

Since

$$\mathcal{P}(B,b,\alpha) = \bigcup_{\mathbf{F}} \breve{\mathcal{P}}(B,b,\alpha;\mathbf{F})$$

where the sum is taken over all subsets $\mathbf{F} \subset \mathbf{T}$ of Lebesgue measure $2\pi\alpha$, it is clear that, for $z \in \mathbf{D}$ fixed,

$$\begin{aligned}\inf(\sup)_{p\in\mathcal{P}(B,b,\alpha)} \operatorname{Re}\{e^{i\lambda}p(z)\} = {}& \inf(\sup)_{\gamma,\mathbf{F}}\{\operatorname{Re}[e^{i\lambda}p(z;\gamma,\mathbf{F})];\\ &\gamma \in \langle -\pi,\pi\rangle,\ \mathbf{F} \subset \mathbf{T},\ m(\mathbf{F}) = 2\pi\alpha\},\end{aligned} \tag{3.2}$$

where we denote by $m(\mathbf{E})$ the Lebesgue measure of a measurable set $\mathbf{E} \subset \mathbf{T}$, $m(\mathbf{T}) = 2\pi$. Since with each function $p \in \mathcal{P}(B,b,\alpha)$ the function $q(z) = p(\varepsilon z)$, $|\varepsilon| = 1$, $z \in \mathbf{D}$, is also contained in $\mathcal{P}(B,b,\alpha)$, it is sufficient to determine

$$\sup_{\gamma,\mathbf{F}}\{\operatorname{Re}[e^{i\lambda}p(r;\gamma,\mathbf{F})]\},\ \inf_{\gamma,\mathbf{F}}\{\operatorname{Re}[e^{i\lambda}p(r;\gamma,\mathbf{F})]\} \quad r \in (0,1). \tag{3.3}$$

For $z = 0$ we have $p(z) = 1$ for each function $p \in \mathcal{P}(B,b,\alpha)$, therefore the case $r = 0$ may be omitted.

Furthermore,

$$\frac{e^{i\gamma}+r}{e^{i\gamma}-r} = P_r(\gamma) + iQ_r(\gamma) \tag{3.4}$$

where

$$P_r(\gamma)=\frac{1-r^2}{1-2r\cos\gamma+r^2},\quad Q_r(\gamma)=-\frac{2r\sin\gamma}{1-2r\cos\gamma+r^2}, \tag{3.5}$$

so, from (3.4) and (3.5) we have

$$\operatorname{Re}\left\{e^{i\lambda}\frac{e^{i\gamma}+r}{e^{i\gamma}-r}\right\}=U_r(\gamma,\lambda),\ \gamma\in\langle-\pi,\pi),\ \lambda\in\langle-\pi,\pi),\ r\in(0,1), \tag{3.6}$$

where

$$U_r(\gamma,\lambda)=P_r(\gamma)\cos\lambda-Q_r(\gamma)\sin\lambda. \tag{3.7}$$

Hence our problem reduces to the following one:
Find

$$\sup_{\gamma\in\langle-\pi,\pi)}U_r(\gamma,\lambda),\quad \inf_{\gamma\in\langle-\pi,\pi)}U_r(\gamma,\lambda) \tag{3.8}$$

and

$$\sup\left\{\int_{\mathbf{F}}U_r(t,\lambda)\,dt;\quad \mathbf{F}\subset\mathbf{T},\ m(\mathbf{F})=2\pi\alpha\right\}, \tag{3.9}$$

$$\inf\left\{\int_{\mathbf{F}}U_r(t,\lambda)\,dt;\quad \mathbf{F}\subset\mathbf{T},\ m(\mathbf{F})=2\pi\alpha\right\}. \tag{3.10}$$

B. We will determine (3.8) first. For this purpose, let us notice that from (3.5)–(3.7) we obtain

$$U(\gamma)=:U_r(\gamma,\lambda)=\frac{c\cos\lambda+a\sin\lambda\cdot\sin\gamma}{1-a\cos\gamma} \tag{3.11}$$

where

$$a=\frac{2r}{1+r^2}\in(0,1),\quad c=\frac{1-r^2}{1+r^2}\in(0,1). \tag{3.12}$$

Consequently,

$$U'(\gamma)=\frac{aL(\gamma)}{(1-a\cos\gamma)^2} \tag{3.13}$$

where

$$L(\gamma) =: L_r(\gamma,\lambda) = (\cos\gamma - a)\sin\lambda - c\sin\gamma\cos\lambda. \tag{3.14}$$

From (3.14) and (3.12) we have

$$(1+r^2)L(\gamma) = r^2\sin(\gamma+\lambda) - 2r\sin\lambda - \sin(\gamma-\lambda). \tag{3.15}$$

Let $\lambda \in (0,\pi)$. Then the right-hand side of (3.15) is treated first as a function of the variable r with the parameters λ and γ. Next, from the results obtained by elementary considerations, we infer the behaviour of the function L of the variable γ. The case $\lambda = 0$ is examined as a limit case of the earlier results. Consequently, we get

Lemma 3.1. *Let $\gamma_1(r,\lambda)$, $\gamma_2(r,\lambda)$, $r \in (0,1)$, $\lambda \in \langle 0,\pi)$, be functions defined by the formulae*

$$\begin{aligned}\gamma_1(r) &=: \gamma_1(r,\lambda) = 2\arctan\left(\frac{1-r}{1+r}\tan\frac{\lambda}{2}\right),\\ \gamma_2(r) &=: \gamma_2(r,\lambda) = -2\arctan\left(\frac{1-r}{1+r}\cot\frac{\lambda}{2}\right)\end{aligned} \tag{3.16}$$

with

$$\begin{aligned}&\gamma_1(r,0)=0,\quad \gamma_1(r,\pi/2) = 2\arctan\frac{1-r}{1+r} = \arccos\frac{2r}{1+r^2},\\ &\lim_{\lambda\to\pi^-}\gamma_1(r,\lambda) = \pi,\quad r\in(0,1)\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}&\lim_{\lambda\to 0^+}\gamma_2(r,\lambda) = -\pi,\quad \gamma_2(r,\pi/2) = -2\arctan\frac{1-r}{1+r},\\ &\lim_{\lambda\to\pi^-}\gamma_2(r,\lambda) = 0,\quad r\in(0,1)\end{aligned} \tag{3.18}$$

whereas $L_r(\gamma,\lambda)$, $\gamma \in \langle -\pi,\pi)$, is the function (3.14)*. Then, for any $r \in (0,1)$ and $\lambda \in \langle 0,\pi)$, we have*

$$L_r(\gamma,\lambda)\begin{cases} <0 & \text{when}\quad -\pi \leqslant \gamma < \gamma_2(r,\lambda)\quad\text{or}\quad \gamma_1(r,\lambda)<\gamma<\pi,\\ =0 & \text{when}\quad \gamma=\gamma_2(r,\lambda)\quad\text{or}\quad \gamma=\gamma_1(r,\lambda),\\ >0 & \text{when}\quad \gamma_2(r,\lambda)<\gamma<\gamma_1(r,\lambda).\end{cases} \tag{3.19}$$

Formulae (3.11), (3.13) and Lemma 3.1 (including (3.19)) imply

Lemma 3.2. *For any $r \in (0,1)$, $\lambda \in \langle 0, \pi)$, the function* (3.11) *is* (i) *decreasing in the intervals $\langle -\pi, \gamma_2(r,\lambda))$, $(\gamma_1(r,\lambda), \pi)$,* (ii) *increasing in the interval $\langle \gamma_2(r,\lambda), \gamma_1(r,\lambda)\rangle$, and*

$$\max U(\gamma) = U_r(\gamma_1(r,\lambda),\lambda) = \frac{(1+r^2)\cos\lambda + 2r}{1-r^2}, \tag{3.20}$$

$$\min U(\gamma) = U_r(\gamma_2(r,\lambda),\lambda) = \frac{(1+r^2)\cos\lambda - 2r}{1-r^2}. \tag{3.21}$$

The functions $\gamma_1(r)$ and $\gamma_2(r)$ are defined by formulae (3.16).

Let $\lambda \in \langle -\pi, 0)$. Then $\widetilde{\lambda} = \lambda + \pi \in \langle 0, \pi)$ and from (3.7) we have

$$U_r(\gamma,\lambda) = -U_r(\gamma,\widetilde{\lambda}), \tag{3.22}$$

whereas from (3.16)

$$\gamma_2(r,\widetilde{\lambda}) = \gamma_1(r,\lambda), \quad \gamma_1(r,\widetilde{\lambda}) = \gamma_2(r,\lambda). \tag{3.23}$$

Consequently, (3.22), (3.23) and Lemma 3.2 imply

Lemma 3.3. *For any $r \in (0,1)$, $\lambda \in \langle -\pi, 0)$ the function* (3.11) *is:* (i) *decreasing in the interval $\langle \gamma_1(r,\lambda), \gamma_2(r,\lambda)\rangle$,* (ii) *increasing in the intervals $\langle -\pi, \gamma_1(r,\lambda))$, $(\gamma_2(r,\lambda), \pi)$, and*

$$\max U(\gamma) = U_r(\gamma_1(r,\lambda),\lambda) = \frac{(1+r^2)\cos\lambda + 2r}{1-r^2}, \tag{3.24}$$

$$\min U(\gamma) = U_r(\gamma_2(r,\lambda),\lambda) = \frac{(1+r^2)\cos\lambda - 2r}{1-r^2}. \tag{3.25}$$

The functions $\gamma_1(r)$ and $\gamma_2(r)$ are defined by formulae (3.23).

The bounds (3.8) have thus been determined.

Remark 3.1. Lemmas 3.2 and 3.3 are certainly known. They can be obtained directly from the fact that the set of values of the functional $H(p) = p(z)$, $0 \neq z \in \mathbf{D}$, $p \in \mathcal{P}$, is the disc $|w - s| \leqslant \varrho$ where $s = (1+r^2)/(1-r^2)$, $\varrho = 2r/(1-r^2)$. In the consideration carried out here, for each $\lambda \in \langle -\pi, \pi)$, the points $\gamma_1(r,\lambda)$ and $\gamma_2(r,\lambda)$ indispensable in further investigations and applications have been determined and examined.

C. We proceed to determine the bound (3.9).

Let $\lambda \in \langle 0, \pi)$. Since $U_r(\gamma)$ has the period 2π, we may suppose that $\mathbf{F} \subset \{e^{it};\ t \in \langle \gamma_2, \gamma_2 + 2\pi)\}$. Denote $\mathbf{F}_1 = \mathbf{F} \cap \{e^{it};\ t \in \langle \gamma_2, \gamma_1\rangle\}$ and $\mathbf{F}_2 = \mathbf{F} \cap \{e^{it};\ t \in (\gamma_1, \gamma_2 + 2\pi)\}$. Since (Lemma 3.2) the function $U_r(\gamma)$ is increasing in $\langle \gamma_2, \gamma_1\rangle$ and decreasing in $\langle \gamma_1, \gamma_2 + 2\pi\rangle$, so, applying the known lemma ([6], L. 4) to the intervals $\langle \gamma_2, \gamma_1\rangle$, $\langle \gamma_1, \gamma_2 + 2\pi\rangle$ and to the sets $\mathbf{F}_1$, $\mathbf{F}_2$, we obtain

$$
\begin{aligned}
\int_{\mathbf{F}_1} U_r(t,\lambda)\,\mathrm{d}t &\leqslant \int_{\gamma_1 - m(\mathbf{F}_1)}^{\gamma_1} U_r(t,\lambda)\,\mathrm{d}t,\\
\int_{\mathbf{F}_2} U_r(t,\lambda)\,\mathrm{d}t &\leqslant \int_{\gamma_1}^{\gamma_1 + m(\mathbf{F}_2)} U_r(t,\lambda)\,\mathrm{d}t,
\end{aligned}
\tag{3.26}
$$

thus by (3.26),

$$
\int_{\mathbf{F}} U_r(t,\lambda)\,\mathrm{d}t \leqslant \int_{\gamma_1 - m(\mathbf{F}_1)}^{\gamma_1 + m(\mathbf{F}_2)} U_r(t,\lambda)\,\mathrm{d}t. \tag{3.27}
$$

Here $m(\mathbf{F}_1) + m(\mathbf{F}_2) = 2\pi\alpha$, $0 \leqslant m(\mathbf{F}_1) \leqslant \gamma_1 - \gamma_2$, $0 \leqslant m(\mathbf{F}_2) \leqslant 2\pi + \gamma_2 - \gamma_1$. Hence denoting $m(\mathbf{F}_1) = 2\pi x$, $m(\mathbf{F}_2) = 2\pi y$ and

$$
\mu(x,y) =: \mu_r(x,y;\lambda) = \int_{\gamma_1 - 2\pi x}^{\gamma_1 + 2\pi y} U_r(t,\lambda)\,\mathrm{d}t, \tag{3.28}
$$

we realize that we have to determine

$$
\max_{(x,y)\in M_r(\alpha,\lambda)} \mu(x,y) \tag{3.29}
$$

where

$$
M_r(\alpha,\lambda) = \Big\{(x,y) \in \mathbf{R}^2;\ 0 \leqslant x \leqslant \frac{\gamma_1 - \gamma_2}{2\pi},\ 0 \leqslant y \leqslant 1 - \frac{\gamma_1 - \gamma_2}{2\pi},\ x + y = \alpha\Big\}. \tag{3.30}
$$

Put

$$
\gamma_0 = \gamma_0(r,\lambda) = \frac{\gamma_1 - \gamma_2}{2}. \tag{3.31}
$$

Then from (3.16)–(3.18) we have

$$
\gamma_0 = \arctan \frac{1 - r^2}{2r\sin\lambda}, \quad r \in (0,1), \quad \lambda \in (0,\pi). \tag{3.32}
$$

Since $r \in (0,1)$, $\lambda \in (0,\pi)$ and therefore $\gamma_0 \in (0,\pi/2)$ is fixed, we easily see that the set (3.30) is given by

$$\begin{aligned} &M_r(\alpha,\lambda) = \{(x,\alpha-x);\, 0 \leqslant x \leqslant \alpha\} \ \text{ if } \ 0<\alpha<\frac{\gamma_0}{\pi}, \\ (3.33)\quad &M_r(\alpha,\lambda) = \Big\{(x,\alpha-x);\, 0 \leqslant x \leqslant \frac{\gamma_0}{\pi}\Big\} \ \text{ if } \ \frac{\gamma_0}{\pi} \leqslant \alpha \leqslant 1-\frac{\gamma_0}{\pi}, \\ &M_r(\alpha,\lambda) = \Big\{(x,\alpha-x);\, \alpha-1+\frac{\gamma_0}{\pi} \leqslant x \leqslant \frac{\gamma_0}{\pi}\Big\} \ \text{ if } \ 1-\frac{\gamma_0}{\pi}<\alpha<1. \end{aligned}$$

Let

$$V(x) = \mu(x,\alpha-x). \tag{3.34}$$

By (3.28) and (3.34),

$$V'(x) = -2\pi[U_r(\gamma_1-2\pi x+2\pi\alpha,\lambda) - U_r(\gamma_1-2\pi x,\lambda)], \tag{3.35}$$

$$V''(x) = 4\pi^2[U_r'(\gamma_1-2\pi x+2\pi\alpha,\lambda) - U_r'(\gamma_1-2\pi x,\lambda)]. \tag{3.36}$$

By (3.33), $0 \leqslant x \leqslant \gamma_0/\pi$ and consequently, $\gamma_1-2\pi x \in \langle\gamma_2,\gamma_1\rangle$, so $U_r'(\gamma_1-2\pi x,\lambda)>0$ for $x \in (0,\gamma_0/\pi)$. From (3.33) we also have $\gamma_1+2\pi(\alpha-x) \in \langle\gamma_1, 2\pi+\gamma_2\rangle$, therefore $U_r'(\gamma_1-2\pi x+2\pi\alpha,\lambda)<0$ for $x\in(0,\gamma_0/\pi)$. Consequently, from (3.36) we infer that $V''(x)<0$, i.e. that $V'(x)$ is a strictly decreasing function in the corresponding interval $\langle 0,\alpha\rangle$ or $\langle 0,\gamma_0/\pi\rangle$ or $\langle\alpha-1+\gamma_0/\pi,\gamma_0/\pi\rangle$. But from (3.35) and (3.31) we have

$$\begin{aligned} V'(0) &= -2\pi[U_r(\gamma_1+2\pi\alpha,\lambda)-U_r(\gamma_1,\lambda)], \\ V'(\alpha) &= -2\pi[U_r(\gamma_1,\lambda)-U_r(\gamma_1-2\pi\alpha,\lambda)], \\ V'(\gamma_0/\pi) &= -2\pi[U_r(\gamma_2+2\pi\alpha,\lambda)-U_r(\gamma_2,\lambda)], \\ V'(\alpha-1+\gamma_0/\pi) &= -2\pi[U_r(\gamma_2+2\pi)-U_r(\gamma_2+2\pi-2\pi\alpha)]. \end{aligned} \tag{3.37}$$

Since $\alpha\in(0,1)$ and the function U_r attains the maximum at the point γ_1 and the minimum at the point γ_2, therefore from (3.37) we have

$$V'(0)>0,\ V'(\alpha)<0,\ V'(\gamma_0/\pi)<0,\ V'(\alpha-1+\gamma_0/\pi)>0. \tag{3.38}$$

From (3.38) we see by (3.33) that, for any $\alpha\in(0,1)$, $\lambda\in(0,\pi)$, $\gamma_0\in(0,\pi/2)$, there exists a unique root x_0 of the equation $V'(x)=0$ with $x_0\in I_r(\alpha,\lambda)$ where

$$I_r(\alpha,\lambda) = \begin{cases} (0,\alpha) \ \text{ if } \ 0<\alpha<\gamma_0/\pi, \\ (0,\gamma_0/\pi) \ \text{ if } \ \gamma_0/\pi\leqslant\alpha\leqslant 1-\gamma_0/\pi, \\ (\alpha-1+\gamma_0/\pi,\gamma_0/\pi) \ \text{ if } \ 1-\gamma_0/\pi<\alpha<1. \end{cases} \tag{3.39}$$

Here γ_0 is from (3.32).

So, by (3.35), $x_0 = x_0(r, \alpha, \lambda)$ is given by the equation

$$U_r(\gamma_1 - 2\pi x + 2\pi\alpha, \lambda) = U_r(\gamma_1 - 2\pi x, \lambda). \tag{3.40}$$

From (3.11), (3.12) and (3.40) we obtain an equation of the form

$$r^2 \sin(\gamma_1 - 2\pi x + \pi\alpha + \lambda) - 2r \sin\lambda \cdot \cos\pi\alpha - \sin(\gamma_1 - 2\pi x + \pi\alpha - \lambda) = 0. \tag{3.41}$$

Since $V'(x)$ is a stricly increasing function in a suitable variability interval of x and $V'(x_0) = 0$ only at the point x_0, therefore

$$\max V(x) = V(x_0). \tag{3.42}$$

In view of (3.34), (3.29), (3.28) and (3.27), we have determined the bound (3.9) for $\lambda \in (0, \pi)$.

If $\lambda = 0$, then from (3.17), (3.18), (3.31), (3.33) and, next, from (3.11), (3.35) and (3.36) we easily conclude that

$$\max V(x) = V(x_0), \quad x_0 = \tfrac{1}{2}\alpha.$$

Hence the case $\lambda = 0$ may be added to the case $\lambda \in (0, \pi)$ considered before.

So we have

Lemma 3.4. *For any $r \in (0, 1)$, $\lambda \in \langle 0, \pi)$ we have*

$$\max_x \mu(x, \alpha - x) = \mu(x_0, \alpha - x_0) \tag{3.43}$$

where $x = x_0$ is the only root of equation (3.41) *belonging to the interval $I_r(\alpha, \lambda)$ defined in* (3.39). *The set $\mathbf{F}^* \subset \mathbf{T}$ for which the function* (3.28) *attains its maximum* (3.43) *is of the form*

$$\mathbf{F}^* = \{e^{it};\, t \in \langle \gamma_1(r, \lambda) - 2\pi x_0, \gamma_1(r, \lambda) + 2\pi\alpha - 2\pi x_0 \rangle\}. \tag{3.44}$$

Let $\lambda \in (-\pi, 0)$. Put $\widetilde{\lambda} = -\lambda$. Then $\widetilde{\lambda} \in (0, \pi)$ and from (3.28) and Lemma 3.4 we get

$$\max_x \mu_r(x, \alpha - x; \widetilde{\lambda}) = \mu_r(x_0, \alpha - x_0, \widetilde{\lambda})$$

where x_0 is the only root of the equation

$$r^2 \sin(\gamma_1(r, \widetilde{\lambda}) - 2\pi x + \pi\alpha + \widetilde{\lambda}) - 2r \sin\widetilde{\lambda} \cdot \cos\alpha\pi - \sin(\gamma_1(r, \widetilde{\lambda}) - 2\pi x + \pi\alpha - \widetilde{\lambda}) = 0$$

belonging to the interval $I_r(\alpha, \widetilde{\lambda})$.

It remains to use the fact that $\widetilde{\lambda} = -\lambda$ in the formulae for the above-mentioned functions. It also turns out that the case $\lambda = -\pi$ may be added to the previous results. So, we have

Lemma 3.5. *For any $r \in (0,1)$, $\lambda \in \langle -\pi, 0)$,*

$$\max_x \mu(x, \alpha - x) = \mu(\alpha - y_0, y_0) \tag{3.45}$$

where $x = y_0$ is the only root of the equation

$$r^2 \sin(\gamma_1(r,\lambda) + 2\pi x - \pi\alpha + \lambda) - 2r \sin\lambda \cdot \cos\pi\alpha - \sin(\gamma_1(r,\lambda) + 2\pi x - \pi\alpha - \lambda) = 0. \tag{3.46}$$

belonging to the interval

$$J_r(\alpha,\lambda) = \begin{cases} (0,\alpha) & \text{if} \quad 0 < \alpha < -\gamma_0/\pi, \\ (0, -\gamma_0/\pi) & \text{if} \quad -\gamma_0/\pi \leqslant \alpha \leqslant 1 + \gamma_0/\pi, \\ (\alpha - 1 - \gamma_0/\pi, -\gamma_0/\pi) & \text{if} \quad 1 + \gamma_0/\pi < \alpha < 1. \end{cases} \tag{3.47}$$

The set $\mathbf{F}^{**} \subset \mathbf{T}$ for which the function (3.28) *attains its maximum* (3.45) *is of the form*

$$\mathbf{F}^{**} = \{\mathrm{e}^{\mathrm{i}t};\, t \in \langle \gamma_1(r,\lambda) - 2\pi\alpha + 2\pi y_0, \gamma_1(r,\lambda) + 2\pi y_0 \rangle\}. \tag{3.48}$$

Consequently, in order to determine (3.9), one should calculate the integral

$$K = K(r,\lambda,t_1,t_2) = \int_{t_1}^{t_2} U_r(t,\lambda)\,\mathrm{d}t, \tag{3.49}$$

where $\langle t_1, t_2\rangle$ is the interval $\langle \gamma_1 - 2\pi x_0, \gamma_1 - \pi x_0 + 2\pi\,\mathrm{d}\rangle$ or $\langle \gamma_1 + 2\pi y_0 - 2\pi\alpha, \gamma_1 + 2\pi y_0\rangle$. It turns out that

$$K = 2\cos\lambda \cdot \arctan\left[c \frac{\sin\frac{t_2 - t_1}{2}}{\cos\frac{t_2-t_1}{2} - a\cos\frac{t_2+t_1}{2}}\right] - \sin\lambda \cdot \log\frac{1 - a\cos t_1}{1 - a\cos t_2}. \tag{3.50}$$

Let us determine $\mu(x_0, \alpha - x_0)$ and $\mu(\alpha - y_0, y_0)$. From (3.28) and (3.49) we get

$$\begin{aligned} \mu(x_0, \alpha - x_0) &= K(r,\lambda,\gamma_1 - 2\pi x_0, \gamma_1 - 2\pi x_0 + 2\pi\alpha), \quad \lambda \in \langle 0, \pi), \\ \mu(\alpha - y_0, y_0) &= K(r,\lambda,\gamma_1 + 2\pi y_0 - 2\pi\alpha, \gamma_1 + 2\pi y_0), \quad \lambda \in \langle -\pi, 0). \end{aligned} \tag{3.51}$$

Consequently, from (3.50) and (3.51) we get

$$
\begin{aligned}
\mu(x_0, \alpha - x_0) = 2\cos\lambda \cdot \arctan\left[\frac{c\sin\pi\alpha}{\cos\pi\alpha - a\cos(\gamma_1 - 2\pi x_0 + \pi\alpha)}\right] \\
- \sin\lambda \cdot \log\frac{1 - a\cos(\gamma_1 - 2\pi x_0)}{1 - a\cos(\gamma_1 - 2\pi x_0 + 2\pi\alpha)},
\end{aligned}
\tag{3.52}
$$

$$
\begin{aligned}
\mu(\alpha - y_0, y_0) = 2\cos\lambda \cdot \arctan\left[\frac{c\sin\pi\alpha}{\cos\pi\alpha - a\cos(\gamma_1 + 2\pi y_0 - \pi\alpha)}\right] \\
- \sin\lambda \cdot \log\frac{1 - a\cos(\gamma_1 + 2\pi y_0 - 2\pi\alpha)}{1 - a\cos(\gamma_1 + 2\pi y_0)}.
\end{aligned}
\tag{3.53}
$$

Summing up, (3.1–3.3), (3.6), (3.27), (3.28) and Lemmas 3.2–3.5 imply

Theorem 3.1. *Let* $p \in \mathcal{P}(B, b, \alpha)$, $0 \neq z \in \mathbf{D}$, $z = re^{i\varphi}$, $\lambda \in \langle -\pi, \pi)$. *Then*

$$
\operatorname{Re}\{e^{i\lambda} p(z)\} \leqslant
\begin{cases}
b\cos\lambda + \frac{B-b}{2\pi}\mu(x_0, \alpha - x_0) + (1-\eta)\frac{(1+r^2)\cos\lambda + 2r}{1-r^2} \\
\text{if } \lambda \in \langle 0, \pi), \\
b\cos\lambda + \frac{B-b}{2\pi}\mu(\alpha - y_0, y_0) + (1-\eta)\frac{(1+r^2)\cos\lambda + 2r}{1-r^2} \\
\text{if } \lambda \in \langle -\pi, 0),
\end{cases}
\tag{3.54}
$$

where $\mu(x_0, \alpha - x_0)$ *and* $\mu(\alpha - y_0, y_0)$ *are functions of the form* (3.52) *and* (3.53), *such that* $x = x_0 \in I_r(\alpha, \lambda)$, $x = y_0 \in J_r(\alpha, \lambda)$ *are the only roots of equations* (3.41) *and* (3.46), *respectively, while*

$$
\begin{aligned}
& a = \frac{2r}{1+r^2}, \quad c = \frac{1-r^2}{1+r^2}, \quad \eta = \alpha B + (1-\alpha)b; \\
& \gamma_1 = \gamma_1(r, \lambda) = 2\arctan\left(\frac{1-r}{1+r}\tan\frac{\lambda}{2}\right), \\
& \gamma_1 \in \langle 0, \pi) \text{ when } \lambda \in \langle 0, \pi), \quad \gamma_1 \in \langle -\pi, 0) \text{ when } \lambda \in \langle -\pi, 0);
\end{aligned}
$$

$$
I_r(\alpha, \lambda) =
\begin{cases}
(0, \alpha) \text{ if } \alpha \in \langle 0, \gamma_0/\pi), \\
(0, \gamma_0/\pi) \text{ if } \alpha \in \langle \gamma_0/\pi, 1 - \gamma_0/\pi\rangle, \\
(\alpha - 1 + \gamma_0/\pi, \gamma_0/\pi) \text{ if } \alpha \in (1 - \gamma_0/\pi, 1),
\end{cases}
\tag{3.55}
$$

$$
\gamma_0 = \arctan\frac{1-r^2}{2r\sin\lambda} \text{ for } \lambda \in \langle 0, \pi) \text{ and } \gamma_0 \in (0, \pi/2\rangle;
$$

$$J_r(\alpha,\lambda)=\begin{cases}(0,\alpha) \text{ if } \alpha\in\langle 0,\eta_0/\pi),\\ (0,\eta_0/\pi) \text{ if } \alpha\in\langle \eta_0/\pi, 1-\eta_0/\pi\rangle,\\ (\alpha-1+\eta_0/\pi,\eta_0/\pi) \text{ if } \alpha\in(1-\eta_0/\pi,1),\end{cases}$$
$$\eta_0=-\gamma_0\in(0,\pi/2\rangle \text{ for } \lambda\in\langle-\pi,0).$$

R e m a r k 3.2. Since the functions (3.1) belong to the class $\mathcal{P}(B,b,\alpha)$, the estimates (3.54) are sharp. If $z=r\in(0,1)$, then in (3.1) one should put $\gamma=\gamma_1$, while for the set $\mathbf{F}\subset\mathbf{T}$ one should take $\mathbf{F}^*$ or $\mathbf{F}^{**}$, respectively (see (3.44) and (3.48)). This follows from (3.20), (3.24) and (3.28), (3.43), (3.45). If $z=re^{i\varphi}$, then one should take into account the argument φ – performing suitable rotations of the number $e^{i\gamma_1}$ as well as of the sets $\mathbf{F}^*$ and $\mathbf{F}^{**}$.

R e m a r k 3.3. Passing $\lambda\to 0$ and $\lambda\to-\pi$ in Theorem 3.1, we obtain the well-known estimate from below and from above of $\operatorname{Re} p(z)$, $p\in\mathcal{P}(B,b,\alpha)$, ([8], Th.3) of the form

$$\begin{aligned}&b+\frac{2(B-b)}{\pi}\arctan\left(\frac{1-r}{1+r}\tan\frac{\pi\alpha}{2}\right)+(1-\eta)\frac{1-r}{1+r}\leqslant \operatorname{Re} p(z)\\ &\leqslant b+\frac{2(B-b)}{\pi}\arctan\left(\frac{1+r}{1-r}\tan\frac{\pi\alpha}{2}\right)+(1-\eta)\frac{1+r}{1-r}.\end{aligned}$$

R e m a r k 3.4. Putting in the above-mentioned theorem $\lambda=\pm\pi/2$, we obtain an estimate of $\operatorname{Im} p(z)$ in the class $\mathcal{P}(B,b,\alpha)$, also well known ([8], Th.3), of the form

$$\begin{aligned}&\frac{B-b}{2\pi}\log\frac{P_r(\gamma_0+2\pi\alpha-2\pi x_0)}{P_r(\gamma_0-2\pi x_0)}-(1-\eta)\frac{2r}{1-r^2}\leqslant \operatorname{Im} p(z)\\ &\leqslant\frac{B-b}{2\pi}\log\frac{P_r(\gamma_0-2\pi x_0)}{P_r(\gamma_0+2\pi\alpha-2\pi x_0)}+(1-\eta)\frac{2r}{1-r^2}\end{aligned}$$

where $x_0\in I_r(\alpha,\pi/2)\subset(0,\gamma_0/\pi)$ is a root of the equation $\cos(\gamma_0-2\pi x_0+\pi\alpha)=\cos\gamma_0\cdot\cos\alpha\pi$, $\cos\gamma_0=2r/(1+r^2)$.

D. From Theorem 3.1 one can, of course, obtain an estimate from below of the functional $\operatorname{Re}(e^{i\lambda}p(z))$, that is, determine the bound (3.10).

Let, for instance, $\lambda\in\langle 0,\pi)$. Put $\widetilde{\lambda}=\lambda-\pi$. Then $\widetilde{\lambda}\in\langle-\pi,0)$. Consequently, from (3.54) we get

$$\operatorname{Re}(e^{i\widetilde{\lambda}}p(z))\leqslant b\cos\widetilde{\lambda}+\frac{B-b}{2\pi}\mu_r(\alpha-y_0,y_0,\widetilde{\lambda})+(1-\eta)\frac{(1+r^2)\cos\widetilde{\lambda}+2r}{1-r^2},$$

thus

$$\operatorname{Re}(\mathrm{e}^{\mathrm{i}\lambda}p(z)) \geqslant b\cos\lambda - \frac{B-b}{2\pi}\mu_r(\alpha - y_0, y_0, \lambda+\pi) + (1-\eta)\frac{(1+r^2)\cos\lambda - 2r}{1-r^2}.$$

Since

$$\begin{aligned}\gamma_1(r,\widetilde{\lambda}) = 2\arctan\left(\frac{1-r}{1+r}\tan\frac{\lambda-\pi}{2}\right) &= -2\arctan\left(\frac{1-r}{1+r}\cot\frac{\lambda}{2}\right)\\ &= \gamma_2(r,\lambda) \in \langle -\pi, 0),\end{aligned}$$

equation (3.46) for $x = y_0$ assumes the form

$$r^2\sin(\gamma_2(r,\lambda) + 2\pi x - \pi\alpha + \lambda) - 2r\sin\lambda\cdot\cos\pi\alpha - \sin(\gamma_2(r,\lambda) + 2\pi x - \pi\alpha - \lambda) = 0.$$

Moreover,

$$\eta_0(r,\widetilde{\lambda}) = -\arctan\frac{1-r^2}{2r\sin(\lambda-\pi)} = \arctan\frac{1-r^2}{2r\sin\lambda} = \gamma_0(r,\lambda) \in (0,\pi/2),$$
$$J_r(\alpha\widetilde{\lambda}) = I_r(\alpha,\lambda).$$

If $\lambda \in \langle -\pi, 0)$, then $\widetilde{\widetilde{\lambda}} =: \pi + \lambda \in \langle 0, \pi)$. So, from (3.54) we have

$$\operatorname{Re}(\mathrm{e}^{\mathrm{i}\lambda}p(z)) \geqslant b\cos\lambda - \frac{B-b}{2\pi}\mu_r(x_0, \alpha - x_0, \widetilde{\widetilde{\lambda}}) + (1-\eta)\frac{(1+r^2)\cos\lambda - 2r}{1-r^2}.$$

Besides,

$$\begin{aligned}\gamma_1(r,\widetilde{\widetilde{\lambda}}) &= \gamma_2(r,\lambda) \in \langle 0,\pi),\\ \gamma_0(r,\widetilde{\widetilde{\lambda}}) &= \eta_0(r,\lambda) \in (0,\pi/2),\end{aligned}$$

therefore equation (3.41) for $x = x_0$ will assume the form

$$r^2\sin(\gamma_2(r,\lambda) - 2\pi x + \pi\alpha + \lambda) - 2r\sin\lambda\cdot\cos\pi\alpha - \sin(\gamma_2(r,\lambda) - 2\pi x + \pi\alpha - \lambda) = 0,$$

and

$$I_r(\alpha,\widetilde{\widetilde{\lambda}}) = J_r(\alpha,\lambda).$$

From the above considerations and (3.52), (3.53) we obtain

Theorem 3.2. *Let* $p \in \mathcal{P}(B,b,\alpha)$, $0 \neq z \in \mathbf{D}$, $z = r\mathrm{e}^{\mathrm{i}\varphi}$, $\lambda \in \langle -\pi, \pi)$. *Then*

$$\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\lambda}p(z)\} \geqslant \begin{cases} b\cos\lambda - \frac{B-b}{2\pi}\widetilde{\mu}(\alpha - y_0, y_0) + (1-\eta)\frac{(1+r^2)\cos\lambda - 2r}{1-r^2} \\ \text{if } \lambda \in \langle 0,\pi), \\ b\cos\lambda - \frac{B-b}{2\pi}\widetilde{\mu}(x_0, \alpha - x_0) + (1-\eta)\frac{(1+r^2)\cos\lambda - 2r}{1-r^2} \\ \text{if } \lambda \in \langle -\pi, 0), \end{cases} \tag{3.56}$$

where

$$\begin{aligned}
\widetilde{\mu}(\alpha-y_0,y_0)= & -2\cos\lambda\cdot\arctan\left[\frac{c\sin\pi\alpha}{\cos\pi\alpha-a\cos(\gamma_2+2\pi y_0-\pi\alpha)}\right]\\
& +\sin\lambda\cdot\log\frac{1-a\cos(\gamma_2+2\pi y_0-2\pi\alpha)}{1-a\cos(\gamma_2+2\pi y_0)},\\
\widetilde{\mu}(x_0,\alpha-x_0)= & -2\cos\lambda\cdot\arctan\left[\frac{c\sin\pi\alpha}{\cos\pi\alpha-a\cos(\gamma_2-2\pi x_0+\pi\alpha)}\right]\\
& +\sin\lambda\cdot\log\frac{1-a\cos(\gamma_2-2\pi x_0)}{1-a\cos(\gamma_2-2\pi x_0+2\pi\alpha)};
\end{aligned}$$

$x=y_0\in I_r(\alpha,\lambda)$ *satisfies the equation*

$$r^2\sin(\gamma_2+2\pi x-\pi\alpha+\lambda)-2r\sin\lambda\cdot\cos\pi\alpha-\sin(\gamma_2+2\pi x-\pi\alpha-\lambda)=0,$$

$x=x_0\in J_r(\alpha,\lambda)$ *satisfies the equation*

$$r^2\sin(\gamma_2-2\pi x+\pi\alpha+\lambda)-2r\sin\lambda\cdot\cos\pi\alpha-\sin(\gamma_2-2\pi x+\pi\alpha-\lambda)=0;$$

$$\gamma_2=\gamma_2(r,\lambda)=-2\arctan\left(\frac{1-r}{1+r}\cot\frac{\lambda}{2}\right),$$

$$\gamma_2\in\langle-\pi,0)\text{ for }\lambda\in\langle0,\pi),\ \ \gamma_2\in\langle0,\pi)\text{ for }\lambda\in\langle-\pi,0);$$

$$\gamma_0=\gamma_0(r,\lambda)=\arctan\frac{1-r^2}{2r\sin\lambda},$$

$$\gamma_0\in(0,\pi/2\rangle\text{ for }\lambda\in\langle0,\pi),\ \ \gamma_0\in\langle-\pi/2,0)\text{ for }\lambda\in\langle-\pi,0),$$

$$\eta_0=\eta_0(r,\lambda)=-\gamma_0;$$

$I_r(\alpha,\lambda)$ *and* $J_r(\alpha,\lambda)$ *are defined by the formulae from* (3.55). *Estimates* (3.56) *are sharp.*

E. Let $B\leqslant 1$ and $p_0(z)\equiv 1$. Then $p_0\in\mathcal{P}(B,b,\alpha)$ for any admissible parameters b and α. As we know, functions (3.1) belong to $\mathcal{P}(B,b,\alpha)$ (of course, for sets $\mathbf{F}\subset\mathbf{T}$ of measure $2\pi\alpha$). It follows directly from the definition that the functions

$$q_\beta(z;\gamma,\mathbf{F})=\beta p(z;\gamma,\mathbf{F})+(1-\beta)p_0(z),\ \ z\in\mathbf{D},\ \ \beta\in(0,1),\tag{3.57}$$

belong to this class, too.

Let

(3.58)

$$\omega(\lambda)=\begin{cases}(b-1)\cos\lambda+\frac{B-b}{2\pi}\mu(x_0,\alpha-x_0)+(1-\eta)\frac{(1+r^2)\cos\lambda+2r}{1-r^2}, & \lambda\in\langle0,\pi),\\ (b-1)\cos\lambda+\frac{B-b}{2\pi}\mu(y_0,\alpha-y_0)+(1-\eta)\frac{(1+r^2)\cos\lambda+2r}{1-r^2}, & \lambda\in\langle-\pi,0),\end{cases}$$

with the notation and conditions from Theorem 3.1 being valid. Making use of the theorem just mentioned, we will determine the set

$$\mathbf{P} = \{p(z) - 1,\ p \in \mathcal{P}(B, b, \alpha)\},\ 0 \neq z \in \mathbf{D},\ B \leqslant 1. \tag{3.59}$$

We have

Theorem 3.3. *The boundary of the set* $\mathbf{P}$ *of values of the functional* $H(p) = p(z) - 1$, $p \in \mathcal{P}(B, b, \alpha)$, $B \leqslant 1$, *is a curve with the equation*

$$w = \omega(\lambda)\mathrm{e}^{-\mathrm{i}\lambda}, \quad \lambda \in \langle -\pi, \pi), \tag{3.60}$$

where ω *is defined by formula* (3.58).

P r o o f. Let $0 \neq z \in \mathbf{D}$, $z = r\mathrm{e}^{\mathrm{i}\varphi}$. In virtue of Theorem 3.1 and (3.58), for any function $p \in \mathcal{P}(B, b, \alpha)$, $\lambda \in \langle -\pi, \pi)$,

$$\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\lambda}(p(z) - 1)\} \leqslant \omega(\lambda), \tag{3.61}$$

so that, in accordance with Remark 3.2, there exists γ^* and a set $\mathbf{F}^* \subset \mathbf{T}$ of measure $2\pi\alpha$ such that, for a function p^* defined, by formula (3.1),

$$\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\lambda}(p^*(z) - 1)\} = \omega(\lambda). \tag{3.62}$$

Since the function $p_0 \in \mathcal{P}(B, b, \alpha)$, therefore $0 \in \mathbf{P}$. Moreover, this and (3.61) imply that $\omega(\lambda) \geqslant 0$ for $\lambda \in \langle -\pi, \pi)$.

If $\omega(\lambda_0) = 0$ for some λ_0, then $\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\lambda_0}\ (p(z) - 1)\} \leqslant 0$ for all functions $p \in \mathcal{P}(B, b, \alpha)$. Hence all points $p(z) - 1$ of the set $\mathbf{P}$ lie in one half-plane passing through the point 0 and inclined under the angle of $\pi/2 - \lambda$ to the real axis. Consequently, the point $\omega(\lambda_0)\mathrm{e}^{\mathrm{i}\lambda_0} = 0$ is a boundary point of the set $\mathbf{P}$.

Let $\omega(\lambda) > 0$ and $p^*(z) - 1 = |p^*(z) - 1|\mathrm{e}^{\mathrm{i}\psi^*}$. Then, by (3.62),

$$p^*(z) - 1 = \omega(\lambda)\mathrm{e}^{-\mathrm{i}\lambda}. \tag{3.63}$$

Since $p^* \in \mathcal{P}(B, b, \alpha)$, the point (3.63) belongs to the set (3.59). Besides, from (3.57) we deduce that, for each $\beta \in (0, 1)$, the function $q^*_\beta = \beta p^* + (1 - \beta)p_0 \in \mathcal{P}(B, b, \alpha)$, therefore the point $q^*_\beta(z) - 1 = \beta(p^*(z) - 1) + (1 - \beta)(p_0(z) - 1) = \beta\omega(\lambda)\mathrm{e}^{-\mathrm{i}\lambda} \in \mathbf{P}$ for each $\beta \in (0, 1)$.

Let p be an arbitrary fixed function of the class $\mathcal{P}(B, b, \alpha)$. Put $p(z) - 1 = |p(z) - 1|\mathrm{e}^{\mathrm{i}\psi}$. Choose $\lambda = -\psi$. From (3.61) we then have $|p(z) - 1| \leqslant \omega(-\psi)$, hence the point $p(z) - 1$ lies on the segment $\langle 0, \omega(-\psi)\mathrm{e}^{\mathrm{i}\psi}\rangle = \langle 0, \omega(\lambda)\mathrm{e}^{-\mathrm{i}\lambda}\rangle \subset \mathbf{P}$.

Since ω is a continuous function, $\omega(\lambda) > 0$, $\omega(-\pi) = \lim\limits_{\lambda \to \pi^-} \omega(\lambda)$, Theorem 3.3 has been proved. $\square$

Corollary 3.1. *If $B \leqslant 1$, then the set* $\mathbf{Q}$ *of values of the function* $p(z)$, $p \in \mathcal{P}(B, b, \alpha)$, *results from the set* $\mathbf{P}$ *by translation by 1.*

Remark 3.5. Separate considerations are needed in the case $B > 1$. In the proof of Theorem 3.3 we were using, among other things, the fact that $p_0 \in \mathcal{P}(B, b, \alpha)$ and we managed to avoid an obstacle caused by the lack of convexity of the class $\mathcal{P}(B, b, \alpha)$.

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