UNIONS OF UNIQUELY COMPLEMENTED LATTICES

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Abstract. In this paper we generalize a result of V. N. Salij concerning direct product decompositions of lattices which are complete and uniquely complemented.

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All lattices under consideration in the present note are assumed to have the least element. When no misunderstanding can occur, this element will be denoted by 0.

Let $\mathcal{U}$ be the class of all uniquely complemented lattices (i.e. lattices having the least and the greatest element in which each element possesses one and only one complement). The importance of the class $\mathcal{U}$ is emphasized by the well-known fact that each lattice can be isomorphically embedded into a lattice belonging to $\mathcal{U}$ (Dilworth [1]).

For a lattice L we denote by $c_0(L)$ the system of all convex sublattices L_i of L with $0 \in L_i$. Let $\mathcal{U}_1$ be the class of all lattices L such that L can be expressed as a union $\bigcup\limits_{i\in I} L_i$, where each L_i $(i \in I)$ is a complete lattice belonging to $\mathcal{U} \cap c_0(L)$.

A lattice L is called a generalized Boolean algebra if for each $0 < x \in L$, the interval $[0, x]$ is a Boolean algebra.

In the present note the following theorem will be proved:

(A) Every lattice L belonging to $\mathcal{U}_1$ is isomorphic to a direct product $A_L \times B_L$ such that A_L is an atomic generalized Boolean algebra and B_L is a lattice which belongs to $\mathcal{U}_1$ and has no atoms.

This generalizes a result of V. N. Salij (which was announced in [2] and published with a complete proof in [3]), namely,

(B) (Salij) Every complete uniquely complemented lattice is isomorphic to a direct product of a complete atomic Boolean algebra and a complete atomless uniquely complemented lattice.

1. Direct product decompositions

Let L be a lattice and let φ be an isomorphism of L onto the direct product $A \times B$ of lattices A and B. It is obvious that the lattice L is complete if and only if both A and B are complete. If $z \in L$ and $\varphi(z) = (z_1, z_2)$, then we denote

$$z_1 = z(A, \varphi), \quad z_2 = z(B, \varphi).$$

When φ is fixed, we sometimes write $z(A)$ and $z(B)$ instead of $z(A, \varphi)$ or $z(B, \varphi)$, respectively.

Under the above notation, let

$$(A_0, \varphi) = \{z \in L \colon z(B, \varphi) = 0\}, \quad (B_0, \varphi) = \{z \in L \colon z(A, \varphi) = 0\}.$$

When no misunderstanding can occur, we write A_0 and B_0 instead of (A_0, φ) and (B_0, φ), respectively. Both A_0 and B_0 are convex sublattices of L and $A_0 \cap B_0 = \{0\}$. The lattice A_0 is isomorphic to A; similarly, B_0 is isomorphic to B. For each $z \in L$ there exists a uniquely determined element z_1' in A_0 such that

$$z_1'(A, \varphi) = z(A, \varphi);$$

similarly, there exists a uniquely determined element z_2' in B_0 with

$$z_2'(B, \varphi) = z(B, \varphi).$$

Denote $\varphi_0(z) = (z_1', z_2')$.

The following lemma is easy to verify.

1.1. Lemma. *Let L, A, B, φ and φ_0 be as above.*

(i) *φ_0 is an isomorphism of the lattice L onto the direct product $A_0 \times B_0$.*

(ii) *For each $z \in L$,*

$$z(A_0) = \max\{t \in A_0 \colon t \leqslant z\}, \quad z(B_0) = \max\{t \in B_0 \colon t \leqslant z\}.$$

(iii) *For each $z \in L$,*

$$z = z(A_0) \vee z(B_0).$$

(iv) *If $z^1 \in A_0$, $z^2 \in B_0$, $z = z^1 \vee z^2$, then $z(A_0) = z^1$ and $z(B_0) = z^2$.*

From (ii) of Lemma 1.1. it follows that for each $z \in L$ we have

$$z \in A_0 \iff z(A_0) = z,$$

and similarly for B_0.

Let $X \subseteq L$. We denote

$$X^\delta = \{y \in L \colon y \wedge x = 0 \text{ for each } x \in X\}.$$

From 1.1. we obtain as a corollary:

1.2. Lemma. *Under the notation as in Lemma* 1.1 *we have*

$$A_0^\delta = B_0, \quad B_0^\delta = A_0, \quad A_0^{\delta\delta} = A_0, \quad B_0^{\delta\delta} = B_0.$$

A lattice is said to be *atomic* (or *atomless*, respectively), if each its nonzero element is a join of atoms (if it has no atom).

If $\psi \colon L \to C \times D$ is another direct product representation of the lattice L, then ψ_0, C_0 and D_0 have an analogous meaning as φ_0, A_0 and B_0 above.

1.3. Lemma. *Let us apply the same assumptions and notation as in Lemma* 1.1. *Suppose that the lattice A is atomic and that the lattice B is atomless. Let P be the set of all atoms in L.*

(i) *$P \subseteq A_0$ and each nonzero element of A_0 is a join of some elements of P.*
(ii) *$B_0 = P^\delta$.*

Proof. We have already remarked that A_0 is isomorphic to A and that B_0 is isomorphic to B. Hence A_0 is atomic and B_0 is atomless. Let $p \in P$. According to (iii) of Lemma 1.1 we have $p = p(A_0) \vee p(B_0)$. Since $A_0 \cap B_0 = \{0\}$, either $p(A_0) = 0$ or $p(B_0) = 0$. If $p(A_0) = 0$, then $p(B_0) = p \in B_0$, thus p is an atom of B_0, which is a contradiction. Therefore $p(B_0) = 0$, whence $p \in A_0$ and so $P \subseteq A_0$. Since A_0 is a convex sublattice of L and $0 \in A_0$, we infer that each atom of A_0 belongs to P. Hence (i) is valid.

If $b \in B_0$ and $p \in P$, then clearly $b \wedge p = 0$. Thus $B_0 \subseteq P^\delta$. Let $0 < z \in P^\delta$. If $0 < z(A_0)$, then in view of (i) there is $p \in P$ with $p \leqslant z(A_0) \leqslant z$, which is a contradiction. Hence $z(A_0) = 0$ and so $z \in B_0$. Therefore $P^\delta \subseteq B_0$. Hence (ii) holds. $\square$

Lema 1.3 yields as a corollary:

1.4. Lemma. *If a lattice L possesses a representation as a direct product of an atomic lattice and an atomless lattice, then this representation is unique in the following sense: if the assumptions from Lemma 1.3 hold and if, moreover, $\psi\colon L \to C \times D$ is an isomorphism such that C is an atomic lattice and D is an atomless lattice, then $C_0 = A_0$ and $D_0 = B_0$.*

2. Proof of theorem (A)

We apply the notation mentioned in the introduction. Let $L \in \mathcal{U}_1$. Hence there are L_i $(i \in I)$ in $\mathcal{U} \cap c_0(L)$ such that

$$L = \bigcup_{i \in I} L_i.$$

Thus for each $z \in L$ there is $x \in L$ having the property that

$$z \in [0, x] = L_i \quad \text{for some} \quad i \in I.$$

In view of Theorem (B) there are lattices $A(x)$ and $B(x)$ such that $A(x)$ is atomic, $B(x)$ is atomless, and there is an isomorphism φ^x of $[0, x]$ onto $A(x) \times B(x)$.

We construct the lattices $A_0(x)$ and $B_0(x)$ and the isomorphism φ_0^x as in Sect. 1 with the distinction that we now have the lattice $[0, x]$ instead of L. Let P be the set of all atoms of L.

2.1. Lemma. *Let z and x be as above, $z > 0$. Then*

(i) $z(A_0(x)) = \sup\{p \in P\colon p \leqslant z\}$,
(ii) $z(B_0(x)) = \max\{t \in P^\delta : t \leqslant z\}$.

Proof. $A_0(x)$ is isomorphic to $A(x)$, hence $A_0(x)$ is atomic. The case $z(A_0(x)) = 0$ is trivial; suppose that $z(A_0(x)) > 0$. Hence $z(A_0(x))$ is the join of some atoms of $A_0(x)$. Since $A_0(x)$ is a convex sublattice of $[0, x]$ and $0 \in A_0(x)$, each atom of $A_0(x)$ belongs to P. Hence (i) holds.

$B_0(x)$ is isomorphic to $B(x)$, hence it is atomless. Next, $B_0(x)$ is a convex sublattice and $0 \in B_0(x)$. Therefore $P \cap B_0(x) = \emptyset$. Thus $b \wedge p = 0$ for each $b \in B_0(x)$ and each $p \in P$. In particular, $z(B_0(x)) \wedge p = 0$ for each $p \in P$ and hence $z(B_0(x)) \in P^\delta$. Let $t \in P^\delta, t \leqslant z$. According to (iii) of 1.1 we have $t = t(A_0(x)) \vee t(B_0(x))$. Moreover, since $A_0(x)$ is atomic, we infer that $t(A_0(x)) = 0$. Hence $t = t(B_0(x))$. In view of $t \leqslant z$ we obtain $t(B_0(x)) \leqslant z(B_0(x))$, whence $t \leqslant z(B_0(x))$. Thus (ii) is valid. $\square$

2.2. Lemma. *Let x and z be as in* 2.1. *Let $j \in I$, $L_j = [0, y]$, $x \leqslant y$. Then (under analogous notation as above) we have*

$$z(A_0(x)) = z(A_0(y)), \quad z(B_0(x)) = z(B_0(y)).$$

Proof. This is an immediate consequence of 2.1. $\square$

2.3. Lemma. *Let x and z be as in* 2.1. *Let $k \in I$, $L_k = [0, t]$, $x \leqslant t$. Then $z(A_0(x)) = z(A_0(t))$ and $z(B_0(x)) = z(B_0(t))$.*

Proof. There exists $j \in I$ such that $x \vee t \in L_j$. Let $L_j = [0, y]$. Now the assertion follows from 2.2.

We denote by A_L the set of all elements of L which can be expressed as joins of elements belonging to P. Next let $B_L = (A_L)^\delta$.

Let $z \in L$. Let x be as above. Put

$$z_1 = z(A_0(x)), \quad z_2 = z(B_0(x)).$$

In view of 2.3, z_1 and z_2 do not depend on the particular choice of x, they are uniquely determined by z. Next, according to 2.1 we have $z_1 \in A_L$ and $z_2 \in B_L$. Denote $\varphi(z) = (z_1, z_2)$. Then φ is a mapping of L into $A_L \times B_L$.

Let $z^1 \in A_L$, $z^2 \in B_L$. There exists $i(1) \in I$ with $L_{i(1)} = [0, x(1)]$ such that $z^1 \vee z^2 \leqslant x(1)$. Put $q = z^1 \vee z^2$. There exists an isomorphism $\varphi^{x(1)}$ of $[0, x(1)]$ onto $A(x(1)) \times B(x(1))$. From 2.3 and 1.1 (iv) we obtain that $q_1 = z^1$ and $q_2 = z^2$. Thus φ is surjective.

Let $s \in L$. There is $i(2) \in I$ with $L_{i(2)} = [0, x(2)]$ such that $z \vee s = x(2)$. By considering the isomorphism $\varphi^{x(2)}$ of $[0, x(2)]$ onto $A(x(2)) \times B(x(2))$ we get that the following conditions are equivalent:

(i) $z \leqslant s$,

(ii) $z_1 \leqslant s_1$ and $z_2 \leqslant s_2$.

Therefore φ is an isomorphism.

If $i \in I$ and $L_i = [0, x]$, then $A(x)$ is a Boolean algebra. Because L_A is the union of all such intervals $A(x)$, we infer that L_A is a generalized Boolean algebra. Moreover, $P \subseteq L_A$. Thus the lattice $B_L = (A_L)^\delta$ is atomless.

If $L_i = [0, x]$ is as above, then $B(x) \in \mathcal{U}$. Since L_B is the union of all such intervals $B(x)$ with $0 \in B(x)$, the relation $L_B \in \mathcal{U}_1$ is valid. This completes the proof of (A). $\square$

Let us remark that if L is complete and if the greatest element of L is denoted by x, then (under the same notation as above) we have $L_A = A(x)$, $L_B = B(x)$, hence both L_A and L_B are complete lattices, L_A is a Boolean algebra and L_B is an atomless lattice belonging to $\mathcal{U}$. Hence (B) is a particular case of (A).

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