DOUBLE n-ARY RELATIONAL STRUCTURES

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(Received January 22, 1996)

Abstract. In [7], V. Novák and M. Novotný studied ternary relational structures by means of pairs of binary structures; they obtained the so-called double binary structures. In this paper, the idea is generalized to relational structures of any finite arity.

Keywords: n-ary relation, n-ary structure, binding relation, double n-ary structure

MSC 1991: 04A05, 08A02

Let G be a set, let $n \geqslant 2$ be an integer. As usual, an *n-ary relation* on G is defined as a set $R \subseteq G^n$. The pair $\mathbf{G} = (G, R)$ is then called an *n-ary relational structure* (or briefly an *n-ary structure*). An n-ary structure $\mathbf{G} = (G, R)$ (and the relation R on G as well) is called

symmetric if $(x_1, x_2, \ldots, x_n) \in R$ implies $(x_n, x_{n-1}, \ldots, x_1) \in R$ for any $x_1, x_2, \ldots, x_{n-1}, x_n \in G$;

asymmetric if $(x_1, x_2, \ldots, x_n) \in R$ implies $(x_n, x_{n-1}, \ldots, x_1) \notin R$ for any $x_1, x_2, \ldots, x_{n-1}, x_n \in G$;

cyclic if $(x_1, x_2, \ldots, x_n) \in R$ implies $(x_2, x_3, \ldots, x_n, x_1) \in R$ for any $x_1, x_2, x_3, \ldots, x_n \in G$;

transitive if $(x_1, x_2, \ldots, x_n) \in R$, $(x_n, x_{n-1}, \ldots, x_2, x_{n+1}) \in R$ imply $(x_1, x_2, \ldots, x_{n-1}, x_{n+1}) \in R$ for any $x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1} \in G$;

weakly transitive if $(x, y, y, \ldots, y) \in R$, $(y, y, \ldots, y, z) \in R$ imply $(x, y, y, \ldots, y, z) \in R$ for any $x, y, z \in G$.

For any $\alpha = (x_1, x_2, \ldots, x_n) \in G^n$, put $\alpha^{-1} = (x_n, x_{n-1}, \ldots, x_1)$, $\alpha' = (x_{n-1}, x_{n-2}, \ldots, x_1, x_n)$.

Let ϱ be an n-ary relation on G, let r be a binary relation on ϱ with the property: If $\alpha = (x_1, x_2, \ldots, x_n) \in \varrho$, $\beta = (y_1, y_2, \ldots, y_n) \in \varrho$, $(\alpha, \beta) \in r$, then $x_{j+1} = y_j$ for $j = 1, 2, \ldots, n-1$. Then r is called a *binding relation* on ϱ.

Let ϱ be an n-ary relation on G, let r be a binding relation on ϱ. Then the triple $\mathbf{G} = (G, \varrho, r)$ is called *a double n-ary relational structure* (or briefly a double n-ary structure). An element $\alpha \in \varrho$ is called *isolated in* G if $(\alpha, \beta) \notin r$ and $(\beta, \alpha) \notin r$ for any $\beta \in \varrho$. The set of all isolated elements in G is denoted by ϱ_i.

A double n-ary structure $\mathbf{G} = (G, \varrho, r)$ (and its binary relation r) is called

inversely symmetric if $(\alpha, \beta) \in r$ implies $(\beta^{-1}, \alpha^{-1}) \in r$ for any α, $\beta \in \varrho$;

inversely asymmetric if $(\alpha, \beta) \in r$ implies $(\beta^{-1}, \alpha^{-1}) \notin r$ for any α, $\beta \in \varrho$;

transferable if $(\alpha, \beta) \in r$ implies the existence of elements $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \varrho$ such that $(\beta, \alpha_1) \in r$, $(\alpha_j, \alpha_{j+1}) \in r$ for $j = 1, 2, \ldots, r-2, (\alpha_{n-1}, \alpha) \in r$ for any α, $\beta \in \varrho$;

reversely transitive if $(\alpha, \beta) \in r$, $(\beta^{-1}, \gamma') \in r$ imply $(\alpha, \gamma) \in r$ for any α, β, $\gamma \in \varrho$.

Let $\mathbf{G} = (G, \varrho, r)$ be a double n-ary structure. Define an $(n+1)$-ary relation R on G as follows:

$(x_1, x_2, \ldots, x_n, x_{n+1}) \in R \iff (x_1, x_2, \ldots, x_n) = \alpha \in \varrho$, $(x_2, x_3, \ldots, x_n, x_{n+1}) = \beta \in \varrho$, $(\alpha, \beta) \in r$ for any $x_1, x_2, x_3, \ldots, x_n, x_{n+1} \in G$. Denote $U(\mathbf{G}) = (G, R)$. Then $U(\mathbf{G})$ is an $(n+1)$-ary structure.

If we denote by ${}_2\mathcal{R}_n$ the class of all double n-ary structures, and by $\mathcal{R}_{n+1}$ the class of all $(n+1)$-ary structures, then U is a map of ${}_2\mathcal{R}_n$ into $\mathcal{R}_{n+1}$.

Now, let $\mathbf{G} = (G, R)$ be an $(n+1)$-ary structure. Define an n-ary relation ϱ on G as follows:

$(x_1, x_2, \ldots, x_n) \in \varrho \iff$ there exists $t \in G$ such that $(x_1, x_2, \ldots, x_n, t) \in R$ or $(t, x_1, x_2, \ldots, x_n) \in R$ for any $x_1, x_2, \ldots, x_n \in G$; further, define a binary relation r on ϱ as follows:

$(\alpha, \beta) \in r \iff \alpha = (x_1, x_2, \ldots, x_n) \in \varrho$, $\beta = (x_2, x_3, \ldots, x_{n+1}) \in \varrho$, $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$ for any $x_1, x_2, \ldots, x_n, x_{n+1} \in G$. Denote $L(\mathbf{G}) = (G, \varrho, r)$. Then $L(\mathbf{G})$ is a double n-ary structure and L is a map of $\mathcal{R}_{n+1}$ into ${}_2\mathcal{R}_n$.

Moreover, denote by ${}_2\mathcal{R}'_n$ the class of all double n-ary structures without isolated elements.

1. Theorem. *Let* $\mathbf{G}$ *be an (n+1)-ary structure. Then* $(U \cdot L)(\mathbf{G}) = \mathbf{G}$, *i.e.* $U \cdot L = \mathrm{id}_{\mathcal{R}_{n+1}}$.

P r o o f. Let $\mathbf{G} = (G, R)$, $L(\mathbf{G}) = (G, \varrho, r)$, $(U \cdot L)(\mathbf{G}) = (G, R')$. Let $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$. By the definition of L, we have $(x_1, x_2, \ldots, x_n) = \alpha \in \varrho$, $(x_2, x_3, \ldots, x_n, x_{n+1}) = \beta \in \varrho$, $(\alpha, \beta) \in r$. By the definition of U, we have $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R'$. Thus $R \subseteq R'$. Let $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R'$. Then, by the definition of U, $(x_1, x_2, \ldots, x_n) = \alpha \in \varrho$, $(x_2, x_3, \ldots, x_n, x_{n+1}) = \beta \in \varrho$, $(\alpha, \beta) \in r$. By the definition of L, $(x_1, x_2, \ldots, x_{n+1}) \in R$. Hence $R' \subseteq R$. Summarizing, we conclude $R = R'$. $\square$

2. Theorem. *Let* $\mathbf{G} = (G, \varrho, r)$ *be a double* n*-ary structure and let* $(L \cdot U)(\mathbf{G}) = (G, \varrho', r')$. *Then* $\varrho' = \varrho - \varrho_i$, $r' = r$, *i.e.* $L \cdot U\big|_2\mathcal{R}'_n = \mathrm{id}_{2\mathcal{R}'_n}$.

P r o o f. Denote $U(\mathbf{G}) = (G, R)$. Let $(x_1, x_2, \ldots, x_n) \in \varrho'$. Then, by the definition of L, there exists $t \in G$ such that $(x_1, x_2, \ldots, x_n, t) \in R$ or $(t, x_1, x_2, \ldots, x_n) \in R$. In the first case, by the definition of U, we have $(x_1, x_2, \ldots, x_n) = \alpha \in \varrho$, $(x_2, x_3, \ldots, x_n, t) = \beta \in \varrho$, $(\alpha, \beta) \in r$, thus the element $\alpha \in \varrho$ is not isolated, so that $\alpha \in \varrho - \varrho_i$. In the second case, $(t, x_1, x_2, \ldots, x_{n-1}) = \alpha \in \varrho$, $(x_1, x_2, \ldots, x_{n-1}, x_n) = \beta \in \varrho$, $(\alpha, \beta) \in r$, hence the element $\beta \in \varrho$ is not isolated and $\beta \in \varrho - \varrho_i$. We have $\varrho' \subseteq \varrho - \varrho_i$. Let, on the contrary, $\alpha = (x_1, x_2, \ldots, x_n) \in \varrho - \varrho_i$. Then there exists $\beta \in \varrho$ such that $(\alpha, \beta) \in r$ or $(\beta, \alpha) \in r$. In the first case we have $\beta = (x_1, x_2, \ldots, x_n, t)$ for some $t \in G$, therefore, by the definition of U, $(x_1, x_2, \ldots, x_n, t) \in R$ and, by the definition of L, $\alpha \in \varrho'$. The second case is analogous. Hence $\varrho - \varrho_i \subseteq \varrho'$. Altogether, we have $\varrho' = \varrho - \varrho_i$.

Let $(\alpha, \beta) \in r'$. By the definition of L, $\alpha = (x_1, x_2, \ldots, x_n)$, $\beta = (x_2, x_3, \ldots, x_n, x_{n+1}) \in R$ for some $x_1, x_2, x_3, \ldots, x_n, x_{n+1} \in G$, $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$. This implies, by the definition of U, $\alpha \in \varrho$, $\beta \in \varrho$, $(\alpha, \beta) \in r$. Thus $r' \subseteq r$. Let $(\alpha, \beta) \in r$. Then $\alpha = (x_1, x_2, \ldots, x_n) \in \varrho$, $\beta = (x_2, x_3, \ldots, x_n, x_{n+1}) \in \varrho$ for some $x_1, x_2, x_3, \ldots, x_n, x_{n+1} \in G$, hence, by the definition of U, we have $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$. Consequently, by the definition of L, $\alpha \in \varrho'$, $\beta \in \varrho'$, $(\alpha, \beta) \in r'$, and $r \subseteq r'$. Summarizing, we obtain $r = r'$.

In the case that G contains no isolated elements, we have $\varrho_i = \emptyset$, thus $\varrho = \varrho'$, $r = r'$, so that $L \cdot U\big|_2\mathcal{R}'_n = \mathrm{id}_{2\mathcal{R}'_n}$.

Denote by ${}_2\mathbf{R}_n$ the category whose class of objects is ${}_2\mathcal{R}_n$ and whose morphisms are maps preserving both relations, i.e., for $\mathbf{G} = (G, \varrho, r)$, $\mathbf{H} = (H, \sigma, s) \in {}_2\mathcal{R}_n$, a map $f : G \longrightarrow H$ is a morphism if $(x_1, x_2, \ldots, x_n) \in \varrho$ implies $\big(f(x_1), f(x_2), \ldots, f(x_n)\big) \in \sigma$, and $\big((x_1, x_2, \ldots, x_n), (x_2, x_3, \ldots, x_{n+1})\big) \in r$ implies $\big((f(x_1), f(x_2), \ldots, f(x_n)), (f(x_2), f(x_3), \ldots, f(x_{n+1}))\big) \in s$ for any $x_1, x_2, x_3, \ldots, x_n, x_{n+1} \in G$.

Further, denote by $\mathbf{R}_{n+1}$ the category whose class of objects is $\mathcal{R}_{n+1}$ and whose morphisms are maps preserving the relation, i.e., for $\mathbf{G} = (G, H)$, $\mathbf{H} = (H, S) \in \mathcal{R}_{n+1}$ a map $f : G \longrightarrow H$ is a morphism if $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$ implies $\big(f(x_1), f(x_2), \ldots, f(x_n), f(x_{n+1})\big) \in S$ for any $x_1, x_2, \ldots, x_n, x_{n+1} \in G$.

Moreover, for any morphism $f \in \mathrm{Hom}_{2\mathcal{R}_n}(\mathbf{G}, \mathbf{H})$, where $\mathbf{G} = (G, \varrho, r)$, $\mathbf{H} = (H, \sigma, s)$, denote $U(f) = f$. Similarly, for any morphism $f \in \mathrm{Hom}_{\mathbf{R}_{n+1}}(\mathbf{G}, \mathbf{H})$, denote $L(f) = f$. $\square$

3. Theorem. U *is a covariant functor from the category* ${}_2\mathbf{R}_n$ *to the category* $\mathbf{R}_{n+1}$, L *is a covariant functor from the category* $\mathbf{R}_{n+1}$ *to the category* ${}_2\mathbf{R}_n$.

P r o o f. Let $f \in \mathrm{Hom}_{2\mathbf{R}_n}(\mathbf{G}, \mathbf{H})$, where $\mathbf{G} = (G, \varrho, r)$, $\mathbf{G} = (G, R)$, $\mathbf{H} = (H, \sigma, s)$, $U(\mathbf{H}) = (H, S)$. Let $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$. Then $(x_1, x_2, \ldots, x_n) \in \varrho$,

$(x_2, x_3, \ldots, x_n, x_{n+1}) \in \varrho$, $\big((x_1, x_2, \ldots, x_n), (x_2, x_3, \ldots, x_n, x_{n+1})\big) \in r$, so that $\big(f(x_1), f(x_2), \ldots, f(x_n)\big) \in \sigma$, $\big(f(x_2), f(x_3), \ldots, f(x_n), f(x_{n+1})\big) \in \sigma$, $\big((f(x_1), f(x_2), \ldots, f(x_n)), (f(x_2), f(x_3), \ldots, f(x_n), f(x_{n+1}))\big) \in s$, thus $\big(f(x_1), f(x_2), \ldots, f(x_n), f(x_{n+1})\big) \in S$ and $U(f) \in \mathrm{Hom}_{\mathbf{R}_{n+1}}\big(U(\mathbf{G}), U(\mathbf{H})\big)$. It is easy to show that $U(\mathrm{id}_{\mathbf{G}}) = \mathrm{id}_{U(\mathbf{G})}$ for any $\mathbf{G} \in {}_2\mathcal{R}_n$ and $U(g \cdot f) = U(g) \cdot U(f)$ for any $f \in \mathrm{Hom}_{{}_2\mathbf{R}_n}(\mathbf{G}, \mathbf{H})$, $g \in \mathrm{Hom}_{{}_2\mathbf{R}_n}(\mathbf{H}, \mathbf{K})$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K} \in {}_2\mathcal{R}_n$. Analogously for L. $\square$

4. Theorem. *Let $\mathbf{G}$ be a double n-ary structure. Then the following assertions hold:*

(i) *$\mathbf{G}$ is inversely symmetric if and only if $U(\mathbf{G})$ is symmetric.*

(ii) *$\mathbf{G}$ is inversely asymmetric if and only if $U(\mathbf{G})$ is asymmetric.*

Proof. Let $\mathbf{G} = (G, \varrho, r)$, $U(\mathbf{G}) = (G, R)$.

(i) Let G be inversely symmetric and let $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$. Then $(x_1, x_2, \ldots, x_n) = \alpha \in \varrho$, $(x_2, x_3, \ldots, x_n, x_{n+1}) = \beta \in \varrho$, $(\alpha, \beta) \in r$. This implies $(\beta^{-1}, \alpha^{-1}) \in r$, thus $\beta^{-1} = (x_{n+1}, x_n, \ldots, x_3, x_2) \in \varrho$, $\alpha^{-1} = (x_n, \ldots, x_2, x_1) \in \varrho$, so that $(x_{n+1}, x_n, \ldots, x_2, x_1) \in R$ and $U(\mathbf{G})$ is symmetric. Let $U(\mathbf{G})$ be symmetric and let $(\alpha, \beta) \in r$. Then there exist elements x_1, x_2, …, x_n, $x_{n+1} \in G$ such that $\alpha = (x_1, x_2, \ldots, x_n) \in \varrho$, $\beta = (x_2, x_3, \ldots, x_n, x_{n+1}) \in \varrho$. This implies $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$, so that $(x_{n+1}, x_n, \ldots, x_2, x_1) \in R$, i.e. $(x_{n+1}, x_n, \ldots, x_3, x_2) = \beta^{-1} \in \varrho$, $(x_n, \ldots, x_2, x_1) = \alpha^{-1} \in \varrho$, hence $(\beta^{-1}, \alpha^{-1}) \in r$ and $\mathbf{G}$ is inversely symmetric.

(ii) Let $\mathbf{G}$ be inversely asymmetric and let $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$. Then again $(x_1, x_2, \ldots, x_n) = \alpha \in \varrho$, $(x_2, x_3, \ldots, x_n, x_{n+1}) = \beta \in \varrho$, $(\alpha, \beta) \in r$. This implies $(\beta^{-1}, \alpha^{-1}) \notin r$. But $\beta^{-1} = (x_{n+1}, x_n, \ldots, x_3, x_2)$, $\alpha^{-1} = (x_n, \ldots, x_2, x_1)$, thus $(x_{n+1}, x_n, \ldots, x_2, x_1) \notin R$ and $U(\mathbf{G})$ is asymmetric. Let $U(\mathbf{G})$ be asymmetric and let $(\alpha, \beta) \in r$. Then there exist elements x_1, x_2, x_3, …, x_n, $x_{n+1} \in G$ such that $(x_1, x_2, \ldots, x_n) = \alpha \in \varrho$, $(x_2, x_3, \ldots, x_n, x_{n+1}) = \beta \in \varrho$. This implies $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$, so that $(x_{n+1}, x_n, \ldots, x_2, x_1) \notin R$. Consequently $(x_{n+1}, x_n, \ldots, x_3, x_2) = \beta^{-1} \notin \varrho$ or $(x_n, \ldots, x_2, x_1) = \alpha^{-1} \notin \varrho$ or β^{-1}, $\alpha^{-1} \in \varrho$, but $(\beta^{-1}, \alpha^{-1}) \notin r$. In all three cases, however, we have $(\beta^{-1}, \alpha^{-1}) \notin r$, and $\mathbf{G}$ is inversely asymmetric. $\square$

5. Theorem. *Let $\mathbf{G}$ be an $(n+1)$-ary structure. Then the following assertions hold:*

(i) *$\mathbf{G}$ is symmetric if and only if $L(\mathbf{G})$ is inversely symmetric.*

(ii) *$\mathbf{G}$ is asymmetric if and only if $L(\mathbf{G})$ is inversely asymmetric.*

Proof. (i) If $L(\mathbf{G})$ is inversely symmetric, then, by 4, $U\big(L(\mathbf{G})\big)$ is symmetric. But, by 1, $U\big(L(\mathbf{G})\big) = \mathbf{G}$. If $\mathbf{G} = U\big(L(\mathbf{G})\big)$ is symmetric, then, by 4, $L(\mathbf{G})$ is inversely symmetric.

(ii) If $L(\mathbf{G})$ is inversely asymmetric, then, by 4, $U(L(\mathbf{G}))$ is asymmetric. But $U(L(\mathbf{G})) = \mathbf{G}$. If $\mathbf{G} = U(L(\mathbf{G}))$ is asymmetric, then, by 4, $L(\mathbf{G})$ is inversely asymmetric. □

6. Theorem. *Let* $\mathbf{G}$ *be a double* n*-ary structure. Then* $\mathbf{G}$ *is transferable if and only if* $U(\mathbf{G})$ *is cyclic.*

Proof. Let $\mathbf{G} = (G, \varrho, r)$, $U(\mathbf{G}) = (G, R)$. Let $\mathbf{G}$ be transferable and let $(x_1, x_2, \dots, x_n, x_{n+1}) \in R$. Then $(x_1, x_2, \dots, x_n) = \alpha \in \varrho$, $(x_2, x_3, \dots, x_n, x_{n+1}) = \beta \in \varrho$, $(\alpha, \beta) \in r$. Thus, there exist α_1, α_2, …, $\alpha_{n-1} \in \varrho$ such that $(\beta, \alpha_1) \in r$, $(\alpha_j, \alpha_{j+1}) \in r$ for $j = 1, 2, \dots, n-2$ and $(\alpha_{n-1}, \alpha) \in r$. Denote $\alpha_0 = \beta$, $\alpha_n = \alpha$. Then we have (α_j, α_{j+1}) for $j = 0, 1, 2, \dots, n-1$. We shall show by induction that $\alpha_j = (x_{j+2}, x_{j+3}, \dots, x_n, x_{n+1}, x_1, x_2, \dots, x_j)$ for $j = 0, 1, 2, \dots, n$. For $j = 0$ it is true. Let $0 < j_0 \leqslant n$. Let the preceding hold for each j, $0 \leqslant j < j_0$. As $(\alpha_{j_0-1}, \alpha_{j_0}) \in r$ and r is binding, there exists $y \in G$ such that $\alpha_{j_0} = (x_{j_0+2}, x_{j_0+3}, \dots, x_1, \dots, x_{j_0-1}, y)$. We shall show by another induction that α_{j_0+k} has y on the $(n-k)$-th position, for $k = 0, 1, 2, \dots, n-j_0$. For $k = 0$ it is true. Let $0 < k_0 \leqslant n - j_0$. As $(\alpha_{j_0+k_0-1}, \alpha_{j_0+k_0}) \in r$, $\alpha_{j_0+k_0-1}$ has y on the $(n-k_0+1)$-th position, and r is binding, $\alpha_{j_0+k_0}$ has y on the $(n-k_0)$-th position. Particularly, α_n has y on the j_0-th position, hence $y = x_{j_0}$. Thus, we have $\beta = (x_2, x_3, \dots, x_n, x_{n+1}) \in \varrho$, $\alpha_1 = (x_3, x_4, \dots, x_n, x_{n+1}, x_1) \in \varrho$, $(\beta, \alpha_1) \in r$, so that $(x_2, x_3, \dots, x_n, x_{n+1}, x_1) \in R$ and $U(\mathbf{G})$ is cyclic.

Let, on the contrary, $U(\mathbf{G})$ be cyclic and let $(\alpha, \beta) \in r$. Then there exist elements x_1, x_2, …, x_n, $x_{n+1} \in G$ such that $\alpha = (x_1, x_2, \dots, x_n) \in \varrho$, $\beta = (x_2, x_3, \dots, x_n, x_{n+1}) \in \varrho$, thus $(x_1, x_2, \dots, x_n, x_{n+1}) \in R$. Hence $(x_2, x_3, \dots, x_n, x_{n+1}, x_1) \in R$, $(x_3, x_4, \dots, x_n, x_{n+1}, x_1, x_2) \in R, \dots$, $(x_{n+1}, x_1, x_2, \dots, x_n) \in R$. Denote $\alpha_1 = (x_3, x_4, \dots, x_n, x_{n+1}, x_1)$, $\alpha_2 = (x_4, x_5, \dots, x_{n+1}, x_1, x_2), \dots$, $\alpha_{n-1} = (x_{n+1}, x_1, x_2, \dots, x_{n-1})$. Then $\alpha_j \in \varrho$ for $j = 1, 2, \dots, n-1$, $(\beta, \alpha_1) \in r$, $(\alpha_j, \alpha_{j+1}) \in r$ for $j = 1, 2, \dots, n-2$, $(\alpha_{n-1}, \alpha) \in r$. Consequently, $\mathbf{G}$ is transferable. □

7. Theorem. *Let* $L(\mathbf{G})$ *be an (n+1)-ary structure. Then* $\mathbf{G}$ *is cyclic if and only if* $L(\mathbf{G})$ *is transferable.*

Proof. Let $L(\mathbf{G})$ be transferable. By 6, $U(L(\mathbf{G}))$ is cyclic. But, by 1, $\mathbf{G} = U(L(\mathbf{G}))$.

Let $\mathbf{G} = U(L(\mathbf{G}))$ be cyclic. Then, by 6, $L(\mathbf{G})$ is transferable. □

8. Theorem. *Let* $\mathbf{G} = (G, \varrho, r)$ *be a double* n*-ary structure. If the binary relation* r *is transitive, then* $U(\mathbf{G})$ *is weakly transitive.*

Proof. Let $U(\mathbf{G}) = (G, R)$ and let $(x, y, y, \dots, y) \in R$, $(y, y, \dots, y, z) \in R$. Then $\alpha = (x, y, y, \dots, y) \in \varrho$, $\beta = (y, y, \dots, y) \in \varrho$, $\gamma = (y, y, \dots, y, z) \in \varrho$, $(\alpha, \beta) \in$

r, $(\beta,\gamma)\in r$. Hence $(\alpha,\gamma)\in r$, so that $(x,y,y,\ \ldots,y,z)\in R$ and $U(\mathbf{G})$ is weakly transitive. $\square$

9. Remark. The converse of 8 does not hold, which can be easily shown by a counterexample.

10. Theorem. *Let* $\mathbf{G}$ *be a double* n*-ary structure. Then* $\mathbf{G}$ *is reversely transitive if and only if* $U(\mathbf{G})$ *is transitive.*

Proof. Let $\mathbf{G}=(G,\varrho,r)$, $U(\mathbf{G})=(G,R)$. Let $\mathbf{G}$ be reversely transitive, let $(x_1,x_2,\ldots,x_n,x_{n+1})\in R$, $(x_{n+1},x_n,\ldots,x_2,x_{n+2})\in R$. Then, by the definition of U, $(x_1,x_2,\ldots,x_n)=\alpha\in\varrho$, $(x_2,x_3,\ldots,x_n,x_{n+1})=\beta\in\varrho$, $(\alpha,\beta)\in r$, $(x_{n+1},x_n,\ldots,x_2)=\beta^{-1}\in\varrho$, $(x_n,x_{n-1},\ldots,x_2,x_{n+2})=\gamma'\in\varrho$, $(\beta^{-1},\gamma')\in r$. As G is reversely transitive, we have $(\alpha,\gamma)\in r$. But $\gamma=(x_2,x_3,\ldots,x_n,x_{n+2})\in\varrho$, hence $(x_1,x_2,\ldots,x_n,x_{n+2})\in R$ and $U(\mathbf{G})$ is transitive.

Let $U(\mathbf{G})$ be transitive and let α, β, $\gamma\in\varrho$, $(\alpha,\beta)\in r$, $(\beta^{-1},\gamma')\in r$. There exist elements $x_1,x_2,\ldots,x_n,x_{n+1},x_{n+2}\in G$ such that $\alpha=(x_1,x_2,\ldots,x_n)$, $\beta=(x_2,x_3,\ldots,x_n,x_{n+1})$ (for r is binding), $\gamma=(x_2,x_3,\ldots,x_n,x_{n+2})$ (for $\beta^{-1}=(x_{n+1},x_n,\ldots,x_3,x_2)$, $\gamma'=(x_n,x_{n-1},\ldots,x_3,x_2,x_{n+2})$ and r is binding). Hence $(x_1,x_2,\ldots,x_n,x_{n+1})\in R$, $(x_{n+1},x_n,\ldots,x_3,x_2,x_{n+2})\in R$, so that $(x_1,x_2,\ldots,x_n,x_{n+2})\in R$, for $U(\mathbf{G})$ is transitive. Consequently, $(\alpha,\gamma)\in r$ and $\mathbf{G}$ is reversely transitive. $\square$

11. Theorem. *Let* $\mathbf{G}$ *be an* $(n+1)$*-ary structure. Then* $\mathbf{G}$ *is transitive if and only if* $L(\mathbf{G})$ *is reversely transitive.*

Proof. By 1, $U\big(L(\mathbf{G})\big)=\mathbf{G}$. Hence $L(\mathbf{G})$ is reversely transitive if and only if $U\big(L(\mathbf{G})\big)=\mathbf{G}$ is transitive, by 10. $\square$

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