FACTORIZATION OF TOEPLITZ AND HANKEL OPERATORS

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Abstract. Using a factorization lemma we obtain improvements and simplifications of results on representation of generalized Toeplitz and Hankel operators as compression of symbols.

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The classical Toeplitz operator may be characterized as a bounded linear operator A on H^2 satisfying the relation $A = S^*AS$ where S is the (forward) shift operator on H^2. To identify the symbol it is possible to proceed as follows: it suffices to represent A as the compression to H^2 of an operator Y on L^2 which satisfies $Y = V^*YV$ where V is the shift operator on L^2. Since Y commutes with the shift, it may be identified with the operator of multiplication by a suitable function $f \in L^\infty$. The dilation Y of A is easily seen to be uniquely determined by the requirement that Y be equal to V^*YV; these observations yield a construction of the symbol in a natural manner. Not long ago, Sz.-Nagy and Foiaş realized that, replacing the shift by a contraction T acting in a certain Hilbert space $\mathscr{H}$, the relation $A = TAT^*$ defines a class of operators on $\mathscr{H}$ with properties analogous to those of Toeplitz operators. In analogy with the classical case it is natural to examine dilations Y of A satisfying the relation $Y = UYU^*$ where U is the minimal isometric dilation of T. The operator Y may then be considered as a natural generalization of the notion of symbol.

A study of the analogous problem for Hankel operators, undertaken by P. Vrbová and the author, revealed a surprising fact. In order to obtain a natural extension of the classical notion of symbol for operators of Hankel type and a corresponding analogy of Nehari's theorem it was necessary to study liftings of intertwining relations of the form $XT_1^* = T_2X$. The classical Hankel operator may be characterized as a bounded linear operator X of H^2 into H^2_- which intertwines the forward shift on H^2 and the backward shift on H^2_-.

$$X(V\big|H^2) = (V^*\big|H^2_-)^*X$$

where V is the shift operator on $L^2 = \mathscr{H}^2 \oplus \mathscr{H}^2_-$. The problem is to construct a dilation $Y\colon \mathscr{K}_1 \to \mathscr{K}_2$ satisfying $YU_1^* = U_2Y$ where U_1 and U_2 are respectively the minimal isometric dilations of T_1 and T_2, acting on $\mathscr{K}_1$ and $\mathscr{K}_2$ respectively. In the case of Toeplitz operators no additional conditions were necessary for the existence of a lifting; surprisingly enough, this is no more true in the case of the relation $XT_1^* = T_2X$; a further restriction must be imposed on X to guarantee the existence of a lifting; the difficulty lies in the fact that this additional condition is trivially satisfied in the classical case so that its meaning only manifests itself in the general situation.

The additional assumption which ensures the validity of an analogy of Nehari's theorem appears in the form of a boundedness condition to be imposed on X:

$$(Xh_1, h_2) \leqslant \beta|P(\mathscr{R}_1)h_1|\,|P(\mathscr{R}_2)h_2|$$

the $P(\mathscr{R}_i)$ being the orthogonal projections in the space $\mathscr{K}_i$ onto the unitary part of $\mathscr{K}_i$ in the Wold decomposition of U_i. The authors of [2] called this condition $\mathscr{R}$-boundedness.

In the present note we intend to sketch an approach to the study of generalized Toeplitz and Hankel operators based on factorization, reducing in this manner the problem to the particular case where T_1^* is an isometry and T_2 a coisometry. (This will become evident after the perusal of the comments following lemma 2.4.) As opposed to [1] and [2] we obtain a considerable simplification of the proofs, in the Toeplitz case explicit formulae for the symbol.

1. The characteristic relations for Toeplitz and Hankel

We begin by replacing the forward and backward shift by an isometry and a coisometry respectively.

Proposition 1.1. *Let $\mathscr{R}$ be a Hilbert space, $\mathscr{P}$ a closed subspace of $\mathscr{R}$. Let V be a coisometry on $\mathscr{R}$ such that $\mathscr{P}$ is invariant with respect to V and $V\big|\mathscr{P}$ is an isometry. Suppose that the smallest V reducing subspace of $\mathscr{R}$ containing $\mathscr{P}$ is $\mathscr{R}$ itself. Denote by P the orthogonal projection of $\mathscr{R}$ onto $\mathscr{P}$. Then*

1° $PV^* = (V\big|\mathscr{P})^*P$

2° *V is unitary*
3° *$(I-P)V^n x \to 0$ for every $x \in \mathscr{R}$.*

P r o o f. For $x \in \mathscr{R}, h \in \mathscr{P}$, we have

$$(V^* x, h) = (x, Vh) = (Px, Vh) = (Px, (V\big|\mathscr{P})h) = ((V\big|\mathscr{P})^* Px, h).$$

Thus V^* is a lifting of $(V\big|\mathscr{P})^*$; this proves 1°. Since $\mathscr{R}$ is the closed linear span of elements of the form $V^{*k}h$, k nonnegative, $h \in \mathscr{P}$, it suffices to prove the identity $V^*Vx = x$ for these elements. For $k = 0$ we use the isometry of $(V\big|\mathscr{P})$. Since $(V\big|\mathscr{P})^*(V\big|\mathscr{P})h = h$ for $h \in \mathscr{P}$ we have $PV^*Vh = h$ for each $h \in \mathscr{P}$. Since $|h| = |PV^*Vh| \leqslant |V^*Vh| \leqslant |h|$ we have the equality $|PV^*Vh| = |V^*Vh|$ whence $PV^*Vh = V^*Vh$ so that $h = PV^*Vh = V^*Vh$. If $k > 0$, we use the identity $VV^* = 1$. Thus $V^*V \cdot V^{*k}h = V^*VV^*V^{*k-1}h = V^{*k}h$. This completes the proof of 2°.

The operators $(I-P)V^n$ being equibounded it suffices to prove 3° for elements from a dense set. If x is of the form $V^{*k}h$ with $h \in \mathscr{P}$ then $V^n x$ will be in $\mathscr{P}$ as soon as $n > k$; thus $(I-P)V^n x = 0$. □

The following proposition shows that an operator $Y \in B(\mathscr{R})$ which commutes with V may be recovered from its compression to $\mathscr{P}$.

Proposition 1.2. *Suppose $Y \in B(\mathscr{R})$ commutes with V. If X is the compression to $\mathscr{P}$ of Y then*

$$Y = \lim V^{*n} X P V^n$$

in the strong operator topology. The compression satisfies the identity

$$(V\big|\mathscr{P})^* X (V\big|\mathscr{P}) = X.$$

P r o o f. The first assertion is the consequence to the following two identities

$$\begin{aligned} Y - V^{*n}PYV^n &= V^{*n}P^{\perp}YV^n = V^{*n}(P^{\perp}V^n)Y \\ V^{*n}PYV^n &= V^{*n}PYPV^n + V^{*n}PY(P^{\perp}V^n). \end{aligned}$$

The second assertion is a consequence of 1° in Proposition (1.1). Indeed,

$$\begin{aligned} (V\big|\mathscr{P})^* X (V\big|\mathscr{P}) &= (V\big|\mathscr{P})^* PYV\big|\mathscr{P} \\ &= PV^*YV\big|\mathscr{P} = PV^*VY\big|\mathscr{P} = X. \end{aligned}$$

□

For the rest of the present chapter we shall consider the following situation.

Suppose we are given two triplets

$$\mathscr{R}_i, \mathscr{P}_i, V_i$$

such that V_i is unitary on $\mathscr{R}_i$ and $\mathscr{P}_i$ is a closed subspace of $\mathscr{R}_i$ invariant with respect to V_i. Furthermore we assume that the smallest V_i reducing subspace of $\mathscr{R}_i$ containing $\mathscr{P}_i$ is the space $\mathscr{R}_i$ itself.

Definition 1.3. An operator $X\colon \mathscr{P}_1 \to \mathscr{P}_2$ is said to be of type T if

$$X = (V_2 \big| \mathscr{P}_2)^* X (V_1 \big| \mathscr{P}_1);$$

observe that $(V_1 \big| \mathscr{P}_1)$ is an isometry and $(V_2 \big| \mathscr{P}_2)^*$ a coisometry.

Proposition 1.4. *Suppose $Y\colon \mathscr{R}_1 \to \mathscr{R}_2$ satisfies $YV_1 = V_2 Y$. Then $X = P_2 Y \big| \mathscr{P}_1$ is of type T and*

$$Y = \lim V_2^{*n} X P_1 V_1^n$$

in the strong operator topology.

P r o o f. Using 1°, we obtain

$$(V_2 \big| \mathscr{P}_2)^* P_2 Y (V_1 \big| \mathscr{P}_1) = P_2 V_2^* Y (V_1 \big| \mathscr{P}_1) = P_2 Y V_1^* (V_1 \big| \mathscr{P}_1) = P_2 Y \big| \mathscr{P}_1.$$

The second assertion is a consequence of the following identities

$$\begin{aligned} Y - V_2^{*n} P_2 Y P_1 V_1^n &= Y - V_2^{*n} P_2 Y V_1^n + V_2^{*n} P_2 Y P_1^{\perp} V_1^n \\ &= V_2^{*n} Y V_1^n - V_2^{*n} P_2 Y V_1^n + V_2^{*n} P_2 Y (P_1^{\perp} V_1^n) \\ &= V_2^{*n} P_2^{\perp} Y V_1^n + V_2^{*n} P_2 Y (P_1^{\perp} V_1^n) \\ &= V_2^{*n} (P_2^{\perp} V_1^n) Y + V_2^{*n} P_2 Y (P_1^{\perp} V_1^n). \end{aligned}$$

□

Proposition 1.5. *Conversely, given $X\colon \mathscr{P}_1 \to \mathscr{P}_2$ of type T, there exists exactly one $Y\colon \mathscr{R}_1 \to \mathscr{R}_2$ such that $YV_1 = V_2 Y$ and $X = P_2 Y \big| \mathscr{P}_1$. The sequence $V_2^{*n} X P_1 V_1^n$ is convergent in the weak operator topology; its limit Y satisfies $YV_1 = V_2 Y$ and $X = P_2 Y \big| \mathscr{P}_1$. Furthermore, $|Y| = |X|$.*

P r o o f. Given nonnegative integers p, q and two elements $h_1 \in \mathscr{P}_1, h_2 \in \mathscr{P}_2$, consider an $n > p + q$.

$$\begin{aligned} (V_2^{*n} X P_1 V_1^n \cdot V_1^{*p} h_1, V_2^{*q} h_2) &= (X P_1 V_1^{n-p} h_1, V_2^{n-q} h_2) \\ &= ((V_2 \big| \mathscr{P}_2)^{*n-q} X (V_1 \big| \mathscr{P}_1)^{n-p} h_1, h_2) \end{aligned}$$

In order to evaluate the last expression we distinguish two cases. If $p-q=t\geqslant 0$ then $n-q=n-p+t$ and the scalar product equals

$$((V_2|\mathscr{P}_2)^{*t}Xh_1,h_2).$$

If $q-p=t>0$ then $n-p=n-q+t$ and we obtain

$$(X(V_1|\mathscr{P}_1)^t h_1,h_2).$$

The operators $V_2^{*n}XP_1V_1^n$ being equibounded and the sequence being stationary for large n, this proves the convergence. The limit operator Y satisfies, for $h_1\in\mathscr{P}_1$, and $h_2\in\mathscr{P}_2$,

$$\begin{aligned}(Yh_1,h_2)&=\lim(V_2^{*n}XP_1V_1^n h_1,h_2)\\&=\lim(XP_1V_1^n h_1,(V_2|\mathscr{P}_2)^n h_2)\\&=\lim((V_2|\mathscr{P}_2)^{*n}XV_1^n h_1,h_2)\\&=(Xh_1,h_2).\end{aligned}$$

□

The preceding propositions establish a one to one correspondence between operators $Y\colon\mathscr{R}_1\to\mathscr{R}_2$ satisfying

$$Y=V_2^*YV_1$$

and operators $X\colon\mathscr{P}_1\to\mathscr{P}_2$ satisfying

$$X=(V_2|\mathscr{P}_2)^*X(V_1|\mathscr{P}_1).$$

This correspondence is linear and isometric: X is the compression of Y

$$X=P_2Y|\mathscr{P}_1$$

and Y is the limit in the strong operator topology of the sequence

$$V_2^{*n}XP_1V_1^n.$$

Definition 1.6. An operator $X\colon\mathscr{P}_1\to\mathscr{P}_2$ is said to be of type H if

$$X(V_1|\mathscr{P}_1)=(V_2|\mathscr{P}_2)^*X.$$

Proposition 1.7. *Suppose $Y\colon\mathscr{R}_1\to\mathscr{R}_2$ satisfies*

$$YV_1=V_2^*Y.$$

Then the compression $X = P_2 Y\big|\mathscr{P}_1$ is an operator of type H. Conversely, for each operator $X\colon \mathscr{P}_1 \to \mathscr{P}_2$ of type H there exists a $Y : \mathscr{R}_1 \to \mathscr{R}_2$ such that $YV_1 = V_2^* Y$, $|Y| = |X|$ and $X = P_2 Y\big|\mathscr{P}_1$.

Proof. Let X be the compression of an operator $Y\colon \mathscr{R}_1 \to \mathscr{R}_2$ with $YV_1 = V_2^* Y$. Then, using 1° of Proposition 1.1,

$$X(V_1\big|\mathscr{P}_1) = P_2 Y V_1\big|\mathscr{P}_1 = P_2 V_2^* Y\big|\mathscr{P}_1 = (V_2\big|\mathscr{P}_2)^* P_2 Y\big|\mathscr{P}_1 = (V_2\big|\mathscr{P}_2)^* X.$$

Conversely suppose $X\colon \mathscr{P}_1 \to \mathscr{P}_2$ satisfies

$$(V_2\big|\mathscr{P}_2)^* X = X(V_1\big|\mathscr{P}_1).$$

The minimal isometric dilation of $(V_2\big|\mathscr{P}_2)^*$ being V_2^* on $\mathscr{R}_2$, the commutant lifting theorem yields the existence of an operator $M\colon \mathscr{P}_1 \to \mathscr{R}_2$ and such that

$$\begin{aligned} M(V_1\big|\mathscr{P}_1) &= V_2^* M, \qquad |M| = |X| \\ P_2 M &= X. \end{aligned}$$

Now consider arbitrary elements $h_0, \ldots, h_n$ in $\mathscr{P}_1$. Then

$$\begin{aligned} \sum V_2^k M h_k &= V_2^n \sum V_2^{*n-k} M h_k \\ &= V_2^n M \sum V_1^{n-k} h_k = V_2^n M V_1^n \sum V_1^{*k} h_k \end{aligned}$$

whence

$$\Big|\sum V_2^k M h_k\Big| \leqslant |M| \Big|\sum V_1^{*k} h_k\Big|.$$

It follows that there exists a linear mapping Y of norm $\leqslant |X|$ such that

$$Y V_1^{*k} h = V_2^k M h$$

for every $k \geqslant 0$ and every $h \in \mathscr{P}_1$. Let us prove that $YV_1 = V_2^* Y$. It suffices to prove

$$Y V_1 V_1^{*k} h = V_2^* Y V_1^{*k} h$$

for every $h \in \mathscr{P}_1$ and every $k \geqslant 0$. For $k = 0$ the relation to be proved reduces to

$$M V_1 h = V_2^* M h.$$

If $k > 0$, we have

$$Y V_1 V_1^{*k} h = Y V_1^{*k-1} h = V_2^{k-1} M h = V_2^* V_2^k M h = V_2^* Y V_1^{*k} h.$$

□

As it could be expected, the correspondence between an operator $Y\colon \mathscr{R}_1 \to \mathscr{R}_2$ satisfying $YV_1 = V_2^*Y$ and its compression is many to one. There is only a considerably weaker analogy of (1.5) linking symbols and their restrictions.

Remark 1.7. There is a one-to-one correspondence between operators $Y\colon \mathscr{R}_1 \to \mathscr{R}_2$ satisfying

$$YV_1 = V_2^*Y$$

and operators $M\colon \mathscr{P}_1 \to \mathscr{R}_2$ satisfying

$$M(V_1 \big| \mathscr{P}_1) = V_2^*M.$$

Y is determined by its restriction to $\mathscr{P}$

$$Y = \lim V_2^n (Y \big| \mathscr{P}_1) P_1 V_1^n$$

in the strong operator topology.

Proof. In view of the equiboundedness of the operators $V_2^n Y P_1 V_1^n$ it suffices to prove the convergence for elements of a dense set. Suppose $x = V_1^{*k} h$ for some $k \geqslant 0$ and $h \in \mathscr{P}_1$. If $n > k$ then

$$\begin{aligned} Yx &= V_2^n Y V_1^n \cdot V_1^{*k} h = V_2^k V_2^{n-k} Y V_1^{n-k} h \\ &= V_2^k Y h = V_2^k (Y \big| \mathscr{P}_1) P_1 V_1^k \cdot V_1^{*k} h \\ &= V_2^k (Y \big| \mathscr{P}_1) P_1 V_1^k x \end{aligned}$$

□

In particular, every operator $M\colon \mathscr{P}_1 \to \mathscr{R}_2$ satisfying $M(V_1 \big| \mathscr{P}_1) = V_2^*M$ admits exactly one extension $Y\colon \mathscr{R}_1 \to \mathscr{R}_2$ for which

$$YV_1 = V_2^*Y.$$

Clearly $|Y| = |X|$.

Definition 2.1. Suppose T_1 and T_2 are two given contractions on the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. A bounded linear operator $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ is said to be generalized Toeplitz if

$$X = T_2 X T_1^*,$$

generalized Hankel if

$$X T_1^* = T_2 X.$$

Denote by U_i (acting on $\mathscr{K}_i$) the minimal isometric dilations of T_i respectively. The problem to be treated in this section is the following: under what conditions may X be represented as the compression

$$X = P(\mathscr{H}_2) Y \big| \mathscr{H}_1$$

of a bounded linear operator $Y\colon \mathscr{K}_1 \to \mathscr{K}_2$ which satisfies

$$Y = U_2 Y U_1^*$$

in the Toeplitz case or

$$Y U_1^* = U_2 Y$$

in the Hankel case.

If we agree to call Y a symbol for X, the following observation shows that, in a manner of speaking, symbols are essentially operators from $\mathscr{R}_1$ into $\mathscr{R}_2$, the $\mathscr{R}_i$ being the unitary part of $\mathscr{K}_i$ in the Wold decomposition of U_i.

Lemma 2.2. *Suppose $Y\colon \mathscr{K}_1 \to \mathscr{K}_2$ satisfies one of the relations*

$$Y = U_2 Y U_1^* \tag{T}$$

$$Y U_1^* = U_2 Y. \tag{H}$$

Then $Y = P(\mathscr{R}_2) Y P(\mathscr{R}_1)$.

Proof. In the case of the relation T we have

$$\begin{aligned} Y U_1 U_1^* &= U_2 Y U_1^* U_1 U_1^* = U_2 Y U_1^* = Y \\ U_2 U_2^* Y &= U_2 U_2^* U_2 Y U_1^* = U_2 Y U_1^* = Y. \end{aligned}$$

For the relation H

$$\begin{aligned}YU_1U_1^* &= U_2^*YU_1^* \cdot U_1U_1^* = U_2^*YU_1^* = Y\\ U_2U_2^*Y &= U_2U_2^*U_2YU_1 = U_2YU_1 = Y.\end{aligned}$$

Replacing, in each of the relations

$$YU_1U_1^* = Y, \quad U_2U_2^*Y = Y,$$

the isometries by their n-th powers and passing to the limit, the proof follows. □

A bounded linear operator $Y\colon \mathscr{K}_1 \to \mathscr{K}_2$ with $Y = P(\mathscr{R}_2)YP(\mathscr{R}_1)$ satisfies the following boundedness condition. For arbitrary $k_1 \in \mathscr{K}_1, k_2 \in \mathscr{K}_2$

$$|(Yk_1, k_2)| \leqslant |Y|\ |P(\mathscr{R}_1)k_1|\,|P(\mathscr{R}_2)k_2|;$$

this estimate is stronger than boundedness in general—this is why the authors of [2] called it $\mathscr{R}$-boundedness.

Lemma 2.3. *Let $\mathscr{H}$ be a Hilbert space, T a contraction on $\mathscr{H}$. We denote by U the minimal isometric dilation of T acting on $\mathscr{K}$ and by $\mathscr{R}$ the U-reducing subspace of $\mathscr{K}$ on which U is unitary. Set $\mathscr{P} = (P(\mathscr{R})\mathscr{H})^-$. Then*

1° $\mathscr{R} = \mathscr{P} \oplus (\mathscr{R} \cap \mathscr{H}^\perp)$

2° $P(\mathscr{P})h = P(\mathscr{R})h$ *for* $h \in \mathscr{H}$

3° $P(\mathscr{H})P(\mathscr{P}) = P(\mathscr{H})P(\mathscr{R})$

4° $U^{*m}P(\mathscr{P})h = P(\mathscr{P})U^{*m}h$ *for* $h \in \mathscr{H}$ *and* $m \geqslant 0$.

5° $\mathscr{P}$ *is invariant with respect to* U^*

6° $\mathscr{R}$ *is the smallest* U *reducing subspace containing* $\mathscr{P}$.

Proof. If $x \in \mathscr{R} \ominus \mathscr{P}$ then $(x,h) = (P(\mathscr{R})x,h) = (x, P(\mathscr{R})h) = 0$ for every $h \in \mathscr{H}$ whence $x \in \mathscr{R} \cap \mathscr{H}^\perp$. It follows that $\mathscr{R} \ominus \mathscr{P} \subset \mathscr{R} \cap \mathscr{H}^\perp$. On the other hand suppose $x \in (\mathscr{R} \cap \mathscr{H}^\perp)$ and $x \perp (\mathscr{R} \ominus \mathscr{P})$. For each $h \in \mathscr{H}$

$$(x, P(\mathscr{R})h) = (P(\mathscr{R})x, h) = (x,h) = 0$$

since $x \in \mathscr{H}^\perp$. It follows that $x \in \mathscr{R},\ x \perp \mathscr{P}$, and $x \perp \mathscr{R} \ominus \mathscr{P}$ whence $x = 0$.

Given $h \in \mathscr{H}$, we have $P(\mathscr{R})h \in \mathscr{P}$. To show that $h - P(\mathscr{R})h \perp \mathscr{P}$ it suffices to show that $(h - P(\mathscr{R})h, P(\mathscr{R})h') = 0$ for every $h' \in \mathscr{H}$. This, however, is obvious.

Using 1°, we have

$$P(\mathscr{H})P(\mathscr{R}) = P(\mathscr{H})(P(\mathscr{P}) + P(\mathscr{R} \cap \mathscr{H}^\perp)) = P(\mathscr{H})P(\mathscr{P})$$

Since $U^{*m}h \in \mathscr{H}$ for $h \in \mathscr{H}$ and $m \geqslant 0$ it follows from 2° that

$$U^{*m}P(\mathscr{P})h = U^{*m}P(\mathscr{R})h = P(\mathscr{R})U^{*m}h = P(\mathscr{P})U^{*m}h.$$

The implication 4° → 5° is immediate. To prove 6°, denote by $\mathscr{M}$ the U reducing subspace of $\mathscr{R}$ containing $\mathscr{P}$. Given any p, we have

$$P(\mathscr{R})U^p h = U^p P(\mathscr{R})h \in \mathscr{M}$$

whence $\mathscr{R} = P(\mathscr{R})\mathscr{K} \subset \mathscr{M}$. □

Lemma 2.4. *Let T be a contraction, U its minimal isometric dilation. Set $A = P(\mathscr{R})\big|\mathscr{H}$, $\mathscr{P} = (P(\mathscr{R})\mathscr{H})^-$. Then*

$$\begin{aligned} AT^* &= (U^*\big|\mathscr{P})A \\ TA^* &= A^*(U^*\big|\mathscr{P})^*. \end{aligned}$$

P r o o f.

$$AT^*h = AU^*h = P(\mathscr{R})U^*h = U^*P(\mathscr{R})h = U^*Ah = (U^*\big|\mathscr{P})Ah.$$

The second relation follows by taking adjoints; it is instructive, however, to prove it directly. For $A^*\colon \mathscr{P} \to \mathscr{H}$ we have $A^* = (P(\mathscr{R})\big|\mathscr{H})^* = P(\mathscr{H})P(\mathscr{R})\big|\mathscr{P} = P(\mathscr{H})\big|\mathscr{P}$. For $p \in \mathscr{P}$ we have

$$\begin{aligned} TA^*p &= TP(\mathscr{H})p = P(\mathscr{H})Up = P(\mathscr{H})UP(\mathscr{R})p \\ &= P(\mathscr{H})P(\mathscr{R})Up = P(\mathscr{H})(P(\mathscr{P}) + P(\mathscr{R} \cap \mathscr{H}^\perp))Up \\ &= P(\mathscr{H})P(\mathscr{P})Up = P(\mathscr{H})(U^*\big|\mathscr{P})^*p \\ &= A^*(U^*\big|\mathscr{P})^*p. \end{aligned}$$

□

Lemma 2.4 makes it possible to reduce the study of generalized Toeplitz and Hankel operators to the case where T_1^* is the isometry $U_1^*\big|\mathscr{P}_1$ and T_2 the coisometry $(U_2^*\big|\mathscr{P}_2)^*$.

We shall need the following factorization lemma. To the best of the author's knowledge this lemma appears first in [2].

Lemma 2.5. *Suppose $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{M}_1$, $\mathscr{M}_2$ are two Hilbert spaces, $T\colon \mathscr{H}_1 \to \mathscr{H}_2$, $B_1\colon \mathscr{H}_1 \to \mathscr{M}_1, B_2\colon \mathscr{H}_2 \to \mathscr{M}_2$ bounded linear operators. Suppose that*

$$|(Th_1, h_2)| \leqslant |B_1h_1|\,|B_2h_2|$$

for all $h_1 \in \mathscr{H}_1, h_2 \in \mathscr{H}_2$. Then there exists a contraction $T_0 \colon \mathscr{M}_1 \to \mathscr{M}_2$ such that

$$T = B_2^* T_0 B_1$$

$$\begin{tikzcd}
\mathscr{H}_1 \arrow[r, "T"] \arrow[d, "B_1"'] & \mathscr{H}_2 \\
\mathscr{M}_1 \arrow[r, "T_0"'] & \mathscr{M}_2 \arrow[u, "B_2^*"']
\end{tikzcd}$$

P r o o f. See [2]. $\square$

Theorem 2.5. *Suppose $T_1 \in B(\mathscr{H}_1)$ and $T_2 \in B(\mathscr{H}_2)$ are two given contractions, U_1, U_2 their minimal isometric dilations acting on $\mathscr{K}_1, \mathscr{K}_2$ respectively. Suppose $X \colon \mathscr{H}_1 \to \mathscr{H}_2$ satisfies*

$$X = T_2 X T_1^*.$$

Then $(Xh_1, h_2) \leqslant |X|\,|P(\mathscr{R}_1)h_1|\ |P(\mathscr{R}_2)h_2|$. There exists exactly one $Y \colon \mathscr{K}_1 \to \mathscr{K}_2$ such that

$$\begin{aligned} &Y = U_2 Y U_1^* \\ &P(\mathscr{H}_2) Y \big| \mathscr{H}_1 = X. \end{aligned}$$

P r o o f. Given $h_1 \in \mathscr{H}_1, h_2 \in \mathscr{H}_2$ and a natural number n, we have

$$(Xh_1, h_2) = (T_2^n X T_1^{*n} h_1, h_2) = (X T_1^{*n} h_1, T_2^{*n} h_2)$$

so that

$$|(Xh_1, h_2)| \leqslant |X|\,|T_1^{*n} h_1|\,|T_2^{*n} h_2| = |X|\,|U_1^n U_1^{*n} h_1|\,|U_2^n U_2^{*n} h_2|$$

and, passing to the limit,

$$|(Xh_1, h_2)| \leqslant |X|\,|P(\mathscr{R}_1)h_1|\,|P(\mathscr{R}_2)h_2|.$$

It follows that there exists $X_0 \colon \mathscr{P}_1 \to \mathscr{P}_2$ such that $X = A_2^* X_0 A_1$ and $|X_0| \leqslant |X|$. The relations stated in the preceding lemma yield the identity

$$A_2^* X_0 A_1 = T_2 A_2^* X_0 A_1 T_1^* = A_2^* (U_2^* \big| \mathscr{P}_2)^* X_0 (U_1^* \big| \mathscr{P}_1) A_1$$

This gives, on $\mathscr{P}_1$, the identity

$$A_2^* X_0 = A_2^* (U_2^* \big| \mathscr{P}_2)^* X_0 (U_1^* \big| \mathscr{P}_1).$$

$$
\begin{tikzcd}
 & \mathscr{P}_1 \arrow[rr, "X_0"] & & \mathscr{P}_2 \arrow[dl, "(U_2^*|\mathscr{P}_2)^*"'] \arrow[dd, "A_2^*"] \\
\mathscr{P}_1 \arrow[ur, "U_1^*|\mathscr{P}_1"] \arrow[rr, "X_0"] & & \mathscr{P}_2 \arrow[dd, "A_2^*"] & \\
 & \mathscr{H}_1 \arrow[uu, "A_1"] \arrow[rr, "X"] & & \mathscr{H}_2 \arrow[dl, "T_2"] \\
\mathscr{H}_1 \arrow[uu, "A_1"] \arrow[ur, "T_1^*"] \arrow[rr, "X"'] & & \mathscr{H}_2 &
\end{tikzcd}
$$

Since A_2^* is injective on $\mathscr{P}_2$, the closure of the range of A_2, this identity implies

$$X_0 = (U_2^* \big| \mathscr{P}_2)^* X_0 (U_1^* \big| \mathscr{P}_1)$$

Setting $V_i = U_i^* \big| \mathscr{R}_i$, we now apply Proposition (1.5).

It follows that there exists an operator $C \colon \mathscr{R}_1 \to \mathscr{R}_2$ such that

$$\begin{aligned}
&C = V_2^* C V_1 \\
&P(\mathscr{P}_2) C \big| \mathscr{P}_1 = X_0 \\
&|C| = |X_0| = |X|.
\end{aligned}$$

Set $Y = CP(\mathscr{R}_1)$. We prove first that $Y = U_2 Y U_1^*$. Indeed,

$$U_2 Y U_1^* = U_2 C P(\mathscr{R}_1) U_1^* = U_2 C U_1^* P(\mathscr{R}_1) = V_2^* C V_1 P(\mathscr{R}_1) = CP(\mathscr{R}_1) = Y.$$

To see that the compression of Y is X, we argue as follows

$$\begin{aligned}
P(\mathscr{H}_2) Y &= P(\mathscr{H}_2) C P(\mathscr{R}_1) = P(\mathscr{H}_2)(P(\mathscr{P}_2) + P(\mathscr{R}_2 \cap \mathscr{H}_2{}^{\perp})) C P(\mathscr{R}_1) \\
&= P(\mathscr{H}_2) P(\mathscr{P}_2) C P(\mathscr{R}_1).
\end{aligned}$$

When applied to an element $h_1 \in \mathscr{H}_1$ this operator identity yields

$$P(\mathscr{H}_2) Y h_1 = P(\mathscr{H}_2) P(\mathscr{P}_2) C A_1 h_1 = P(\mathscr{H}_2) X_0 A_1 h_1 = A_2^* X_0 A_1 h_1 = X h_1.$$

Uniqueness follows from the fact that an operator satisfying the Toeplitz relation $Y = U_2 Y U_1^*$ is fully determined by its compression to $\mathscr{P}_1, \mathscr{P}_2$. $\square$

Theorem 2.6. *Suppose $X \colon \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator. Then these are equivalent:*

1° *there exists a bounded linear operator* $Y\colon \mathscr{K}_1 \to \mathscr{K}_2$ *such that* $YU_1^* = U_2Y$, $|Y| \leqslant \beta$ *and* $X = P(\mathscr{H}_2)Y\big|\mathscr{H}_1$

2° $XT_1^* = T_2X$ *and* $|(Xh_1, h_2)| \leqslant \beta|P(\mathscr{R}_1)h_1|\,|P(\mathscr{R}_2)h_2|$ *for all* $h_1 \in \mathscr{H}_1$, $h_2 \in \mathscr{H}_2$.

P r o o f. To see that 1° implies 2°, use lemma (2.2). Since $Y = P(\mathscr{R}_2)YP(\mathscr{R}_1)$ we have $(Xh_1, h_2) = (Yh_1, h_2) = (P(\mathscr{R}_2)YP(\mathscr{R}_1)\ h_1, h_2) = (YP(\mathscr{R}_1)\ h_1,\ P(\mathscr{R}_2)h_2)$ whence

$$|(Xh_1, h_2)| \leqslant \beta|P(\mathscr{R}_1)h_1|\,|P(\mathscr{R}_2)h_2|.$$

Furthermore, $XT_1^* = P(\mathscr{H}_2)YU_1^*\big|\mathscr{H}_1 = P(\mathscr{H}_2)U_2Y\big|\mathscr{H}_1 = T_2P(\mathscr{H}_2)Y\big|\mathscr{H}_1 = T_2X$.

Now assume 2°. The second assumption together with lemma (2.6) imply that there exists a $C_0\colon \mathscr{P}_1 \to \mathscr{P}_2$ such that $X = A_2^*C_0A_1$ and $|C_0| \leqslant \beta$. Furthermore,

$$A_2^*C_0(U_1^*\big|\mathscr{P}_1)A_1 = A_2^*C_0A_1T_1^* = XT_1^* = T_2X = T_2A_2^*C_0A_1 = A_2^*(U_2^*\big|\mathscr{P}_2)^*C_0A_1.$$

We have thus, on $\mathscr{P}_1$, the identity

$$A_2^*C_0(U_1^*\big|\mathscr{P}_1) = A_2^*(U_2^*\big|\mathscr{P}_2)^*C_0.$$

Since A_2^* is injective on the closure of the range of A_2, in other words on $\mathscr{P}_2$, it follows that

$$C_0(U_1^*\big|\mathscr{P}_1) = (U_2^*\big|\mathscr{P}_2)^*C_0.$$

$$\begin{tikzcd}
 & \mathscr{P}_1 \arrow[rr, "C_0"] & & \mathscr{P}_2 \arrow[dd, "A_2^*"] \\
\mathscr{P}_1 \arrow[ur, "U_1^*|\mathscr{P}_1"] \arrow[rr, "C_0"'] & & \mathscr{P}_2 \arrow[ur, "(U_2^*|\mathscr{P}_2)^*"] \arrow[dd, "A_2^*"] & \\
 & \mathscr{H}_1 \arrow[uu, "A_1"'] \arrow[rr, "X"'] & & \mathscr{H}_2 \\
\mathscr{H}_1 \arrow[uu, "A_1"] \arrow[ur, "T_1^*"] \arrow[rr, "X"'] & & \mathscr{H}_2 \arrow[ur, "T_2"'] &
\end{tikzcd}$$

Since C_0 is of type H, Proposition 1.7 yields the existence of an operator C: $\mathscr{R}_1 \to \mathscr{R}_2$ such that $C(U_1^*\big|\mathscr{R}_1) = U_2C$ on $\mathscr{R}_1$, $|C| = |C_0|$ and $P(\mathscr{P}_2)C\big|\mathscr{P}_1 = C_0$.

Now we prove an operator identity:

$$\begin{aligned}
&P(\mathscr{H}_2)CP(\mathscr{R}_1) = P(\mathscr{H}_2)P(\mathscr{R}_2)CP(\mathscr{R}_1)\\
&= P(\mathscr{H}_2)(P(\mathscr{P}_2) + P(\mathscr{R}_2 \cap \mathscr{H}_2^{\perp}))CP(\mathscr{R}_1)\\
&= P(\mathscr{H}_2)P(\mathscr{P}_2)CP(\mathscr{R}_1).
\end{aligned}$$

When applied to $h \in \mathscr{H}_1$ this operator identity yields

$$P(\mathscr{H}_2)CP(\mathscr{R}_1)h = P(\mathscr{H}_2)P(\mathscr{P}_2)CP(\mathscr{R}_1)h = P(\mathscr{H}_2)C_0A_1h = A_2^*C_0A_1h = Xh.$$

Now define $Y\colon \mathscr{K}_1 \to \mathscr{R}_2$ by the formula $Y = CP(\mathscr{R}_1)$. The preceding identity shows that the compression of Y is X. It remains to show that $U_2Y = YU_1^*$. This is immediate, since

$$U_2Y = U_2CP(\mathscr{R}_1) = CU_1^*P(\mathscr{R}_1) = CP(\mathscr{R}_1)U_1^*.$$

The norm it Y is bounded by β since

$$|Y| = |CP(\mathscr{R}_1)| \leqslant |C| = |C_0| \leqslant \beta.$$

$\square$

We conclude with a few comments on the condition of $\mathscr{R}$-boundedness. In the case of the classical Hankel operator, we have

$$\mathscr{H}_1 = H^2, \quad T_1 = S^*$$
$$\mathscr{H}_2 = H^2_-, \quad T_2 = (V^*\big|H^2_-)^*$$

where V denotes the multiplication by z on L^2 and $S = P_+V\big|H^2$. Thus $U_1 = V^*$ on L^2 and $U_2 = V$ on L^2. It follows that both U_1 and U_2 are unitary so that both $P(\mathscr{R}_1)$ and $P(\mathscr{R}_2)$ are identities on L^2 and the condition of $\mathscr{R}$-boundedness is automatically fulfilled.

To show that an operator X satisfying $XT_1^* = T_2X$ may fail to possess a dilation Y with $YU_1^* = U_2Y$ it suffices, by Theorem (2.6), to produce an example of a nonzero X with $XT_1^* = T_2X$ and such that either $\mathscr{R}_1$ or $\mathscr{R}_2$ is zero.

If T is the zero operator on $\mathscr{H}$ then U is the shift operator on $H^2(\mathscr{H})$ so that $\mathscr{R} = 0$. It follows that, in the case that T_1 and T_2 are both zero operators on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, any operator $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ is generalized Hankel and X cannot be $\mathscr{R}$-bounded unless $X = 0$.

E x a m p l e. Consider two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{M}$ and define T_1 to be zero on $\mathscr{H}_1$ so that U_1 is the shift operator S_1 on $H^2(\mathscr{H}_1)$.

Denote by S the shift operator on $H^2(\mathscr{M})$ and set $\mathscr{H}_2 = H^2(\mathscr{M})$ and $T_2 = S^*$. Thus U_2 equals V_2^* on $L^2(\mathscr{M})$. Let X be any nonzero operator from $\mathscr{H}_1$ into $\mathscr{H}_2$ with range within $\mathscr{M}$. It follows that

$$T_2X = S^*X = 0 \quad \text{so that} \quad T_2X = XT_1.$$

If $Y\colon H^2(\mathscr{H}_1) \to L^2(\mathscr{M})$ satisfies $Y = V_2YS_1^*$ then, for each $y \in H^2(\mathscr{H}_1)$,

$$|Yy| = |V_2^nYS_1^{*n}y| \leqslant |Y|\,|S_1^{*n}y| \to 0.$$

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